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Stable ample 2-vector bundles on Hirzebruch surfaces

by
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Abstract

We discuss stability conditions for all rank-2 ample vector bundles on Hirzebruch surfaces with the second Chern class less than 7.

Key Words: Rank 2-vector bundles, Hirzebruch surface, stability conditions, Chern classes.
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1 Introduction

Ample vector bundles with small Chern numbers on surfaces have been studied by several authors; see [BL], [LS], [N] and [Is]. Ishihara studied ample vector bundles of rank 2 on a Hirzebruch surface. On the other hand, the notion of stability has proved to be crucial in the construction of moduli spaces of vector bundles with given numerical invariants.

In this paper, we discuss the stability of rank-2 ample vector bundles on Hirzebruch surfaces with the second Chern class less than 7. We mainly rely on a classification theorem obtained by Ishihara [Is] and a stability criterion proved by Aprodu and Brînzănescu [A-Br1].

2 Hirzebruch surfaces

We shall use classical notations and facts on Hirzebruch surfaces, and we refer to [Har] or [A-Br1] for more details.

Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \stackrel{\pi}{\to} \mathbb{P}^1$ be a Hirzebruch surface with invariant $e \geq 0$. Let C_0 be a section of X with $\mathcal{O}_X(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$ and let f_0 be a fixed fibre. Then ${C_0}^2 = -e$, $C_0 f_0 = 1$ and ${f_0}^2 = 0$.

Let E be a rank-2 algebraic vector bundle on X with fixed numerical Chern classes $c_1 = (\alpha, \beta) \in H^2(X, \mathbb{Z}) \cong \mathbb{Z}^2$ and $c_2 = \gamma \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$, where $\alpha, \beta, \gamma \in \mathbb{Z}$. Since the fibres of π are isomorphic to $\mathbb{P}^1$, we can speak about the generic splitting type of E; for a general fibre f of π, we have

$$E|_f \cong \mathcal{O}_f(d) \oplus \mathcal{O}_f(d'),$$

with $d' \leq d$ and $d + d' = \alpha$. We will call d the first numerical invariant of E, and we define the second invariant of E (see [Br]) to be:

$$-r = inf\{\, l \,|\, \exists\, L \in Pic(\mathbb{P}^1),\, \deg L = l,\ s.t.\ H^0(X, E(-dC_0) \oplus \pi^* L) \neq 0\}.$$

Since $Pic_0(\mathbb{P}^1)$ is trivial, the following result (see [Br]) takes the form:

Theorem 1. *(see [Br]) For every E rank-2 vector bundle on a Hirzebruch surface X, with fixed Chern classes $c_1 = (\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$, $c_2 \in \mathbb{Z}$ and invariants d and r, there exist $Y \subset X$ a locally complete intersection of codimension 2 in X, or the empty set, such that E is given by the extension:*

$$0 \to \mathcal{O}_X(dC_0 + rf_0) \to E \to \mathcal{O}_X(d'C_0 + sf_0) \otimes I_Y \to 0, \quad (\bigstar)$$

where $d + d' = \alpha$, $r + s = \beta$ and $\deg(Y) = c_2 + \alpha(de - r) - \beta d + 2dr - d^2 e \geq 0$.

3 Ampleness on Hirzebruch surfaces

The aim of this section is to present some generalities of ample vector bundles on Hirzebruch surface (see [Har] and [Is]).

Given a divisor $D = aC_0 + bf_0$ on a Hirzebruch surface, it is well known that D is ample if and only if it is very ample, if and only if $a > 0$ and $b > ae$. We also know that a divisor $D = a'C_0 + b'f_0$ is effective if and only if $a' \geq 0$ and $b' \geq 0$.

Definition 1. *Let E be a vector bundle on X, $\mathbb{P}(E)$ the associated projective bundle and H_E the tautological line bundle on $\mathbb{P}(E)$. We say that E is ample if H_E is ample.*

Ample rank-2 vector bundles on Hirzebruch surfaces $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ were classified by Ishihara in [Is]. Since the decomposable bundles are not stable, we express the results of Ishihara only for indecomposables bundles:

Theorem 2. *(see [Is]) Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ be a Hirzebruch surface and let E be an indecomposable ample vector bundle of rank-2 on X. Then $c_2(E) \geq e + 5$. Moreover, if $c_2(E) \leq e + 6$ then one of the following cases occurs:*

1. *$e = 1$, $c_2 = 6$ and E is a non-trivial extension*
   $0 \to \mathcal{O}(2C_0 + 2f_0) \to E \to \mathcal{O}(C_0 + 3f_0) \to 0$;

2. *$e = 0$, $c_2 = 6$ and E is a non-trivial extension*
   $0 \to \mathcal{O}(2C_0) \to E \to \mathcal{O}(C_0 + 3f_0) \to 0$;

3. *$e = 1$, $c_2 = 7$ and E is a non-trivial extension*
   $0 \to \mathcal{O}(2C_0 + f_0) \to E \to \mathcal{O}(C_0 + 4f_0) \to 0$;

4. *$e = 2$, $c_2 = 8$ and E is a non-trivial extension*
   $0 \to \mathcal{O}(2C_0 + 4f_0) \to E \to \mathcal{O}(C_0 + 4f_0) \to 0$.

4 Stability

In this section we recall some facts concerning the stability in the sense of Mumford-Takemoto of vector bundles on the Hirzebruch surfaces X. For more details we refer to [A-Br1], [Mr] and [Q2].

Definition 2. *Let H be an ample line bundle on a smooth projective surface X. For a torsion free sheaf E on X we set:*

$$\mu(E) = \mu_H(E) := \frac{c_1(E)H}{rk(E)}.$$

The sheaf E is called semistable with respect to H if

$$\mu_H(G) \leq \mu_H(E)$$

for all non-zero subsheaves $G \subset E$ with $rk(G) < rk(E)$. If strict inequality holds, then E is said to be stable with respect to H.

We shall also use the description of Qin (see [Q1] or [Fr]) for walls and chambers.
A very useful criterion for stability in the case of rank-2 vector bundles over ruled surfaces was found by Aprodu and Brînzanescu in [A-Br1]. We remind now this criterion in the case of Hirzebruch surfaces, which will be used later.

Theorem 3. *(see [A-Br1]) Let E be a rank two vector bundle over a Hirzebruch surface, with numerical invariants as in theorem 1. Then, there exists an ample line bundle H such that E is H stable if and only if $2r < \beta$ and the extension ($\bigstar$) of E is non-splitting.*

5 Main Result

Using the tools presented in the above sections we are now able to state and prove our main result:

Theorem 4. *Let $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ be a Hirzebruch surface. Then, the only rank-2 ample stable vector bundles on X with $c_2(E) \leq 6$ or $c_2(E) \leq 7$ and $e \geq 1$ are given by the following exact sequences:*

1. $0 \to \mathcal{O}(2C_0 + 2f_0) \to E \to \mathcal{O}(C_0 + 3f_0) \to 0$ *for* $e = 1$, $(c_2(E) = 6)$;
2. $0 \to \mathcal{O}(2C_0) \to E \to \mathcal{O}(C_0 + 3f_0) \to 0$ *for* $e = 0$ $(c_2(E) = 6)$.
3. $0 \to \mathcal{O}(2C_0 + f_0) \to E \to \mathcal{O}(C_0 + 4f_0) \to 0$ *for* $e = 1$ $(c_2(E) = 7)$.

Proof: According to theorem 2 (or see [Is]), there are four cases to discuss.
Following the idea of the proof of remark 1 in [Br], it is easy to see that all the four extensions from theorem 2 coincide with the extension ($\bigstar$) of the corresponding 2-vector bundles E's. We show this fact only for the first case, the arguments used in the proofs of other cases being quite similar.
Case A. The ample rank-2 vector bundles E constructed from the extension

$$0 \to \mathcal{O}(2C_0 + 2f_0) \to E \to \mathcal{O}(C_0 + 3f_0) \to 0, \tag{1}$$

are non-splitting, since $Ext^1(\mathcal{O}(C_0+3f_0),\mathcal{O}(2C_0+2f_0)) \neq 0$. In this case, we know that $e=1$, $c_1(E)=3C_0+5f_0$ and $c_2(E)=6$. Obviously, $\alpha=3, \beta=5$ and $\gamma=6$. By restricting the exact sequence (1) to a general fibre f we obtain

$$0 \longrightarrow \mathcal{O}_f(2) \longrightarrow E|_f \longrightarrow \mathcal{O}_f(1) \longrightarrow 0,$$

and since $H^1(f,\mathcal{O}_f(1))=0$, it follows that $E|_f=\mathcal{O}_f(2)\oplus\mathcal{O}_f(1)$, and so using the notations from theorem 1, we get $d=2$ and $d'=1$.
Considering the equivalent form of (1):

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E(-2C_0-2f_0) \longrightarrow \mathcal{O}_X(-C_0+f_0) \longrightarrow 0 \qquad (2)$$

and looking to the corresponding long exact sequence of cohomology we obtain the injective map:

$$0 \longrightarrow H^0(X,\mathcal{O}_X) \longrightarrow H^0(X,E(-2C_0-2f_0)).$$

Thus, $H^0(X,E(-2C_0-2f_0)) \neq 0$. To show that the second invariant of E is $r=2$, we must verify that $H^0(X,E(-2C_0-(2+m)f_0))=0$ for every $m \geq 1$. Tensorising (2) by $\mathcal{O}_X(-mf_0)$ it follows:
$0 \longrightarrow H^0(X,\mathcal{O}_X(-mf_0)) \longrightarrow H^0(X,E(-2C_0-(2+m)f_0)) \longrightarrow$
$\longrightarrow H^0(X,\mathcal{O}_X(-C_0+(1-m)f_0)) \longrightarrow \dots$.
By the projection formula:

$$H^0(X,\mathcal{O}_X(-C_0+(1-m)f_0)) \cong H^0(\mathbb{P}^1,\pi_*(\mathcal{O}_X(-C_0))\otimes\mathcal{O}_{\mathbb{P}^1}(1-m)f_0))=0,$$

since $\pi_*(\mathcal{O}_X(-C_0))=0$.
Therefore,

$$H^0(X,E(-2C_0-(2+m)f_0)) \cong H^0(X,\mathcal{O}_X(-mf_0)) \cong H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-m))=0,$$

since $m \geq 1$. We can deduce now that $r=2$ and $s=3$.
Moreover, for all these invariants it follows that $\deg Y=0$, and thus $Y=\emptyset$.
We conclude that E sits in an extension of type (★).
According to theorem 3, E will be stable if and only if will verify:

$$2r < \beta,$$

which in our case this give $4<5$. Hence E est stable.
We will proceed in the same way with the other 3 cases who left.
Case B. Consider the extension:

$$0 \to \mathcal{O}(2C_0) \to E \to \mathcal{O}(C_0+3f_0) \to 0. \qquad (3)$$

In this case:

$$e=0,\ c_1(E)=3C_0+3f_0,\ c_2(E)=6$$

and the coefficients are:

$$d=2,\ r=0,\ d'=1,\ s=3,\ \alpha=3,\ \beta=3.$$

An easy computation shows that $Ext^1(\mathcal{O}(C_0+3f_0), \mathcal{O}(2C_0)) \neq 0$, so there are non-splitt bundles given by sequence (3). Since $2r = 0$ and $\beta = 3$, it follows that E is stable.
Case C. For E gived by the non-trivial extension:

$$0 \to \mathcal{O}(2C_0 + f_0) \to E \to \mathcal{O}(C_0 + 4f_0) \to 0,$$

we have:

$$e = 1,\ c_1(E) = 3C_0 + 5f_0,\ c_2(E) = 7,$$

$$d = 2,\ r = 1,\ d' = 1,\ s = 4,\ \alpha = 3,\ \beta = 5.$$

Again, the inequality $2r < \beta$ holds, and so E is stable.
Case D. In the last case E is given by the non-splitting extension:

$$0 \to \mathcal{O}(2C_0 + 4f_0) \to E \to \mathcal{O}(C_0 + 4f_0) \to 0.$$

For this case

$$e = 2,\ c_1(E) = 3C_0 + 8f_0,\ c_2(E) = 8,$$

$$d = 2,\ r = 4,\ d' = 1,\ s = 4,\ \alpha = 3,\ \beta = 8.$$

Since $2r = \beta = 8$, we conclude that E is not stable. □

The next result describe the polarizations for which the ample rank-2 vector bundles from the above theorem are stable.

Theorem 5. *Let $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ be a Hirzebruch surface and $H = aC_0 + bf_0$ an ample line bundle.*

1. *If $e = 1$ and $a < b < 2a$, then the ample rank-2 vector bundle E given by the non-trivial extension:*

   $$0 \to \mathcal{O}(2C_0 + 2f_0) \to E \to \mathcal{O}(C_0 + 3f_0) \to 0$$

   is H stable.

2. *If $e = 0$ and $b < 3a$, then the ample rank-2 vector bundle E given by the non-trivial extension*

   $$0 \to \mathcal{O}(2C_0) \to E \to \mathcal{O}(C_0 + 3f_0) \to 0$$

   is H stable.

3. *If $e = 1$ and $a < b < 4a$, then the ample rank-2 vector bundle E given by the non-trivial extension*

   $$0 \to \mathcal{O}(2C_0 + f_0) \to E \to \mathcal{O}(C_0 + 4f_0) \to 0;$$

   is H stable.

Proof: According to Theorem 1.2.3 from Qin (see [Q1]), we know that if E is a rank-2 vector bundle given by a non-trivial extension of the form

$$0 \to \mathcal{O}_X(G) \to E \to \mathcal{O}_X(c_1 - G) \otimes I_Y \to 0,$$

and if we choose a wall $\zeta := 2G - c_1$ then:
E will be H-stable for every ample line bundle H who verify $H.\zeta < 0$;

1. In this case $\zeta = C_0 - f_0$, so

$$H.\zeta < 0 \Longleftrightarrow (aC_0 + bf_0)(C_0 - f_0) < 0 \Longleftrightarrow$$
$$a{C_0}^2 - aC_0 f_0 + bf_0 C_0 - b{f_0}^2 < 0 \Longleftrightarrow -2a + b < 0,$$
$$\text{since } C_0^2 = -1,\ C_0 f_0 = 1 \text{ and } {f_0}^2 = 0.$$

2. For the second case $\zeta = C_0 - 3f_0$ and hence:

$$H.\zeta < 0 \Longleftrightarrow (aC_0 + bf_0)(C_0 - 3f_0) < 0 \Longleftrightarrow$$
$$a{C_0}^2 - 3aC_0 f_0 + bf_0 C_0 - 3b{f_0}^2 < 0 \Longleftrightarrow -3a + b < 0,$$
$$\text{since } C_0^2 = 0,\ C_0 f_0 = 1 \text{ and } {f_0}^2 = 0.$$

3. In the third case $\zeta = C_0 - 3f_0$, so we get:

$$H.\zeta < 0 \Longleftrightarrow (aC_0 + bf_0)(C_0 - 3f_0) < 0 \Longleftrightarrow$$
$$a{C_0}^2 - 3aC_0 f_0 + bf_0 C_0 - 3b{f_0}^2 < 0 \Longleftrightarrow -4a + b < 0,$$
$$\text{since } C_0^2 = -1,\ C_0 f_0 = 1 \text{ and } {f_0}^2 = 0.$$

□

In the next paragraph we verify directly the stability for the first case of vector bundles from theorem 5 with respect to a fixed polarization:
Example. Let X be a Hirzebruch surface with $e = 1$, and $H = 2C_0 + 3f_0$ an ample line bundle on X. We will check directly that the nonsplit ample rank-2 vector bundles E on X which sit in the non-trivial extension:

$$0 \to \mathcal{O}(2C_0 + 2f_0) \to E \to \mathcal{O}(C_0 + 3f_0) \to 0 \ (\dagger)$$

are μ stable with respect to H. That is for any rank 1 subbundle $\mathcal{O}_X(D)$ of E we have

$$c_1(\mathcal{O}_X(D))H < \frac{c_1(E)H}{2} = \frac{(3C_0 + 5f_0)(2C_0 + 3f_0)}{2} = \frac{13}{2}.$$

Since E is given by the exact sequence (†), we have:

1. $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}(2C_0 + 2f_0)$ or

2. $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}(C_0 + 3f_0)$.

In the first case $(2C_0 - 2f_0) - D \geq 0$ (i.e. effective divisor). Since H is an ample line bundle, it's clear that $(2C_0 - 2f_0 - D)H \geq 0$ and hence we have $c_1(\mathcal{O}_X(D))H = DH \leq (2C_0 + 2f_0)H = (2C_0 + 2f_0)(2C_0 + 3f_0) = 6 < \frac{13}{2} = \frac{c_1(E)H}{2}$.
In the second case, we consider $D = \alpha C_0 + \beta f_0$.
Since $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}(C_0 + 3f_0)$, it results that $(C_0 + 3f_0) - D = (1 - \alpha)C_0 + (3 - \beta)$ is an effective divisor, and we get $\alpha \leq 1$ and $\beta \leq 3$. But since E is given by an extension which does not split, this implies that $(\alpha, \beta) \neq (1, 3)$.
In conclusion, we have:

$$(\alpha C_0 + \beta f_0)(2C_0 + 3f_0) = \alpha + 2\beta < \frac{13}{2} = \frac{c_1(E)H}{2}.$$

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