Bull. Math. Soc. Sci. Math. Roumanie
Tome 55(103) No. 3, 2012, 327–335

On Douglas Surfaces

by
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Abstract

In this paper, we consider one of open problems in Finsler geometry which presented by Chen-Shen on Douglas surfaces. First, we prove that a Finsler surface has isotropic Berwald curvature if and only if it is of isotropic mean Berwald and relatively isotropic Landsberg curvature. Then we solve the open problem and show that on Douglas surfaces, a Finsler metric has isotropic mean Berwald curvature if and only if it has relatively isotropic Landsberg curvature

Key Words: Finsler surface, Douglas metric, Landsberg metric.
2010 Mathematics Subject Classification: Primary 53C60; Secondary 53C25.

1 Introduction

The geodesic curves of a Finsler metric $F = F(x, y)$ on a smooth manifold M, are determined by the system of second order differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the spray coefficients, and given by $G^i = \frac{1}{4}g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}$. A Finsler metric F is called a Berwald metric, if G^i are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently the Berwald curvature ${B^i}_{jkl} := [G^i]_{y^j y^k y^l}$ vanishes. There is an extension of Berwald metrics which introduced by Chen-Shen and called isotropic Berwald metrics [7]. A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i{}_{jkl} = c\{F_{y^j y^k}\delta^i{}_l + F_{y^k y^l}\delta^i{}_j + F_{y^l y^j}\delta^i{}_k + F_{y^j y^k y^l} y^i\}, \tag{1}$$

where $c = c(x)$ is a scalar function on M. Funk metrics are non-trivial isotropic Berwald metrics [13][18].

Other than isotropic Berwald curvature, there are many forms of curvatures in Finsler geometry. Let (M, F) be a Finsler manifold. The third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is the symmetric trilinear forms $\mathbf{C}_y$on T_xM, which called the Cartan torsion. The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM. F is called of relatively isotropic Landsberg curvature if it satisfies $\mathbf{L} + cF\mathbf{C} = 0$, where $c = c(x)$ is a scalar function on M. Every isotropic Berwald metric is of relatively isotropic Landsberg curvature.

Every Finsler metric of isotropic Berwald curvature is of isotropic mean Berwald curvature and relatively isotropic Landsberg curvature. In this paper, we show that for every Finsler surfaces the converse of this fact is true. More precisely, we prove the following.

Theorem 1.1. *Let (M, F) be a Finsler surface. Then the following are equivalent:*

(a) F has isotropic Berwald curvature;

(b) F has isotropic mean Berwald and relatively isotropic Landsberg curvature.

On the other hand, the Douglas metrics are another extension of Berwald metrics, which introduced by Douglas as a projective invariant in Finsler geometry [8]. A Finsler metric is called a Douglas metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x,y)y^i$. The study shows that the above mentioned quantities are closely related to the Douglas metrics, namely Bácsó-Matsumoto proved that every Douglas metric with vanishing Landsberg curvature is a Berwald metric [2].

In [7], Chen-Shen proved that on a Douglas manifold with dimension $n \geq 3$, a Finsler metric has isotropic mean Berwald curvature if and only if it has relatively isotropic Landsberg curvature. But this fact is unsolved for the case of two-dimensional Finsler manifolds, yet. In this paper, we prove the following.

Theorem 1.2. *Let (M, F) be a Douglas surface. Then the following are equivalent:*

(a) F has isotropic mean Berwald curvature;

(b) F has relatively isotropic Landsberg curvature.

There are many connections in Finsler geometry [5][6] [11][12]. In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2 Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian [10].

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|s}y^s$, $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L=0}$ [15]. Thus the quotient $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along Finslerian geodesics. F is said to be relatively isotropic Landsberg metric if $\mathbf{L} + cF\mathbf{C} = \mathbf{0}$, where $c = c(x)$ is a scalar function on M.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i := \frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\}$, $y \in T_xM$. $\mathbf{G}$ is called the spray associated to (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^jv^k$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2}B^m{}_{jkm}.$$

The $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$, respectively [10][16].

A Finsler metric F is said to be isotropic Berwald metric and isotropic mean Berwald metric if its Berwald curvature and mean Berwald curvature is in the following form, respectively

$$B^i{}_{jkl} = c\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\}, \tag{2}$$

$$E_{ij} = \frac{1}{2}(n + 1)cF^{-1}h_{ij} \tag{3}$$

where $c = c(x)$ is a scalar function on M and h_{ij} is the angular metric [14].

Define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{D}_y(u, v, w) := D^i{}_{jkl}(y)u^iv^jw^k\frac{\partial}{\partial x^i}|_x$ where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y\in TM_0}$ the Douglas curvature. A Finsler metric with $\mathbf{D} = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by B*á*cs*ó*-Matsumoto as a generalization of Berwald metrics [3]. The Douglas tensor $\mathcal{D}$ is a projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$.

3 Proof of Theorem 1.1

The special and useful Berwald frame was introduced and developed by Berwald [4]. Let (M, F) be a two-dimensional Finsler manifold. We study two dimensional Finsler space and define a local field of orthonormal frame (ℓ^i, m^i) called the Berwald frame, where $\ell^i = y^i/F(y)$, m^i is the unit vector with $\ell_i m^i = 0$, $\ell_i = g_{ij}\ell^i$ and g_{ij} is defined by $g_{ij} = \ell_i\ell_j + m_i m_j$.

Lemma 3.1. *Let (M, F) be a Finsler surface. Then F is of isotropic Berwald curvature*

$$B^i{}_{jkl} = c\{h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk} + 2C_{jkl}y^i\} \tag{4}$$

if an only if the main scalar of F satisfies

$$I_{,1} + cI = 0 \quad \text{and} \quad I_2 = 3c, \tag{5}$$

where $c = c(x)$ is a scalar function on M.

Proof: The Berwald curvature of Finsler surfaces is given by

$$B^i{}_{jkl} = F^{-1}(-2I_{,1}\ell^i + I_2 m^i)m_j m_k m_l, \tag{6}$$

where I is 0-homogeneous function called the main scalar of Finsler metric and $I_2 = I_{,2} + I_{,1|2}$ (see page 689 in [1]). On the other hand, the Cartan tensor of F is in the following form

$$C_{ijk} = F^{-1} I m_i m_j m_k. \tag{7}$$

By (6) and (7), we have

$$B^i{}_{jkl} = -\frac{2I_{,1}}{I}C_{jkl}\ell^i + \frac{I_2}{3F}\{h_{jk}h^i_l + h_{kl}h^i_j + h_{lj}h^i_k\}, \tag{8}$$

Then for a Finsler surface, the Berwald curvature can be written as follows

$$B^i{}_{jkl} = \mu C_{jkl}\ell^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{9}$$

where $\mu := -\frac{2I_{,1}}{I}$ and $\lambda := \frac{I_2}{3F}$ are homogeneous functions on TM of degrees 0 and -1 with respect to y, respectively. Thus if we put $\mu = 2c$ and $\lambda = cF^{-1}$, where $c = c(x)$ is a scalar function on M, then F reduces to a isotropic Berwald metric. Indeed, F is of isotropic Berwald curvature if and only if $I_{,1} + cI = 0$ and $I_2 = 3c$, where $c = c(x)$ is a scalar function on M. $\Box$

Lemma 3.2. *Let (M, F) be a Finsler surface. Then F is of isotropic mean Berwald curvature if an only if the main scalar of F satisfies $I_2 = 3c$, where $c = c(x)$ is a scalar function on M.*

Proof: Taking a trace of (9) yields

$$E_{jk} = \frac{3}{2}\lambda h_{jk}. \tag{10}$$

Thus F has isotropic mean Berwald curvature $E_{ij} = \frac{3}{2}cF^{-1}h_{ij}$ if and only if $I_2 = 3c$, where $c = c(x)$ is a scalar function on M. ∎

Lemma 3.3. *Let (M, F) be a Finsler surface. Then F is of relatively isotropic Landsberg curvature if an only if the main scalar of F satisfies $I_{,1} + cI = 0$, where $c = c(x)$ is a scalar function on M.*

Proof: Contracting (9) with y_i implies that

$$L_{jkl} + \frac{\mu}{2}FC_{jkl} = 0. \tag{11}$$

Thus F has relatively isotropic Landsberg curvature $L_{jkl}+cFC_{jkl} = 0$ if and only if $I_{,1}+cI = 0$, where $c = c(x)$ is a scalar function on M. ∎

Proof of the Theorem 1.1: By Lemmas 3.1, 3.2 and 3.3, we get the proof. ∎

Corollary 3.1. ([10]) *Let (M, F) be a Finsler surface. Then $\mathbf{B} = 0$ if an only if $\mathbf{E} = 0$ and $\mathbf{L} = 0$.*

4 Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2. First we find the necessary and sufficient condition for a Finsler surface to be a Douglas surface. More precisely, we prove the following.

Lemma 4.1. *Let (M, F) be a Finsler surface. Then F is a Douglas metric if and only if it satisfies*

$$\mathbf{L} + F^2\lambda\mathbf{C} = 0. \tag{12}$$

The equation (12) is equal to $II_2 - 3I_{,1} = 0$.

Proof: By assumption, we have

$$B^i{}_{jkl} = \mu C_{jkl}\ell^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{13}$$

By (10) we get

$$B^i{}_{jkl} = \mu C_{jkl}\ell^i + \frac{2}{3}(E_{jk}h^i_l + E_{kl}h^i_j + E_{jl}h^i_k). \tag{14}$$

Taking (11) in (14), yields

$$B^i{}_{jkl} = -2F^{-1}L_{jkl}\ell^i + \frac{2}{n+1}(E_{jk}h^i_l + E_{kl}h^i_j + E_{jl}h^i_k). \tag{15}$$

On the other hand, we have

$$h_{ij,k} = 2C_{ijk} - F^{-2}(y_j h_{ik} + y_i h_{jk}), \tag{16}$$

which implies that

$$2E_{jk,l} = 3\lambda_{,l}h_{jk} + 3\lambda\Big\{2C_{jkl} - F^{-2}(y_k h_{jl} + y_j h_{kl})\Big\}. \tag{17}$$

The Douglas tensor is given by

$$D^i{}_{jkl} = B^i{}_{jkl} - \frac{2}{3}\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\}. \tag{18}$$

Putting (10), (15) and (17) in (18) yields

$$D^i{}_{jkl} = -2\{F^{-2}L_{jkl} + \lambda C_{jkl}\}y^i - (\lambda y_l F^{-2} + \lambda_{,l})h_{jk}y^i. \tag{19}$$

Since $D^i{}_{jkl} = D^i{}_{jlk}$, then

$$\lambda y_l F^{-2} + \lambda_{,l} = 0. \tag{20}$$

From (19) and (20), we deduce that

$$D^i{}_{jkl} = -2\{F^{-2}L_{jkl} + \lambda C_{jkl}\}y^i. \tag{21}$$

Therefore F is a Douglas metric if and only if it satisfies $L_{jkl} + F^2\lambda C_{jkl} = 0$. By considering (11), F is a Douglas metric if and only if it satisfies $II_2 - 3I_{,1} = 0$. This completes the proof. □

Lemma 4.2. *Let (M, F) be a non-Riemannian Douglas surface. Suppose that F has isotropic mean Berwald curvature. Then F has relatively isotropic Landsberg curvature.*

Proof: By Lemma 3.2, $E_{jk} = \frac{3}{2}\lambda h_{jk}$. By assumption, F has isotropic mean Berwald curvature $E_{jk} = \frac{3}{2}cF^{-1}h_{jk}$, thus $\lambda = cF^{-1}$. Since F is a Douglas metric then by Lemma 4.1, we have $L_{jkl} = -F^2\lambda C_{jkl}$. It conclude that F has isotropic Landsberg curvature $L_{jkl} + cFC_{jkl} = 0$. □

Lemma 4.3. *Let (M, F) be a non-Riemannian Douglas surface. Suppose that F has isotropic relatively Landsberg curvature. Then F has isotropic mean Berwald curvature.*

Proof: F has relatively isotropic Landsberg curvature $L_{jkl} + cF\lambda C_{jkl} = 0$. By Lemma 4.1, we have $L_{jkl} = -F^2\lambda C_{jkl}$. Then we have $\lambda = cF^{-1}$. By (10), we get $E_{jk} = \frac{3}{2}\lambda h_{jk}$. Therefore F has isotropic mean Berwald curvature $E_{jk} = \frac{3}{2}cF^{-1}h_{jk}$. ∎

Proof of the Theorem 1.2: By Lemmas 4.2 and 4.3, we get the proof. ∎

Finally, we can prove the following.

Corollary 4.1. *Let (M, F) be a Finsler surface. The following are equivalent.*

(a) F is of isotropic Berwald curvature;

(b) F is a Douglas metric with isotropic mean Berwald curvature;

(c) F is a Douglas metric with relatively isotropic Landsberg curvature.

Proof: Every isotropic Berwald metric is a Douglas metric with isotropic mean Berwald curvature and relatively isotropic Landsberg curvature. By Theorem 1.2, (a) and (b) are equivalent. Thus it is sufficient to prove $(c) \Rightarrow (a)$. Let F is a Douglas metric with relatively isotropic Landsberg curvature $\mathbf{L} + cF\mathbf{C} = 0$. Since F is a Douglas metric, then by Lemma 4.1, $\mathbf{L} + F^2\lambda\mathbf{C} = 0$, which implies that $\lambda = cF^{-1}$. By Lemma 3.3, we have $\mathbf{L} + \frac{\mu}{2}F\mathbf{C} = 0$. This deduce that $\mu = 2c$. Thus by the Lemma 3.1, we conclude that F is of isotropic Berwald curvature. ∎

Corollary 4.2. ([2]) *Let (M, F) be a Douglas surface. Suppose that $\mathbf{L} = 0$. Then F is a Berwald metric.*

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Received: 05.07.2011, Accepted: 10.03.2012.

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