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A Schreier domain type condition

by
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Abstract

We study the integral domains D satisfying the following condition: whenever $I \supseteq AB$ with I, A, B nonzero ideals, there exist ideals $A' \supseteq A$ and $B' \supseteq B$ such that $I = A'B'$.

Key Words: Schreier domain, pseudo-Dedekind domain, Prüfer domain.
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In [6], Cohn introduced the notion of Schreier domain. A domain D is said to be a *Schreier domain* if (1) D is integrally closed and (2) whenever I, J_1, J_2 are principal ideals of D and $I \supseteq J_1J_2$, then $I = I_1I_2$ for some principal ideals I_1, I_2 of D with $I_i \supseteq J_i$ for $i = 1, 2$. The study of Schreier domains was continued in [13] and [17] (where a domain was called a *pre-Schreier domain* if it satisfies condition (2) above). In [8] and [3], an extension of the class of pre-Schreier domains was studied. A domain D was called a *quasi-Schreier domain* if whenever I, J_1, J_2 are invertible ideals of D and $I \supseteq J_1J_2$, then $I = I_1I_2$ for some (invertible) ideals I_1, I_2 of D with $I_i \supseteq J_i$ for $i = 1, 2$.

In this paper we study the domains satisfying a Schreier-like condition for all nonzero ideals. Since this class of domains turns out to be rather narrow, we use an ad hoc name for it.

Definition 1 *We call a domain D a* sharp domain *if whenever $I \supseteq AB$ with I, A, B nonzero ideals of D, there exist ideals $A' \supseteq A$ and $B' \supseteq B$ such that $I = A'B'$.*

If the domain D is Noetherian or Krull, then D is sharp if and only if D is a Dedekind domain (Corollaries 2 and 12). In Proposition 4, we show that a sharp domain is pseudo-Dedekind. In particular, a sharp domain is a completely integrally closed GGCD domain. The ring E of entire functions is pseudo-Dedekind but not sharp (Example 8). Recall (cf. [16] and [4]) that a domain D is called a *pseudo-Dedekind domain* (the name used in [16] was *generalized Dedekind domain*) if the v-closure of each nonzero ideal of D is invertible. Also, recall from [2] that a domain D is called a *generalized GCD domain (GGCD domain)* if the v-closure of each nonzero finitely generated ideal of D is invertible. The definition of the v-closure is recalled below. In Proposition 6, we show that a valuation domain is sharp if and only if the value group of D is a complete subgroup of the reals.

The main results of this paper are Theorems 11 and 15. In Theorem 11, we show that the localizations of a sharp domain at the maximal ideals are valuation domains with value group a complete subgroup of the reals. In particular, a sharp domain is a Prüfer domain of dimension ≤ 1. A key point in proving Theorem 11 is the fact that if D is a sharp domain and $x, y \in D - \{0\}$ such that $xD \cap yD = xyD$, then $xD + yD = D$ (Proposition 10). The converse of Theorem 11 is not true (Example 13). In Theorem 15, we prove the converse of Theorem 11 for the domains of finite character (i.e., domains whose every nonzero element is contained in only finitely many maximal ideals). The problem whether a sharp domain is of finite character is left open. A countable sharp domain is a Dedekind domain (Corollary 17).

For reader's convenience, we recall the following facts. Let D be a domain with quotient field K and I a nonzero fractional ideal of D. The *v-closure* of I is the fractional ideal $I_v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in K |\ xI \subseteq D\}$, and I is called a *v-ideal* if $I = I_v$. The *t-closure* of I is the fractional ideal I_t which is the union of the v-closures of the finitely generated nonzero subideals of I. Moreover, I is called a *t-ideal* if $I = I_t$. In general, we have $I \subseteq I_t \subseteq I_v$. A nonzero prime ideal P of D is called *t-prime* if $P = P_t$. For basic facts and terminology not recalled in this paper, our references are [10] and [11]. Throughout this paper, all rings are domains, that is, commutative, unitary and without zero-divisors.

We begin with a characterization of the sharp domains. If I, H are ideals of a domain D, we denote by $I : H$ the ideal $\{x \in D \mid xH \subseteq I\}$.

Proposition 2 *A domain D is sharp if and only if for every two nonzero ideals I, H we have $I = [I : (I : H)](I : H)$.*

Proof: ($\Rightarrow$). Let I, H be nonzero ideals of D. Set $A = I : (I : H)$ and $B = I : H$. Note that $AB \subseteq I$, $A = I : B$ and $I : A = I : (I : (I : H)) = I : H = B$. As D is sharp, there exists a factorization $I = A'B'$ with A',B' ideals such that $A' \supseteq A$ and $B' \supseteq B$. Then $A \subseteq A' \subseteq I : B' \subseteq I : B = A$, so $A = A'$. Similarly, we get $B = B'$. ($\Leftarrow$). Let I, A, B be nonzero ideals of D such that $AB \subseteq I$. By our assumption, we get $I = [I : (I : A)](I : A)$. Note that $A \subseteq I : (I : A)$ and $B \subseteq I : A$. □

Corollary 3 *A Dedekind domain is a sharp domain.*

Proof: Let D be a Dedekind domain and I, H nonzero ideals of D. Since $I : H$ is an invertible ideal, it follows easily that $I : (I : H) = I(I : H)^{-1}$. Hence $[I : (I : H)](I : H) = I(I : H)^{-1}(I : H) = I$. Apply Proposition 2. □

Proposition 4 *Every sharp domain is pseudo-Dedekind. In particular, a sharp domain is a completely integrally closed GGCD domain.*

Proof: Let D be a sharp domain, A a nonzero ideal of D and $0 \neq b \in A$. By Proposition 2, $bD = [bD : (bD : A)](bD : A)$. It follows that $bD : A = bA^{-1}$ is an invertible ideal, so A^{-1} and A_v are invertible ideals. Thus D is pseudo-Dedekind. By [16, Corollaries 1.4 and 1.5], a pseudo-Dedekind domain is a completely integrally closed GGCD domain. □

We show that for a pseudo-Dedekind domain D it suffices to test the condition in Definition 1 only for ideals I with $I_v = D$.

Proposition 5 *A pseudo-Dedekind domain D is sharp if and only if for all nonzero ideals I,A,B of D such that $I \supseteq AB$ and $I_v = D$, there exist ideals $A' \supseteq A$ and $B' \supseteq B$ such that $I = A'B'$.*

Proof: We prove the nontrivial implication. Let I, A, B be nonzero ideals of D such that $I \supseteq AB$. Then $I_v \supseteq A_v B_v$ and I_v, A_v, B_v are invertible ideals, because D is pseudo-Dedekind. A pseudo-Dedekind domain is a GGCD domain, cf. [16, Corollary 1.5], and a GGCD domain is quasi-Schreier, cf. [8, Proposition 2.3]. So there exist invertible ideals $A_1 \supseteq A_v$ and $B_1 \supseteq B_v$ such that $I_v = A_1 B_1$. We have $I^{-1} = A_1^{-1} B_1^{-1}$, so $II^{-1} \supseteq (AA_1^{-1})(BB_1^{-1})$ and AA_1^{-1}, BB_1^{-1} are integral ideals. Since I_v is invertible, $(II^{-1})_v = D$. By our hypothesis, there exist ideals $A_2 \supseteq AA_1^{-1}$ and $B_2 \supseteq BB_1^{-1}$ such that $II^{-1} = A_2 B_2$. Hence $I = (A_1A_2)(B_1B_2)$ and $A_1A_2 \supseteq A$, $B_1B_2 \supseteq B$. □

Next, we characterize the sharp valuation domains. Recall [5, Exercise 21, page 551], that a *pseudo-principal domain* is a domain whose v-ideals are principal. Clearly, a quasi-local domain is pseudo-Dedekind if and only if it is pseudo-principal.

Proposition 6 *For a valuation domain D, the following assertions are equivalent:*

(a) *D is sharp.*

(b) *D is pseudo-Dedekind.*

(c) *the value group of D is a complete subgroup of the reals.*

In particular, a sharp valuation domain has dimension ≤ 1.

Proof: $(b) \Leftrightarrow (c)$ is given in [4] at the bottom of pages 325 and 327 and $(a) \Rightarrow (b)$ follows from Proposition 4. We prove that (b) and (c) imply (a). By Corollary 3, we may assume that the value group of D is the whole group of real numbers. By Proposition 5, D is sharp, because the maximal ideal is the only proper ideal of D whose v-closure is D. The "in particular" assertion follows from the well-known fact that a valuation domain has dimension ≤ 1 if and only if its value group is a subgroup of the reals (see [18, page 45]). □

Proposition 7 *If D is a sharp domain, then every fraction ring D_S of D is also a sharp domain.*

Proof: Let I, A, B be nonzero ideals of D such that $ID_S \supseteq ABD_S$. Then $H = ID_S \cap D \supseteq AB$. As D is sharp, we get $H = A'B'$ with A', B' ideals of D such that $A' \supseteq A$ and $B' \supseteq B$. Then $ID_S = HD_S = A'B'D_S$. □

Example 8 *The ring E of entire functions is pseudo-Dedekind but some localization of E is not pseudo-Dedekind, cf. [16, Example 2.1]. By Proposition 7, E is not a sharp domain.*

Proposition 9 *If D is a sharp domain and P is a t-prime ideal of D, then D_P is a valuation domain whose value group is a complete subgroup of the reals. In particular, in a sharp domain every t-prime ideal of D has height one.*

Proof: By Proposition 4, D is a GGCD domain. By [2, page 218], [14, Corollary 4.3] and Proposition 7, D_P is a sharp valuation domain. Apply Proposition 6. ∎

Recall that two nonzero elements x, y of a domain D are called *v-coprime* if $(xD + yD)_v = D$ (equivalently $xD \cap yD = xyD$, equivalently $xD : yD = xD$).

Proposition 10 *Let D be a sharp domain and x, y two nonzero v-coprime elements. Then $xD + yD = D$.*

Proof: We have $(x, y)^2 \subseteq (x^2, y)$, so $(x^2, y) = AB$ with A, B ideals such that $A, B \supseteq (x, y)$. Note that $(x^2, y) : (x, y) = (x, y)$. Indeed, if $a \in (x^2, y) : (x, y)$, then $ax = bx^2 + cy$ for some $b, c \in D$, so $c \in xD : yD = xD$, hence $a = bx + (c/x)y$ belongs to (x, y). From $(x^2, y) = AB$, we get $A \subseteq (x^2, y) : B \subseteq (x^2, y) : (x, y) = (x, y)$, so $A = (x, y)$. Similarly, we get $B = (x, y)$. Then $(x^2, y) = (x, y)^2$. So $y = fx + gy^2$ for some $f, g \in D$, hence $f \in yD : xD = yD$, thus $1 = (f/y)x + gy$, that is, $xD + yD = D$. ∎

Theorems 11 and 15 are the main results of this paper.

Theorem 11 *If D is a sharp domain, then D_M is a valuation domain with value group a complete subgroup of the reals, for each maximal ideal M of D. In particular, a sharp domain is a Prüfer domain of dimension ≤ 1.*

Proof: By Proposition 7, we may assume that D is quasi-local with nonzero maximal ideal M. Suppose that the height of M is ≥ 2. By Proposition 4, D is a quasi-local GGCD domain, hence a GCD domain, cf. [2, Corollary 1]. By Proposition 9, M is not a t-ideal, so $M_t = D$. Since D is a GCD domain, there exist two v-coprime elements $x, y \in M$ (see the paragraph before Theorem 4.8 in [1]). But this contradicts Proposition 10. It remains that M has height one, hence it is a t-prime, cf. [11, Proposition 6.6]. Now apply Proposition 9 to conclude. The "in particular" assertion is clear. ∎

According to [12], a *TV domain* is a domain in which every t-ideal is a v-ideal. Noetherian domains and Krull domains are TV domains, cf. [12, page 291].

Corollary 12 *If D is a sharp TV domain, then D is a Dedekind domain. In particular, if a sharp domain is Noetherian or Krull, then it is a Dedekind domain.*

Proof: Let D be a sharp TV domain. By Theorem 11, D is a Prüfer domain, so every nonzero ideal of D is a t-ideal, hence a v-ideal, because D is a TV domain. Since D is also a pseudo-Dedekind domain (cf. Proposition 4), it follows that every nonzero ideal of D is invertible. Thus D is a Dedekind domain. ∎

The converse of Theorem 11 is not true. Recall [10, page 434] that a domain D is said to be *almost Dedekind* if D_M is a discrete (Noetherian) valuation domain for each maximal ideal M of D. We exhibit an almost Dedekind domain which is not a sharp domain (not even pseudo-Dedekind).

Example 13 *Let D be the almost Dedekind domain constructed in the proof of [7, Proposition 7]. We recall some properties of D proved there. The maximal ideals of D are the principal ideals $(p_iD)_{i\geq 1}$ and the ideal $M = (q_0, q_1, ..., q_n, ...)$. Here $(q_i)_{i\geq 0}$ are nonzero elements of D such that $q_{i-1} = p_iq_i$ and p_i does not divide q_i for all $i \geq 1$. Note that M is not finitely generated, because it is the union of the strictly ascending chain of principal ideals $(q_iD)_{i\geq 0}$. We claim that D is not pseudo-Dedekind, so it is not a sharp domain (cf. Proposition 5). For that, it suffices to prove that the v-ideal $\cap_{i\geq 1}p_{2i-1}D$ equals the union of the strictly ascending chain of principal ideals $p_1q_2D \subset p_1p_3q_4D \subset p_1p_3p_5q_6D \cdots$, so it is not finitely generated. Indeed, the inclusion $\supseteq$ is clear. Conversely, let $x \in \cap_{i\geq 1}p_{2i-1}D$. If $x \notin M$, then $1 = ax + bq_{2n}$ for some $a, b \in D$ and $n \geq 0$. But this is a contradiction, because p_{2n+1} divides both x and q_{2n}. So $x \in M$, say $x = cq_{2n}$ for some $c \in D$ and $n \geq 1$. Since $x \in \cap_{i\geq 1}p_{2i-1}D$ and q_{2n} is not divisible by p_1,p_3,...,p_{2n-1}, we get that $x \in p_1p_3 \cdots p_{2n-1}q_{2n}D$.* •

We give a partial converse of Theorem 11. Recall that a domain D is said to be of *finite character* if every nonzero element is contained in only finitely many maximal ideals. And D is said to be *h-local* if D is of finite character and every nonzero prime ideal of D is contained in a unique maximal ideal of D. It is easy to see that a one-dimensional domain of finite character is h-local. The next lemma was implicit in [15, Proposition 3.1].

Lemma 14 *Let D be a h-local domain, A, B nonzero ideals of D and $M \in Max(D)$. Then $(A : B)D_M = AD_M : BD_M$.*

Proof: Let K denote the quotient field of D. The inclusion $(\subseteq)$ is clear. Conversely, let $x \in AD_M : BD_M$. We may assume that $x \in D$. Pick $a \in A - \{0\}$. Since D is h-local, we have $[M]D_M = K$ where $[M] = \cap\{D_N \mid N \in Max(D)$ and $N \neq M\}$, cf. [15, Proposition 3.1]. Consequently, there exist $y \in [M]$ and $s \in D - M$ such that $x/a = y/s$. So $sx = ay$. Note that $ayB \subseteq AD_N$ for each $N \in Max(D) - \{M\}$. So $ayBD_Q = sxBD_Q \subseteq AD_Q$ for each $Q \in Max(D)$, hence $sx \in A : B$. Thus $x \in (A : B)D_M$. □

We show that the converse of Theorem 11 is true for a domain of finite character.

Theorem 15 *Let D be a domain of finite character such that D_M is a valuation domain with value group a complete subgroup of the reals for each $M \in Max(D)$. Then D is a sharp domain.*

Proof: Let I, A be nonzero ideals of D. By Proposition 2, it suffices to check locally that $(I : A)[I : (I : A)] = I$. Let M be a maximal ideal of D. Since D is one-dimensional of finite character, it is h-local. By Lemma 14, we have $(I : A)[I : (I : A)]D_M = (ID_M : AD_M)[ID_M : (ID_M : AD_M)] = ID_M$, where the last equality follows from Propositions 6 and 2. □

We do not know if a sharp domain is necessarily of finite character. A connected question, which is up to our knowleadge not solved, is whether a pseudo-Dedekind almost Dedekind domain is necessarily a Dedekind domain. We end our paper with two results for countable domains.

Proposition 16 *If D is a countable pseudo-Dedekind Prüfer domain, then D is of finite character.*

Proof: Assume that D is not of finite character. By [9, Corollary 7], there exists a nonzero element z and an infinite family $(I_n)_{n\geq 1}$ of invertible proper mutually comaximal ideals containing z. For each nonempty set of natural numbers Λ, consider the v-ideal $I_\Lambda = \cap_{n\in\Lambda} I_n$ (note that I_Λ contains z). As D is pseudo-Dedekind, I_Λ is invertible. We claim that $I_\Lambda \neq I_{\Lambda'}$ whenever Λ, Λ' are distinct nonempty sets of natural numbers. Deny. Then there exists a nonempty set of natural numbers Γ and some $k \notin \Gamma$ such that $I_k \supseteq I_\Gamma$. Consider the ideal $H = I_k^{-1} I_\Gamma \supseteq I_\Gamma$. If $n \in \Gamma$, then $I_n \supseteq I_\Gamma = I_k H$, so $I_n \supseteq H$, because $I_n + I_k = D$. It follows that $I_\Gamma \supseteq H$, so $I_\Gamma = H = I_k^{-1} I_\Gamma$. Since I_Γ is invertible, we get $I_k = D$, a contradiction. Thus the claim is proved. But then it follows that $\{I_\Lambda \mid \emptyset \neq \Lambda \subseteq \mathbb{N}\}$ is an uncountable set of invertible ideals. This leads to a contradiction, because D being countable, it has countably many finitely generated ideals. □

Corollary 17 *If D is a countable sharp domain, then D is a Dedekind domain.*

Proof: We may assume that D is not a field. By Theorem 11, D is a Prüfer domain. Now Propositions 4 and 16 show that D is of finite character. Let M be a maximal ideal of D. By Theorem 11, D_M is a countable valuation domain with value group $\mathbb{Z}$ or $\mathbb{R}$, so D_M is a DVR. Thus D is a Dedekind domain, cf. [10, Theorem 37.2]. □

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