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Existence and prolongation of analytic solutions of non-local differential equations

by
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Abstract

In this paper, we prove the general invertibility theorem for non-local pseudo-differential operators and using this, we establish the theorems of analytic continuation and of existence of holomorphic solutions to non-local differential equations. We also obtain a concrete method to construct a holomorphic solutions to constant coefficient equations.

Key Words: pseudo-differential operators, partial differential equations of infinite ordre, non-local equations, differential-difference equations

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1 Introduction

In their monumental work [18], Sato, Kawai and Kashiwara introduced the micro-local analysis to establish the local theory of linear partial differential equations. Especially, the notion of pseudo-differential operators as micro-localization of diffrerential operators has allowed them to obtain quite general theorems on system of partial differential equations. Ever since, local pseudo-differential equations have been studied by many authors (e.g. Aoki [3], Aoki, Kataoka and Yamazaki [4], Kashiwara and Kawai [10], Kashiwara and Schapira [11] or Ishimura [5] and so on).

As for the non-local equations which include the linear differential-difference equations, mainely the convolution equations have been studied in the complex domain (e.g. Malgrange [17], Korobeĭnik [13], Krivosheev [14], Kawai [12], Lelong and Gruman [15], Sébbar [19] or Ishimura and Okada [9]).

In the prededing article [6], we proposed to introduce the non-local pseudo-differential operators to be a natural generalization of both the (micro-local) pseudo-differential operators and the convolution operators and proved the invertibility theorem for such operators under some conditions (see also [7] and [8]).

In the present article, we employ the exponential calculus developped by Aoki [1], [2] and applying it to the non-local situation, we generalize the invertivility theorem. By using this theorem, we first prove the analytic continuation theorem of solutions to such equations and then we establish an existence theorem for non-local pseudo-differential equations in the complex domain.

2 Notations and recall

In this section, we recall some notations and definitions which we will need later.

Let $S^{2n-1}_\infty := (\mathbb{C}^n \setminus \{0\})/\mathbb{R}_+$ be the sphere at infinity and $\mathbb{D}^{2n} := \mathbb{C}^n \sqcup S^{2n-1}_\infty$ be the compactification by directions of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. For any $\zeta \in \mathbb{C}^n \setminus \{0\}$, we denote by $\zeta\infty \in S^{2n-1}_\infty$ the class defined by ζ: $\zeta\infty :=$ (the closure of $\mathbb{R}_+ \cdot \zeta$ in $\mathbb{D}^{2n}$) $\cap$ S^{2n-1}_∞. For any $A \subset \mathbb{C}^n$, we set $A\infty := \{\zeta\infty \in S^{2n-1}_\infty \mid \mathbb{R}_+ \cdot \zeta \subset A\}$ and for any $\Omega \subset \mathbb{C}^n \times S^{2n-1}_\infty \simeq S^*\mathbb{C}^n$ (the co-sphere bundle of $\mathbb{C}^n$), we also write

$$\Omega\infty := \{(z,\zeta)\infty := (z,\zeta\infty) \in S^*\mathbb{C}^n \mid \{z\} \times (\mathbb{R}_+ \cdot \zeta) \subset \Omega\}.$$

In this paper, we take a fixed point $p = (x_0, \xi_0) \in T^*\mathbb{C}^n$. For any compact convex set $M \subset \mathbb{C}^n$, we use the following two *supporting functions*:

$$H_M(\zeta) := \sup_{z\in M} \mathrm{Re}\langle z,\zeta\rangle, \quad I_M(\zeta) := \inf_{z\in M} \mathrm{Re}\langle z,\zeta\rangle \tag{2.1}$$

where $\langle z,\zeta\rangle := \sum_{j=1}^n z_j\zeta_j$ with $z = (z_1, z_2, \cdots, z_n)$ and $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)$. For any $A, B \subset \mathbb{C}^n$, set $A + B := \{a + b \mid a \in A, b \in B\}$ and $A - B := \{a - b \mid a \in A, b \in B\}$.

For any $\delta > 0$, consider the cones

$$\Gamma_{1,\delta} = \Gamma_1 := \{z \in \mathbb{C}^n \mid \delta|\mathrm{Im} z_1| \leqslant -\mathrm{Re} z_1\}, \tag{2.2}$$

$$\Gamma_{j,\delta} = \Gamma_j := \{z \in \mathbb{C}^n \mid \delta|z_j| \leqslant |z_1|\} \quad (2 \leqslant j \leqslant n). \tag{2.3}$$

For a compact convex set $M \subset \mathbb{C}^n$, we set

$$G = G_\delta = G(M) := \cap_{k=1}^n (\Gamma_k + M). \tag{2.4}$$

An open set $W \subset \mathbb{C}^n$ is said to be *G-open* if there exists a point $a \in \mathbb{C}^n$ such that $W + (G + a) \subset W$. An open set $D \subset \mathbb{C}^n$ is said to be a *G-round* if there exists a point $a \in \mathbb{C}^n$ s.t. $\{y \in \mathbb{C}^n \mid (x, y) \in G + a, (y, z) \in G + a, x \in D, z \in D+M+M\} \subset D+M$ (see [11]). We will denote by $\mathcal{O}_{\mathbb{C}^n}$ the sheaf of holomorphic functions in $\mathbb{C}^n$.

Definition 2.1. Let $M \subset \mathbb{C}^n$ be a compact convex set. By a rotation, we may assume $\xi_0 = (1, 0, \cdots, 0)$. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ and $G = G(M)$ be as in (2.2) -

(2.4) for some $\delta > 0$ and M. We set $Z := \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^n \mid y - x \in G = \cap_{k=1}^n(\Gamma_k + M)\}$. For any G-round open set D with $x_0 \in D$, we set

$$\mathcal{E}(G;D) := H_Z^n(D \times (D+M), \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}^{(0,n)}),$$

here $\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}^{(0,n)}$ is the sheaf of holomorphic $(0,n)$-forms on $\mathbb{C}^n \times \mathbb{C}^n$. We call any $P \in \mathcal{E}(G;D)$ with some $\delta > 0$ and D, a *non-local pseudo-differential operator* defined near the point p, (or defined near the direction ξ_0), carried by the compact convex set M. We set

$$\mathcal{E}_{[M]}^{\mathbb{R}}(D)_{\xi_0} := \varinjlim_{\delta} \mathcal{E}(G_\delta; D), \quad \mathcal{E}_{[M]p}^{\mathbb{R}} := \varinjlim_{D \ni x_0} \mathcal{E}_{[M]}^{\mathbb{R}}(D)_{\xi_0}.$$

In [6], we established the compositon of two operators belonging to $\mathcal{E}_{[M]p}^{\mathbb{R}}$ ([6], Proposition 3.5).

Let D be a convex open set. In the sequel, for the simplicity, we will often write

$$\tilde{D} := D + M.$$

We take the following holomorphically convex domains:

$$V_1 := D \times \tilde{D} \setminus \{(x,y) \mid z = y - x \in (\Gamma_1 + M)\}, \tag{2.5}$$
$$V_j := D \times \tilde{D} \setminus \{(x,y) \mid z = y - x \in (\Gamma_j + M)\} \quad (2 \leqslant j \leqslant n). \tag{2.6}$$

The family $\mathcal{V} := \{V_1, V_2, \cdots, V_n\}$ is an open covering of $D \times \tilde{D} \setminus Z$. Write $V := \cap_{k=1}^n V_k$ and $V_{\hat{k}} := \cap_{l\neq k} V_l$ for $k = 1, 2, \cdots, n$. We have $V = D \times \tilde{D} \setminus \cup_{k=1}^n\{(x,y) \mid z = y - x \in (\Gamma_k + M)\}$. We easily see that

$$\begin{aligned} H_Z^n(D \times \tilde{D}, \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}^{(0,n)}) &\simeq H^{n-1}(D \times \tilde{D} \setminus Z, \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}^{(0,n)}) \\ &\simeq \frac{\mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}^{(0,n)}(V)}{\sum_{k=1}^n \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}^{(0,n)}(V_{\hat{k}})}. \end{aligned} \tag{2.7}$$

So to any $P \in \mathcal{E}(G;D)$, a $(0,n)$-form $K(x,z)dy = K(x,y)dy \in \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}^{(0,n)}(V)$ determined uniquely modulo the denominator $\sum_{k=1}^n \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}^{(0,n)}(V_{\hat{k}})$ corresponds. We call $K(x,y)dy$ *the kernel* of P. We define also the space of *non-local differential operators carried by* M:

$$\mathcal{D}_{[M]}^\infty(\mathbb{C}^n) := H_{\Delta_M}^n(\mathbb{C}^n \times \mathbb{C}^n; \mathcal{O}_{\mathbb{C}^n\times\mathbb{C}^n}^{(0,n)}).$$

Here we set $\Delta_M := \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^n \mid y - x \in M\}$.

Let Ω be a conic neighbourhood of $p = (x_0, \xi_0) \in T^*\mathbb{C}^n$. For any $r > 0$, we set $\Omega(r) := \{(x,\xi) \in \Omega \mid |\xi| > r\}$ and $\Omega[r] := \{(x,\xi) \in \Omega \mid |\xi| \geqslant r\}$.

Definition 2.2. For a compact convex set M, set

$$\begin{aligned} S^M(\Omega) \quad := \quad & \bigcup_{r>0} \{P(x,\xi) \in \mathcal{O}(\Omega(r)) \mid \forall \Omega' \Subset \Omega \text{ conic}, \forall r' > r, \forall \varepsilon > 0, \exists C_\varepsilon > 0 \\ & \text{ s.t.} \\ & |P(x,\xi)| \leqslant C_\varepsilon \mathrm{e}^{H_M(\xi)+\varepsilon|\xi|} \quad (\forall (x,\xi) \in \Omega'[r'])\}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} N^M(\Omega) \quad := \quad & \bigcup_{r>0} \{P(x,\xi) \in S^M(\Omega) \mid P(x,\xi) \in \mathcal{O}(\Omega(r)), \forall \Omega' \Subset \Omega \text{ conic}, \forall r' > r, \\ & \exists \varepsilon_0 > 0, \exists C > 0 \text{ s.t.} \\ & |P(x,\xi)| \leqslant C \mathrm{e}^{I_M(\xi)-\varepsilon_0|\xi|} \quad (\forall (x,\xi) \in \Omega'[r'])\}. \end{aligned} \tag{2.9}$$

We call any $P(x,\xi) \in S^M(\Omega)$ a *symbol* on Ω with *support* M and $P(x,\xi) \in N^M(\Omega)$ a *nul symbol*. In this paper, we set also

$$\begin{aligned} T^M(\Omega) \quad := \quad & \bigcup_{r>0} \{P(x,\xi) \in S^M(\Omega) \mid P(x,\xi) \in \mathcal{O}(\Omega(r)), \forall \Omega' \Subset \Omega \text{ conic}, \\ & \forall r' > r, \exists \varepsilon_0 > 0, \exists C > 0 \text{ s.t.} \\ & |P(x,\xi)| \leqslant C_\varepsilon \mathrm{e}^{I_M(\xi)+\varepsilon|\xi|} \quad (\forall (x,\xi) \in \Omega'[r']) \\ & (\text{ instead of } (2.8))\}. \end{aligned} \tag{2.10}$$

Definition 2.3. In the situation of Definition 2.3, let $P(x,\xi) \in S^M(\Omega)$ (resp. $P(x,\xi) \in T^M(\Omega)$) and suppose (2.8) (resp. suppose (2.10)). Any point $p \in \Omega$ is said to be *non M-characteristic* or *non $\underline{M}$-characteristic* with respect to the corresponding non-local pseudo-differential operator P, or P is *non M-characteristic* (abbreviatory *non M-char*) or *non $\underline{M}$-characteristic* (abbreviatory *non $\underline{M}$-char*) at p, if there exist an open conic neighbourhood $\Omega' \subset \Omega$ of p and $r' > r$ so that we have:

$$\begin{aligned} & \forall \varepsilon > 0, \exists C_\varepsilon > 0 \text{ s.t. } \forall (x,\xi) \in \Omega'(r') \\ & |P(x,\xi)| \geqslant C_\varepsilon e^{H_M(\xi)-\varepsilon|\xi|} \\ & \text{ or} \\ & |P(x,\xi)| \geqslant C_\varepsilon e^{I_M(\xi)-\varepsilon|\xi|}. \end{aligned} \tag{2.11}$$

We define

$$\begin{aligned} Car^M_\infty(P) \quad &:= \quad \{(x,\xi\infty) \in \Omega\infty \mid P \text{ is not non } M\text{-char at } (x,\xi)\}, \\ \underline{Car}^M_\infty(P) \quad &:= \quad \{(x,\xi\infty) \in \Omega\infty \mid P \text{ is not non} \underline{M}\text{-char at} (x,\xi)\}. \end{aligned} \tag{2.12}$$

A point $q \in \Omega$ is also said to be *M-characteristic* (resp. *$\underline{M}$-characteristic*) with respect to P if $q\infty \in Car^M_\infty(P)$ (resp. if $q\infty \in \underline{Car}^M_\infty(P)$).

In this article, we will set $\mathbb{Z}_+ := \{\nu \in \mathbb{Z} \mid \nu \geqslant 0\}$. We have to consider also the formal symbols:

Definition 2.4. Let $M \subset \mathbb{C}^n$ be a compact convex set and $\Omega \subset T^*\mathbb{C}^n$ a conic open set.

$$\begin{aligned}
\hat{S}^M(\Omega) \quad := \quad & \{P(t;x,\xi) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(x,\xi) \mid \exists r > 0 \text{ s.t.} \\
& P_\nu(x,\xi) \in \mathcal{O}(\Omega((\nu+1)r)) \ (\forall \nu \in \mathbb{Z}_+), \\
& \forall \Omega' \Subset \Omega \text{ conic}, \exists d > r, 0 < \exists A < 1 : \ \forall \varepsilon > 0, \exists C_\varepsilon > 0 \qquad (2.13) \\
& |P_\nu(x,\xi)| \leqslant C_\varepsilon A^\nu \mathrm{e}^{H_M(\xi)+\varepsilon|\xi|} \quad (\forall \nu \in \mathbb{Z}_+, \forall (x,\xi) \in \Omega'[(\nu+1)d])\}
\end{aligned}$$

and

$$\begin{aligned}
\hat{N}^M(\Omega) \quad := \quad & \{P(t;x,\xi) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(x,\xi) \in \hat{S}^M(\Omega) \mid \exists r > 0 \text{ s.t.} \\
& P_\nu(x,\xi) \in \mathcal{O}(\Omega((\nu+1)r)), \\
& \forall q = (y,\eta) \in \Omega(r), \exists \Omega' \Subset \Omega \text{ a conic neighbourhood of } q, \\
& \exists d > r, 0 < \exists A < 1 : \\
& \quad 0 \leqslant \mu := H_M(\eta/|\eta|) - I_M(\eta/|\eta|) < -\frac{\ln A}{d} \qquad (2.14) \\
& \quad \forall \varepsilon > 0, \exists C_\varepsilon > 0 \text{ s.t.} \qquad (2.15) \\
& \quad |\sum_{\nu=0}^{m-1} P_\nu(x,\xi)| \leqslant C_\varepsilon A^m \mathrm{e}^{H_M(\xi)+\varepsilon|\xi|} \ (\forall m \in \mathbb{Z}_+, \forall (x,\xi) \in \Omega'[md])\}.
\end{aligned}$$

We call any $P(t;x,\xi) \in \hat{S}^M(\Omega)$ a *formal symbol* and $P(t;x,\xi) \in \hat{N}^M(\Omega)$ a *formal nul symbol.* We set also

$$\begin{aligned}
\hat{T}^M(\Omega) \quad := \quad & \{P(t;x,\xi) \in \hat{S}^M(\Omega) \mid \exists r > 0 \text{ s.t.} \\
& P_\nu(x,\xi) \in \mathcal{O}(\Omega((\nu+1)r)) \ (\forall \nu \in \mathbb{Z}_+), \forall \ \Omega' \Subset \Omega \text{ conic}, \\
& \exists d > r, 0 < \exists A < 1 : \forall \varepsilon > 0, \exists C_\varepsilon > 0, \qquad (2.16) \\
& |P_\nu(x,\xi)| \leqslant C_\varepsilon A^\nu \mathrm{e}^{I_M(\xi)+\varepsilon|\xi|} \quad (\forall \nu \geqslant 0, \forall (x,\xi) \in \Omega'[(\nu+1)d]) \\
& (\text{instead of } (2.13))\},
\end{aligned}$$

$$\begin{aligned}
\hat{s}^M(\Omega) \quad := \quad & \{P(t;x,\xi) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(x,\xi) \mid \exists r > 0 \text{ s.t.} \\
& P_\nu(x,\xi) \in \mathcal{O}(\Omega((\nu+1)r)) \ (\forall \nu \in \mathbb{Z}_+), \\
& \forall \Omega' \Subset \Omega \text{ conic}, \forall h > 0, \exists d > r : \forall \varepsilon > 0, \exists C_\varepsilon > 0, \qquad (2.17) \\
& |P_\nu(x,\xi)| \leqslant C_\varepsilon h^\nu \mathrm{e}^{H_M(\xi)+\varepsilon|\xi|} \quad \forall \nu \in \mathbb{Z}_+, \forall (x,\xi) \in \Omega'[(\nu+1)d])\},
\end{aligned}$$

$$\begin{aligned}\hat{t}^M(\Omega) \quad := \quad & \{P(t;x,\xi) \in \hat{s}^M(\Omega) \mid \exists r > 0 \text{ s.t.} \\ & P_\nu(x,\xi) \in \mathcal{O}(\Omega((\nu+1)r)) \ (\forall \nu \in \mathbb{Z}_+), \\ & \forall \Omega' \Subset \Omega \text{ conic}, \forall h > 0, \exists d > r : \forall \varepsilon > 0, \exists C_\varepsilon > 0, \\ & |P_\nu(x,\xi)| \leqslant C_\varepsilon h^\nu \mathrm{e}^{I_M(\xi)+\varepsilon|\xi|} \quad (\forall \nu \in \mathbb{Z}_+, \forall (x,\xi) \in \Omega'[(\nu+1)d]) \\ & (\text{instead of } (2.17))\}. \end{aligned} \tag{2.18}$$

In the preceding paper [6] (Propositions 4.2, 4.3, 6.5, 6.7 and 6.8), we have proved the following:

Theorem 2.5. *There is a natural isomorphism*

$$S^M(\Omega)/N^M(\Omega) \xrightarrow{\sim} \hat{S}^M(\Omega)/\hat{N}^M(\Omega).$$

And there is a natural isomorphism

$$\sigma : \mathcal{E}^{\mathbb{R}}_{[M],\xi_0} \xrightarrow{\sim} \varinjlim_{\Omega \ni p} S^M(\Omega)/N^M(\Omega).$$

Remark 2.6. By the proof of Propositions 6.5, 6.7 and 6.8 [6], we can also prove that if $P(t;x,\xi) \in \hat{T}^M(\Omega)$, there exists a symbol $P(x,\xi) \in T^M(\Omega)$ corresponding to $P(t;x,\xi)$.

In this paper, we will give a generalization of the main theorem of [6]:

Theorem 2.7. *Let $M \subset \mathbb{C}^n$ be a compact convex set and $\Omega \subset T^*\mathbb{C}^n$ a conic open set of the form $\mathbb{C}^n \times \omega$ with an open cone ω with vertex at 0. Let $P(x,\xi) \in S^M(\Omega)$ (resp. $P(x,\xi) \in T^M(\Omega)$) be a symbol satisfying (2.5) for any $\Omega' \subset \Omega$ of the form $\mathbb{C}^n \times \omega'$ with $\omega' \Subset \omega$). Suppose that the corresponding non-local pseudo-differential operator P carried by M is non M-characteristic (resp. non $\underline{M}$-characteristic) on Ω. Then there exists a non-local pseudo-differential operator, which we denote by P^{-1}, defined on $\Omega_1 := \mathbb{C}^n \times \omega_1$ with an open cone $\omega_1 \Subset \omega$, with a symbol in $\hat{T}^{-M}(\Omega_1)$ (reap. in $\hat{S}^{-M}(\Omega_1)$) such that we have $P \circ P^{-1} = P^{-1} \circ P = id$. Here id is the identity operator of the space of non-local pseudo-differential operators carried by the compact convex set $M - M$.*

We will see briefly the proof of this theorem by using the exponential calculus developped by Aoki [2].

Remark 2.8. We remark also that in the theorem, the inverse operator P^{-1} has a formal symbol $P^{-1}(t;x,\xi)$ which, in fact, belongs to $\hat{t}^{-M}(\Omega_1)$ (resp. to $\hat{s}^{-M}(\Omega_1)$) (see the proof).

3 Exponential calculus for non-local pseudo-differential operators

In this section, we review the exponential calculus for the formal symbols of non-local pseudo-differential operators. The methods employed in this section are those developed by Aoki [1], [2] for (micro-)local pseudo-differential operators (see also Aoki, Kataoka and Yamazaki [4]).

Definition 3.1. Suppose $\Omega = U \times \omega$ with an open set $U \subset \mathbb{C}^n$ and an open cone $\omega \subset \mathbb{C}^n$ with vertex at 0. A formal symbol $p(t;x,\xi) = \sum_{j=0}^{\infty} t^j p_j(x,\xi) \in \hat{S}^{\{0\}}(\Omega)$ satisfying (2.13) with $M = \{0\}$, is said to be *of type* M^+ (resp. M^-) when we have the following conditions:

for any conic set $\Omega' \Subset \Omega$ and any $\delta > 0$, there exists $d > r$ such that for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ so that we have

$$\begin{aligned} &\mathrm{Re}\; p_0(x,\xi) \leqslant H_M(\xi) + \varepsilon|\xi| + C_\varepsilon \\ &(\text{ resp. } \mathrm{Re}\; p_0(x,\xi) \leqslant I_M(\xi) + \varepsilon|\xi| + C_\varepsilon), && (3.1) \\ &|p_0(x,\xi)| \leqslant H \cdot |\xi| + \varepsilon|\xi| + C_\varepsilon \qquad \textit{on } \Omega'[d], && (3.2) \\ &\textit{and} \\ &|p_j(x,\xi)| \leqslant \delta^j(H \cdot |\xi| + \varepsilon|\xi| + C_\varepsilon) \quad \textit{on } \Omega'[(j+1)d] \; (j = 1,2,3,\cdots) && (3.3) \end{aligned}$$

By a translation on $\mathbb{C}^n$, we may asssume that $H_M(\xi) \geqslant 0$ on ω. Then if we take $\delta > 0$ smaller, the estimate (3.3) can be replaced by

$$|p_j(x,\xi)| \leqslant \delta^j(H_M(\xi) + \varepsilon|\xi| + C_\varepsilon) \tag{3.4}$$

or rather by

$$|p_j(x,\xi)| \leqslant \delta^j(\varepsilon|\xi| + C_\varepsilon) \tag{3.5}$$

on $\Omega[(j+1)d]$ $(\forall j \geqslant 1)$.

Proposition 3.2. *If* $p(t;x,\xi) \in \hat{S}^{\{0\}}(\Omega)$ *is of type* M^+ *(resp.* M^-*), then* $P(t;x,\xi) := \mathrm{e}^{p(t;x,\xi)}$ *which is calculated formally, belongs to* $\hat{s}^M(\Omega)$ *(resp. to* $\hat{t}^M(\Omega)$*).*

Proof: We assume that $p(t;x,\xi) = \sum_{j=0}^{\infty} t^j p_j(x,\xi)$ satisfies (3.1) – (3.3). Formally we have

$$\begin{aligned} \mathrm{e}^{p(t;x,\xi)} &= \sum_{k=0}^{\infty} \frac{1}{k!} p(t;x,\xi)^k = \sum_{k=0}^{\infty} \frac{1}{k!} (\sum_{j=0}^{\infty} t^j p_j(x,\xi))^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1,j_2,\cdots,j_k=0}^{\infty} t^{j_1+j_2+\cdots+j_k} p_{j_1}(x,\xi) p_{j_2}(x,\xi) \cdots p_{j_k}(x,\xi). \end{aligned}$$

And this must be formally equal to $P(t;x,\xi) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(x,\xi)$, that is, for any $\nu \in \mathbb{Z}_+$, we have

$$P_\nu(x,\xi) = \begin{cases} 1 + \displaystyle\sum_{k=1}^{\infty} \frac{1}{k!} p_0(x,\xi)^k = e^{p_0(x,\xi)} & (\nu = 0), \\ \displaystyle\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1+j_2+\cdots+j_k=\nu} p_{j_1}(x,\xi) p_{j_2}(x,\xi) \cdots p_{j_k}(x,\xi) & (\nu \geqslant 1). \end{cases}$$

By (3.1), for any $\varepsilon > 0$, there exists $C = C_\varepsilon > 0$ such that we have $|P_0(x,\xi)| = \mathrm{e}^{\mathrm{Re} p_0(x,\xi)} \leqslant C\mathrm{e}^{H_M(\xi)+\varepsilon|\xi|}$ (resp. $|P_0(x,\xi)| \leqslant C\mathrm{e}^{I_M(\xi)+\varepsilon|\xi|}$). By (3.1) – (3.3), for any $\nu \geqslant 1$ and any $(x,\xi) \in \Omega'[(\nu+1)d]$, we have the following: $\forall\varepsilon > 0$, $\exists C' = C'_\varepsilon > 0$ s.t.

$$\begin{aligned} |P_\nu(x,\xi)| &\leqslant \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1+j_2+\cdots+j_k=\nu} \delta^\nu (H_M(\xi)+\varepsilon|\xi|+C')^k \\ &= \delta^\nu \sum_{k=1}^{\infty} \frac{1}{k!} \binom{\nu+k-1}{\nu} (H_M(\xi)+\varepsilon|\xi|+C')^k. \end{aligned} \tag{3.6}$$

By a formula due to Aoki ([1], P.234), setting $C_1 := \mathrm{e}^{C'}$, (3.6) is equal to

$$\begin{aligned} &= \delta^\nu \sum_{l=1}^{\nu} \frac{(\nu-1)!}{l!(l-1)!(\nu-l)!} (H_M(\xi)+\varepsilon|\xi|+C')^l \mathrm{e}^{H_M(\xi)+\varepsilon|\xi|+C'} \\ &= C_1\delta^\nu \sum_{l=1}^{\nu} \binom{\nu-1}{l-1} \frac{|\xi|^l}{l!} (H_M(\frac{\xi}{|\xi|})+\varepsilon+\frac{C'}{|\xi|})^l \mathrm{e}^{H_M(\xi)+\varepsilon|\xi|}. \end{aligned}$$

There exists $C_0 > 0$ independent of $\varepsilon > 0$ such that $H_M(\frac{\xi}{|\xi|})+\varepsilon+\frac{C'}{|\xi|} \leqslant C_0$ for any ξ with $|\xi| \geqslant (\nu+1)d$. So taking ε smaller than 1, this last is estimated as:

$$\begin{aligned} &\leqslant C_1\delta^\nu \sum_{l=1}^{\nu} \binom{\nu-1}{l-1} \frac{C_0^l|\xi|^l}{l!} \mathrm{e}^{H_M(\xi)+\varepsilon|\xi|} \leqslant C_1\delta^\nu \sum_{l=1}^{\nu} \binom{\nu-1}{l-1} (\frac{1}{\varepsilon})^l \mathrm{e}^{H_M(\xi)+\varepsilon(1+C_0)|\xi|} \\ &= C_1\delta^\nu (1+\frac{1}{\varepsilon})^{\nu-1} \frac{1}{\varepsilon} \mathrm{e}^{H_M(\xi)+\varepsilon(1+C_0)|\xi|} \leqslant C_1[\delta\frac{1+\varepsilon}{\varepsilon}]^\nu \mathrm{e}^{H_M(\xi)+\varepsilon(1+C_0)|\xi|}. \end{aligned}$$

Therefore, for any $h > 0$, taking $\delta > 0$ so small that we have $0 < \delta < \frac{h\varepsilon}{1+\varepsilon}$, we have $|P_\nu(x,\xi)| \leqslant C_1 h^\nu \mathrm{e}^{H_M(\xi)+\varepsilon(1+C_0)|\xi|}$ on $\Omega'[(\nu+1)d]$. Thus $P(t;x,\xi) \in \hat{s}^M(\Omega)$. The proof of the case of type M^- is similar. □

By the similar way with Aoki [2], Théorème 2.1, we have

Proposition 3.3. *We assume $\Omega = \mathbb{C}^n \times \omega$ with an open cone $\omega \subset \mathbb{C}^n$. Let formal symbols $p(t;x,\xi), r(t;x,\xi) \in \hat{S}^{\{0\}}(\Omega)$ be respectively of type $M^+, \{0\}^+$ and let $q(t;x,\xi) \in \hat{S}^{\{0\}}(\Omega)$. Suppose, in the situation of Definition 3.1, for any conic set $\Omega' \Subset \Omega$, there exist $d' > d$ such that $p(t;x,\xi)$ and $r(t;x,\xi)$ satisfy the following: $\forall\varepsilon > 0$, $\exists C > 0$ s.t.*

$$\begin{cases} \mathrm{Re}\ p_0(x,\xi) \geqslant H_M(\xi)-\varepsilon|\xi|-C, \\ \mathrm{Re}\ r_0(x,\xi) \geqslant -\varepsilon|\xi|-C \end{cases} \tag{3.7}$$

on $\Omega'[d']$. Set $P(t;x,\xi) := \mathrm{e}^{p(t;x,\xi)}, Q(t;x,\xi) := \mathrm{e}^{q(t;x,\xi)}, R(t;x,\xi) := \mathrm{e}^{r(t;x,\xi)}$ and respectively, we denote by P,Q,R corresponding non-local pseudo-differential operators. Suppose that we have $\sigma(P\circ Q) = \sigma(R)$ formally, then $q(t;x,\xi)$ is of type $(-M)^-$.

By using these propositions, as Aoki [1], Théorème 5.1, we have Theorem 2.7.

4 Continuation of holomorphic solutions of non-local pseudo-differential equations

First let $M \subset \mathbb{C}^n$ be a compact convex set and $\Omega \subset \mathbb{C}^n \times \mathbb{C}^n$ an open conic set. In the sequel, we assume $\Omega = \mathbb{C}^n \times \omega$ with an open cone $\omega \subset \mathbb{C}^n$. We recall without proof, a way to define concretely the action of non-local pseudo-differential operators on holomorphic functions (see [6], section 4, in particular, Proposition 4.3):

For $\eta' = (\eta_2, \eta_3, \cdots, \eta_n)$ with $|\eta'| \ll 1$, set $\eta := (1, \eta') := (1, \eta_2, \eta_3, \cdots, \eta_n)$. For any small $\varepsilon > 0$ and z with $z_j \neq 0$ $(2 \leqslant j \leqslant n)$, define the $(n-1)$-chain

$$\tilde{\beta} := [0, \frac{\varepsilon}{z_2}] \times [0, \frac{\varepsilon}{z_3}] \times \cdots [0, \frac{\varepsilon}{z_n}], \tag{4.1}$$

where we set $[0, w] := \{cw \mid 0 \leqslant c \leqslant 1\}$ for any $w \in \mathbb{C}$. Then there exist cones $\Gamma_{1,\delta}, \Gamma_{2,\delta}, \cdots, \Gamma_{n,\delta}$ of the form (2.2) and (2.3) with some $\delta > 0$ and a convex $G := \cap_{k=1}^n (\Gamma_{k,\delta} + M)$-round open set D such that we can associate to a symbol $P(x, \xi)$, its kernel $K(x, y)$, by the following formula: let $\xi := (1, \eta')$ and take $\tilde{\beta}$ for $z := y - x$. For any $x \in U$ and $\rho \in \mathbb{C}$ so that $|\rho| \gg 1$ and $\rho\xi \in \omega$, set

$$K(x, y) := \frac{1}{(2\pi\sqrt{-1})^n} \int_{\tilde{\beta}} d\eta' \int_{\rho}^{\infty} \mathrm{e}^{-\tau(y-x)\cdot\xi} \tau^{n-1} P(x, \tau\xi) d\tau \tag{4.2}$$

where the path of integration for $d\tau$ is taken in the direction ρ. In the notations of (2.7), we can prove that $K(x, y)$ is uniquely determined by $P(x, \xi)$ in $\dfrac{\mathcal{O}^{(0,n)}_{\mathbb{C}^n \times \mathbb{C}^n}(V)}{\sum_{k=1}^n \mathcal{O}^{(0,n)}_{\mathbb{C}^n \times \mathbb{C}^n}(V_{\hat{k}})}$ and thus, by (2.7), determines an operator $P \in \mathcal{E}(G; D) = H^n_Z(D \times (D + M), \mathcal{O}^{(0,n)}_{\mathbb{C}^n \times \mathbb{C}^n})$ so that $K(x, y)$ is a kernel of P. Thus any symbol $P(x, \xi)$ defines a non-local pseudo-differential operator $P \in \mathcal{E}(G; D)$. Furthermore P is equal to 0 if the symbol $P(x, \xi)$ is contained in $N^M(\Omega)$.

Thanks of the G-roundness, D satisfies the following: let π_k be the k-th projection on $\mathbb{C}^n$ to $\mathbb{C}$. Set $A_k := \pi_k(\{z \in \mathbb{C}^n \mid \mathrm{Re}\langle z, \xi_0\rangle \geqslant I_M(\xi_0)\} \cap G)$, $B_1 := \pi_1(G)$ and $C_j := conv A_j$ $(2 \leqslant j \leqslant n)$ (here $conv E$ being the convex-hull of a set $E \subset \mathbb{C}$). We remark that

$$A_1 = \{z_1 = w_1 + x_1 \mid \delta|\mathrm{Im} w_1| \leqslant -\mathrm{Re} w_1, x \in M\} \cap \{z \in \mathbb{C}^n \mid \mathrm{Re}\langle z, \xi_0\rangle \geqslant I_M(\xi_0)\}.$$

There exists an open set $U_0 \subset D, U_0 \neq \emptyset$ such that we have

$$U_0 + [(\partial A_1 \cap \partial B_1) \times \partial C_2 \times \cdots \times \partial C_n] \subset D + M,$$

where ∂C is the boundary set of a set $C \subset \mathbb{C}$. Let define an integral path γ in $\mathbb{C}^n$ such that for an open set $U \Subset U_0, U \neq \emptyset$, the set $\{(x, x + z) \mid x \in U,\ z \in \gamma\}$ is included in the domain $V = \cap_{k=1}^n V_k$ where $K(x, y)dy$ is defined: there are two points $a, b \in \mathbb{C}$ near ∂A_1 such that Re a, Re $b < I_M(\xi_0)$ and Im $a < \inf_{z \in M} \mathrm{Im} z_1$, Im $b > \sup_{z \in M} \mathrm{Im} z_1$. Let take an oriented smooth Jordan path $\gamma_1 \subset \mathbb{C}$ having a_1 as the start point and b_1 as the terminal point so that $\tilde{\gamma}_1 := \{(x, x + (z_1, z_2, \cdots, z_n)) \mid x \in U, z_1 \in \gamma_1, z_j \in \mathbb{C} \ \ (2 \leqslant j \leqslant n)\} \subset V_1$. For $z_1 \in \gamma_1$ and $j \geqslant 2$, let $\gamma_j = \gamma_j(z_1) \subset \mathbb{C}$ be an oriented smooth Jordan closed curve with positive orientation in $\mathbb{C}$ so that $\tilde{\gamma}_j := \{(x, x + (z_1, z_2, \cdots, z_n)) \mid x \in U, z_j \in \gamma_j, z_i \in \mathbb{C} \ \ (i \neq 1, j)\} \subset V_j$. (These are possible if we take U small enough). We remark that γ_j may be taken independently of z_1.

Let $W, W_0 \subset \mathbb{C}^n$ be two G-open sets (G being defined in (2.4)) such that $W \supset W_0$ and $W \setminus W_0 \subset D$. We showed in the Corollary 3.7, [6] that we have the operation

$$P : \mathcal{O}(\tilde{D} \cap W_0)/\mathcal{O}(\tilde{D} \cap W) \to \mathcal{O}(D \cap W_0)/\mathcal{O}(D \cap W).$$

Setting $\gamma := \gamma_1 \times \gamma_2 \times \cdots \times \gamma_n$, this action of P on a holomorphic function $f(x) \in \mathcal{O}(\tilde{D} \cap W_0)$, which is determined modulo $\mathcal{O}(D \cap W)$, is given by

$$Pf(x) = \int_\gamma K(x, x+z) f(x+z) dz. \tag{4.3}$$

We remark here, W and W_0 being G-open, $W + M = W, \ W_0 + M = W_0$.

Theorem 4.1. *Take $p = (x_0, \xi_0)$ with $\xi_0 = (1, 0, \cdots, 0)$ and fix it. Let M be a compact convex set, $\Omega = \mathbb{C}^n \times \omega$ with an open cone $\omega \subset \mathbb{C}^n$ with $\xi_0 \in \omega$ and $P(x, \xi) \in S^M(\Omega)$ a symbol. As stated above, there are cones $\Gamma_{1,\delta}, \Gamma_{2,\delta}, \cdots, \Gamma_{n,\delta}$ as in (2.2) - (2.3) and, setting $G := \cap_{k=1}^n (\Gamma_{k,\delta} + M)$, a non-local pseudo-differential operator $P \in \mathcal{E}(G; \mathbb{C}^n)$, carried by M, with the symbol $P(x, \xi)$. Suppose that P is non M-characteristic in Ω. Let $W, W_0 \subset \mathbb{C}^n$ be two G-open sets such that $W_0 \subset W$ and $W \setminus W_0$ is relatively compact. If $u(x) \in \mathcal{O}(W_0 - M)$ satisfies to the equation $Pu(x) \equiv 0 \mod \mathcal{O}(W - M)$, then $u(x) \in \mathcal{O}(W)$, that is, $u(x)$ can be analytically continued to $W \cup (W_0 - M)$.*

Here we remark $W \subset W - M$.

Proof: By Theorem 2.7, there exists $\omega' \subset \omega$ such that P has the inverse P^{-1} in $\Omega' := \mathbb{C}^n \times \omega'$. By Remark 2.7, taking Ω smaller, we may suppose that P^{-1} has a symbol $Q(x, \xi) \in T^{-M}(\Omega)$. For any ξ, we set $\mathcal{H}_\xi^+ := \{z \in \mathbb{C}^n \mid I_{-M}(\xi) < \mathrm{Re}\, z \cdot \xi\}$ and we define

$$Y := \bigcup_{|\mathrm{Im}\tau| < \delta \mathrm{Re}\tau, \ 0 < \delta' < \delta} \Big(\bigcap_{|\eta'| \leqslant \delta'} \mathcal{H}_{\tau\xi}^+ \Big)^\circ \tag{4.4}$$

where $\xi = (1, \eta')$ and A° means the interior of A. We may assume $\omega = \{\xi \mid |\mathrm{Im}\, \xi_1| < \delta \ \mathrm{Re}\, \xi_1, \ |\xi_j| < \delta |\xi_1| \ (2 \leqslant j \leqslant n)\}$. Then, in this case, we will prove $K(x, y)$ defined in (4.2) for $Q(x, \xi)$ is analytically continued to the open set $Z := \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid z := y - x \in Y\}$ as follows: in fact, for any $z \in Y$, there exists τ with $|\mathrm{Im}\tau| < \delta \mathrm{Re}\tau$ and δ' with $0 < \delta' < \delta$ such that for any $\xi = (1, \eta')$ with $|\eta'| \leqslant \delta'$, we have $I_{-M}(\tau\xi) < \mathrm{Re}\ \tau\xi \cdot z$. Then there exists $\varepsilon > 0$ so small enough that we have

$$I_{-M}(\tau\xi) < \mathrm{Re}\ \tau\xi \cdot z + \varepsilon |\tau\xi| \tag{4.5}$$

for any $\xi = (1, \eta')$ with $|\eta'| \leqslant \delta'$. For such τ and ξ, we have $\tau\xi = (\tau, \tau\xi') \in \omega$. Let consider the integral (4.2). By (2.10), for any $\varepsilon' > 0$ with $\varepsilon' < \varepsilon$, there is $C > 0$ such that

$$\begin{aligned} |\mathrm{e}^{-\tau z\cdot\xi}\tau^{n-1}Q(x,\tau\xi)| &\leqslant C|\tau|^{n-1}\mathrm{e}^{-\mathrm{Re}\ \tau z\cdot\xi + I_{-M}(\tau\xi)+\varepsilon'|\tau\xi|} \\ &\leqslant C|\tau|^{n-1}\mathrm{e}^{-(\varepsilon-\varepsilon')|\tau\xi|}. \end{aligned}$$

This last is estimated by a constant independent of τ and ξ. Therefore the integral (4.2) is holomorphic for any $(x, y) \in \mathbb{C}^n \times \mathbb{C}^n$ with $z := y - x \in Y$.

We remark that

$$Y \Supset \{z \mid \mathrm{Re}z_1 > I_{-M}(\xi_0)\} \tag{4.6}$$

where $\xi_0 := (1, 0, \cdots, 0)$: in fact, let z be so that $\mathrm{Re}z_1 > I_{-M}(\xi_0)$. If $z' = 0$, then we have clearly $z \in Y$. So we assume $z' \neq 0$. There exists $\varepsilon > 0$ such that $\mathrm{Re}z_1 - I_{-M}(\xi_0) > 2\varepsilon|z'|$ and $|\eta'| \max_{w\in -M}|w'| < \varepsilon|z'|$. For any $\delta' > 0$ with $\delta' < \min(\delta, \varepsilon)$ and any η' with $|\eta'| \leqslant \delta'$, we have

$$\mathrm{Re}z\cdot\xi > \mathrm{Re}z_1 - \mathrm{Re}z'\cdot\eta' > I_{-M}(\xi_0) + 2\varepsilon|z'| - \mathrm{Re}z'\cdot\eta' \geqslant I_{-M}(\xi_0) + \varepsilon|z'|.$$

Then we have for $\xi = (1, \eta')$,

$$\begin{aligned} I_{-M}(\xi) &= \inf_{w\in -M}\mathrm{Re}w_1 + \mathrm{Re}w'\cdot\eta' \leqslant \inf_{w\in -M}\mathrm{Re}w_1 + \max_{w\in -M}|w'||\eta'| \\ &< \inf_{w\in -M}\mathrm{Re}w_1 + \varepsilon|z'| = I_{-M}(\xi_0) + \varepsilon|z'| \leqslant \mathrm{Re}z\cdot\xi \end{aligned}$$

that means $z \in Y$.

Let $f(x) \in \mathcal{O}(W - M)$, and by (4.3) for $-M$ and $Q(x, \xi)$, we set $g(x) := P^{-1}f(x)$. Then $g(x)$ is holomorphic on W (and this is also true for W_0 instead of W): in fact, for any $x \in W$, thanks to (4.6), we may change, in (4.3) for $-M$ and $Q(x, \xi)$, the integral path γ to be in an arbitrary small neighbourhood of any point $-m \in -M$. Then for any $z \in \gamma$, we have $x + z \in W - M$ and thus $g(x)$ is well-defined and holomorphic in the set W. The proof for W_0 is similar.

Now set $f(x) := Pu(x)$ which is, by the hypothesis, holomorphic in $W - M$. Then the function $v(x)$ defined by (4.3) for $-M$ and $Q(x, \xi)$ with $f(x)$, is holomorphic in W. Therefore we have

$$u(x) \equiv P^{-1}Pu(x) = P^{-1}f(x) = v(x) \in \mathcal{O}(W)$$

mod $\mathcal{O}(W)$ and thus $u(x)$ is continued analytically to W. □

Example 4.2. Let consider the difference operator $P(D) = 1 + e^D$ on $\mathbb{C}$ and we restrict ourselves at $\xi_0 = -1$. In this case, it has the unique support $M = [0, 1]$ (because $n = 1$). The characteristic set of the corresponding operator $1 + e^D$ is, as easily proved, the imaginary axis directions $\{\pm\sqrt{-1}\infty \in S^1_\infty\}$. For any $\delta > 0$, set the closed cone $\Gamma := \{z \mid \mathrm{Re}z \geqslant \delta|\mathrm{Im}z|\}$. Let W, W_0 be two $\Gamma =$

$(\Gamma+[0,1])$-open sets with $W \supset W_0$ and so that $W \setminus W_0$ is relatively compact and $k := [\inf\{\mathrm{Re}z \mid z \in W\} - \inf\{\mathrm{Re}z \mid z \in W_0\}]$, where $[\cdot]$ is the Gauss' notation. In this case, $P(D)$ has the inverse $Q(D) = 1/(1+e^D)$. We remark here that there exists $C > 0$ such that for any ξ, we have

$$|\frac{1}{1+e^{\xi}}| \leqslant Ce^{-H_{[0,1]}(\xi)} = Ce^{I_{[-1,0]}(\xi)} \leqslant e^{H_{[-1,0]}(\xi)} = Ce^{H_{-M}(\xi)}$$

(c.f. Theorem 2.7). Fix any $\rho > 0$, recalling (4.2), we set

$$G(z) := \int_{\rho}^{\infty} \frac{e^{-\tau z}}{1+e^{\tau}} d\tau. \tag{4.7}$$

It is easy to see that we have for any $l \in \mathbb{Z}_+$,

$$G(z) = \frac{e^{-\rho z}}{z} - \cdots + (-1)^l \frac{e^{-\rho(z-l)}}{z-l} + (-1)^{l+1} G(z-l-1); \tag{4.8}$$

thus $G(z)$ is continued analytically to $\mathbb{C} \setminus \mathbb{Z}_+$ and has pole of ordre 1 at any $l \in \mathbb{Z}_+$.

Let $f(z)$ be a holomorphic function on $W - [0,1] = W + [-1,0]$ and $u(z)$ a holomorphic function on $W_0 - [0,1]$ so that $(1+e^D)u(z) = u(z) + u(z+1) = f(z)$ on W_0. If we take a positively oriented Jordan closed path γ including $0, 1, 2, \cdots, k+1$ inside, but not other point of $\mathbb{Z}$, we have $\mod \ \mathcal{O}(W)$

$$\begin{aligned}
u(x) &\equiv \frac{1}{1+e^D} f(x) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} G(z) f(x+z) dz \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \frac{e^{-\rho z}}{z} f(x+z) dz - \cdots + \frac{(-1)^{k+1}}{2\pi\sqrt{-1}} \int_{\gamma} \frac{e^{-\rho(z-k-1)}}{z-k-1} f(x+z) dz \\
&+ \frac{(-1)^{k+2}}{2\pi\sqrt{-1}} \int_{\gamma} G(z-k-2) f(x+z) dz \\
&= f(x) - f(x+1) + \cdots + (-1)^{k+1} f(x+k+1)
\end{aligned}$$

but this last is holomorphic in $W - [0,1]$. We remark also that this last satisfies the equation $u(x) + u(x+1) \equiv f(x) \mod \ \mathcal{O}(W - [0, k+2])$.

5 Existence of holomorphic solutions of some non-local differential equations of infinite ordre

In this section, according to Aoki [3], we will give a sufficient condition for the existence of holomorphic solutions to a non-local differential equation, which will be the generalization of the Theorem 4.1.2 of [3] to our situation. Let $P \in \mathcal{D}^{\infty}_{[M]}(\mathbb{C}^n)$ be a non-local differential operator carried by M. In this section, we make use of the following assumption: there exists $\delta > 0$ with $\delta < 1$ such that we have

$$Car^M_{\infty}(P) \subset \mathbb{C}^n \times \{\xi\infty \in S^{2n-1}_{\infty} \mid \mathrm{Re}\ \xi_1 \geqslant \delta|\mathrm{Im}\ \xi_1| \ or \ |\xi'| \geqslant \delta|\xi_1|\} \tag{5.1}$$

where $\xi' := (\xi_2, \cdots, \xi_n)$ when $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$. Then by Propositions 2.8, Theorem 2.9 and Remark 2.10, for any open conic set $\Omega \subset \mathbb{C}^n \times \{\xi \in \mathbb{C}^n \mid \text{Re } \xi_1 < \delta|\text{Im } \xi_1|, |\xi'| < \delta|\xi_1|\}$, we have a non-local pseudo-differential operator Q, inverse to P, carried by $-M$ with a formal symbol $Q(t; x, \xi) = \sum_{\nu=0}^{\infty} t^\nu Q_\nu(x, \xi) \in \hat{t}^{-M}(\Omega)$ where each $Q_\nu(x, \xi) \in \mathcal{O}(\Omega((\nu + 1)r))$ with $r > 0$ and satisfy the following:

for any open conic set $\Omega' \Subset \Omega$ and any $h > 0$, there exists $d > r$ such that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ so that we have

$$|Q_\nu(x, \xi)| \leqslant C_\varepsilon h^\nu e^{I_{-M}(\xi)+\varepsilon|\xi|} \quad \text{on } \Omega'[(\nu + 1)d]. \tag{5.2}$$

For any $\delta' > 0$ with $\delta' < \delta$, take $\delta_1 > 0$ with $0 < \delta' < \delta_1 < \delta$ and define the path as follows: set $r_1 := d, r_2 := \delta d$ and for any $\nu \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^n$, set $B_{k,\nu,\alpha} := (\nu + |\alpha| + 1)r_k \quad (k = 1, 2)$,

$$\lambda^{\pm}_{1,\nu,\alpha} := \{\xi_1 \in \mathbb{C} \mid \text{Re } \xi_1 = \pm\delta_1 \text{Im } \xi_1, |\xi_1| \geqslant B_{1,\nu,\alpha}\}, \tag{5.3}$$

$$\lambda^{0}_{1,\nu,\alpha} := \{\xi_1 \in \mathbb{C} \mid \text{Re } \xi_1 \leqslant \delta_1|\text{Im } \xi_1|, |\xi_1| = B_{1,\nu,\alpha}\} \tag{5.4}$$

and $\lambda_{1,\nu,\alpha} := \lambda^0_{1,\nu,\alpha} + \lambda^+_{1,\nu,\alpha} + \lambda^-_{1,\nu,\alpha}$. And also for any $i = 2, 3, \cdots, n$, we set

$$\lambda_{i,\nu,\alpha} := \{\xi_i \in \mathbb{C} \mid |\xi_i| = B_{2,\nu,\alpha}\}. \tag{5.5}$$

We set $\lambda_{\nu,\alpha} := \lambda_{1,\nu,\alpha} \times \lambda_{2,\nu,\alpha} \times \cdots \times \lambda_{n,\nu,\alpha}$ and also $\lambda^0_{\nu,\alpha} := \lambda^0_{1,\nu,\alpha} \times \lambda_{2,\nu,\alpha} \times \cdots \times \lambda_{n,\nu,\alpha}$ and $\lambda^{\pm}_{\nu,\alpha} := \lambda^{\pm}_{1,\nu,\alpha} \times \lambda_{2,\nu,\alpha} \times \cdots \times \lambda_{n,\nu,\alpha}$. Then we have $\lambda_{\nu,\alpha} = \lambda^0_{\nu,\alpha} + \lambda^+_{\nu,\alpha} + \lambda^-_{\nu,\alpha}$.

For any $\rho > 0$ and $c \in \mathbb{C}^n$, we set the polydisk $\Delta(a; \rho) := \{x \in \mathbb{C}^n \mid |x_j - c_j| < \rho \quad (1 \leqslant j \leqslant n)\}$. Now let $U, V \subset \mathbb{C}^n$ be open sets with $V \Subset U$ so small that we have the following: there exist $b \in U$, $m \in M$ and $\rho > 0$ such that, setting $a := b - m$, we have

$$\begin{aligned} &V \subset \Delta(b; \ln(d\rho)/\sqrt{n}d) \cap [\{x \in \mathbb{C}^n \mid \delta\text{Re } x_1 < |\text{Im } x_1|, \delta|x'| < |x_1|\} + a], \\ &\Delta(a; \rho) \subset U + M - M. \end{aligned} \tag{5.6}$$

Then for small $\varepsilon > 0$ and $x \in V$, we have

$$\begin{aligned} \text{Re } (x - a) \cdot \xi + I_{-M}(\xi) + \varepsilon|\xi| &= \text{Re } (x - b) \cdot \xi + (I_{-M}(\xi) - \text{Re } (-m) \cdot \xi) + \varepsilon|\xi| \\ &\leqslant \text{Re } (x - b) \cdot \xi + \varepsilon|\xi|. \end{aligned}$$

By (5.6), taking $\varepsilon > 0$ small enough, we have $\text{Re } (x - b) \cdot \xi + \varepsilon|\xi| < \frac{\ln(d\rho)}{\sqrt{n}d}|\xi|$ and on $\lambda^{\pm}_{\nu,\alpha}$, we have $\text{Re } (x - b) \cdot \xi + \varepsilon|\xi| < 0$. Now let $f(x) \in \mathcal{O}_{\mathbb{C}^n}(U + M - M)$ and expand $f(x)$ into the Taylor series at a: $f(x) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha (x - a)^\alpha$. We define

$$u(x) := \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\nu \in \mathbb{Z}_+, \alpha \in \mathbb{Z}_+^n} c_\alpha \int_{\lambda_{\nu,\alpha}} Q_\nu(x, \xi) \frac{\alpha!}{\xi^{\alpha+1}} e^{(x-a)\cdot\xi} d\xi \tag{5.7}$$

with $\mathbf{1} := (1, 1, \cdots, 1)$ and we will prove the right-hand side converges for $x \in V$ ($x \in V$ being arbitrary, this means $u(x) \in \mathcal{O}_{\mathbb{C}^n}(V)$): At first, by (5.6), we remark that there exists $C_1 > 0$ such that $|c_\alpha| \leqslant C_1 \rho^{-|\alpha|}$. Then by (5.2), we have

$$|c_\alpha Q_\nu(x,\xi) \frac{\alpha!}{\xi^{\alpha+\mathbf{1}}} e^{(x-a)\cdot\xi}| \leqslant C_1 C_\varepsilon h^\nu \frac{\rho^n \alpha!}{[\rho(\nu+|\alpha|+1)d]^{|\alpha|+n}} e^{\mathrm{Re}(x-a)\cdot\xi + I_{-M}(\xi)+\varepsilon|\xi|}.$$

On $\lambda^{\pm}_{\nu,\alpha}$, the right-hand side is estimated as $< C_1 C_\varepsilon h^\nu \frac{\rho^n \alpha!}{[\rho(\nu+|\alpha|+1)d]^{|\alpha|+n}}$ and on $\lambda^0_{\nu,\alpha}$, the right-hand side is estimated as $< C_1 C_\varepsilon (hd\rho)^\nu \rho d^{1-n}$. Thus, taking $h > 0$ small enough, (5.7) gives well-defined holomorphic function on V. We are ready to prove:

Theorem 5.1. *Let $M \subset \mathbb{C}^n$ be a comact convex set and P a non-local differential operator carried by M. We suppose $Car^M_\infty(P) \subset \mathbb{C}^n \times \{\xi\infty \in S^{2n-1}_\infty \mid \mathrm{Re}\ \xi_1 \geqslant \delta|\mathrm{Im}\ \xi_1|$ or $|\xi'| \geqslant \delta|\xi_1|\}$. Then for any open sets $U \subset \mathbb{C}^n$, any point $a \in U$ and $f(x) \in \mathcal{O}(U + M - M)$, we can find an open neighbourhood $V \subset U$ of a such that we have a solution $u(x) \in \mathcal{O}(V)$ of the equation $Pu(x) = f(x)$ on V.*

Proof: We continue to work in the above situation. Denote by $P(x,\xi)$ the symbol of P. Then we have

$$Pu(x) = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\nu\in\mathbb{Z}_+,\alpha\in\mathbb{Z}^n_+} c_\alpha \int_{\lambda_{\nu,\alpha}} \frac{\alpha!}{\xi^{\alpha+\mathbf{1}}} P(Q_\nu(x,\xi)e^{(x-a)\cdot\xi}) d\xi$$

$$= \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\nu\in\mathbb{Z}_+,\alpha\in\mathbb{Z}^n_+} c_\alpha \int_{\lambda_{\nu,\alpha}} \frac{\alpha!}{\xi^{\alpha+\mathbf{1}}} e^{(x-a)\cdot\xi} \sum_\beta \frac{1}{\beta!} \partial^\beta_\xi P(x,\xi) \partial^\beta_x Q_\nu(x,\xi) d\xi$$

$$= \frac{1}{(2\pi\sqrt{-1})^n} \sum_\gamma c_\gamma \int_{\lambda_{\nu,\alpha}} \frac{\gamma!}{\xi^{\gamma+\mathbf{1}}} e^{(x-a)\cdot\xi} d\xi = \sum_\gamma c_\gamma (x-a)^\gamma = f(x).$$

□

Example 5.2. Let $n = 1$ and consider the operator $P = e^{cxD}$ with $c \in \mathbb{C}$. We know for any $\delta > 0$, its symbol $P(x,\xi) = e^{cx\xi}$ is contained in $S^{\overline{B}(0;|c|\delta)}(B(0;\delta) \times \mathbb{C})$. As we can see easily that we have $\underline{Car}^{\overline{B}(0;|c|\delta)}_\infty(e^{cxD}) = \emptyset$, but supposing $\xi \neq 0, c \neq 0$, we have $Car^{\overline{B}(0;|c|\delta)}_\infty(e^{cxD}) \neq \emptyset$. The equation $e^{cxD}u(x) = u((c+1)x) = f(x)$ has for any $c \neq -1$, unique solution $u(x) = f(x/(c+1))$ and no solution for $c = -1$ in general (the equation being $u(0) = f(x)$ in this case). Thus for $c \neq -1$, the non-local operator e^{cxD} has the inverse operator $e^{-\frac{c}{c+1}xD}$ having the symbol $e^{-\frac{c}{c+1}x\xi} \in S^{\overline{B}(0;|c|\delta)}(B(0;|c+1|\delta) \times \mathbb{C})$.

Remark 5.3. The proof of Theorem 5.1 is not true with the assumption of non $\underline{M}$ -charactericity instead of the non M-charactericity. Above example is not an application of Theorem 5.1 but we may conclude that the definition of non M-charactericity is better than the non $\underline{M}$-charactericity.

6 Construction of entire holomorphic solutions to non-local differential equations with constant coefficients

In section 5, for a non-local differential equation $Pu(x) = f(x)$, we gave a sufficient condition - call it *Aoki's condition* - for the existence of a holomorphic solution. But in many case, in fact, this condition is not satisfied. For example, we recall the following difference equation in one variable: $u(x)+u(x+1) = f(x)$. The characteristic set of the corresponding operator $1 + e^D$ being the imaginary axis directions $\mathbb{C} \times \{\pm\sqrt{-1}\infty \in S^1_\infty\}$, the Aoki's condition (5.1) could not be satisfied.

However, we will present another way to construct concretely a holomorphic solution to non-local differential equations in the case of constant coeffocients. We emphasize that in such method, we do not assume any non-characterisity condition. This make us possible to calculate many examples, for example, we will present an operational calculus for constant coefficients differential-differnce equations. Let $P \in \mathcal{D}^\infty_{[M]}(\mathbb{C}^n)$ be a non-local differential operator with constant coefficient symbol $P(\xi) \in S^M(\mathbb{C}^n \times \mathbb{C}^n)$ satisfying (2.8) with $r = 0$ (and in particular $P(\xi) \in \mathcal{O}(\mathbb{C}^n)$), and we define the *zero divisor* of its symbol

$$Z_P := \{\xi \in \mathbb{C}^n \mid P(\xi) = 0\}. \tag{6.1}$$

(Remark that in the differential operator case, the symbol is uniquely determined). So, in the above example, we have $Z_{(1+e^D)} = \{(2k+1)\pi\sqrt{-1} \mid k \in \mathbb{Z}\}$.

Instead of the Aoki's condition, we impose the following conditions:

(I) There exists a sequence (ρ_ν) with $\rho_\nu > 0$ such that the following limit exists:

$$c := \lim_{\nu\to\infty} \frac{\rho_\nu}{\nu} > 0. \tag{6.2}$$

(II) There exist $\delta > 0$, $b > 0$ and positively oriented Jordan closed curves C^j_ν $(1 \leqslant j \leqslant n)$ in $\mathbb{C}$ arround 0, depending continuously on x, such that, setting $C_\nu := C^1_\nu \times C^2_\nu \times \cdots \times C^n_\nu$, for any $\nu \in \mathbb{Z}_+$, we have

$$b\rho_\nu \leqslant |\xi| \leqslant b^{-1}\rho_\nu \qquad (\forall \xi \in C_\nu), \tag{6.3}$$

$$dist(Z_P, C_\nu) \geqslant \delta. \tag{6.4}$$

By a translation, we may assume that we have $P(0) \neq 0$. For any $\xi \in C_\nu$, set $\eta := \dfrac{\xi}{|\xi|}$. Recall $H = \max_{|\zeta|=1} H_M(\zeta)$. Then, setting $K := \max_{|\zeta|=2\mathrm{e}} H_M(\zeta)+2\mathrm{e}$, there exists $C > 0$ such that we have for any $R > 0$

$$\begin{aligned}\max_{|\tau|=2\mathrm{e}R} \ln|P(\tau\eta)| &\leqslant \max_{|\tau|=2\mathrm{e}R}(C + H_M(\tau\eta) + |\tau\eta|)\\ &\leqslant C + R\max_{|\tau|=2\mathrm{e}R}(H_M(\frac{\tau\eta}{R}) + |\frac{\tau\eta}{R}|) \leqslant C + RK.\end{aligned}$$

By Levin [16], p.21 Theorem 11, for $0 < \forall\varepsilon < 3\mathrm{e}/2$, there exist $a_1, a_2, \cdots, a_N \in B(0;R)$ and $r_1, r_2, \cdots, r_N > 0$ such that we have

$$r_1 + r_2 + \cdots + r_N < 4\varepsilon R,\ B(a_j;r_j) \cap \{\tau \mid P(\tau\eta) = 0\} \neq \emptyset \quad (1 \leqslant j \leqslant N),$$
$$(\overline{B}(0;R) \setminus \cup_{j=1}^N B(a_j;r_j)) \cap \{\tau \mid P(\tau\eta) = 0\} = \emptyset$$

and that we have $\ln|P(\tau\eta)| > -h(\varepsilon)(C + RK) + (h(\varepsilon)+1)\ln|P(0)|$ for any $\tau \in \overline{B}(0;R) \setminus \cup_{j=1}^N B(a_j;r_j)$. Now for $\xi \in C_\nu$, take $R := b^{-1}\rho_\nu + \delta$ and $\tau := |\xi|$: we have $\tau \leqslant |\xi| \leqslant R$. There is j with $dist(\tau, B(a_j;r_j)) = \min_{1\leqslant k\leqslant N} dist(\tau, B(a_k;r_k))$. So $\exists\tau' \in B(a_j;r_j) \cap \{\tau''|P(\tau''\eta) = 0\}$ s. t.

$$|\tau - \tau'| = \min_{\tau'' \in B(a_j;r_j)\cap\{\tau''|P(\tau''\eta)=0\}} |\tau - \tau''|.$$

Set $\xi' := \tau'\eta$, then $\xi' \in Z_P$. By (6.4), we have $\delta < |\xi - \xi'| = |\tau - \tau'| \cdot |\eta| = |\tau - \tau'|$. Taking $\varepsilon > 0$ so small that we have $\delta > 16\varepsilon R > 2r_j$, we have $|\tau - \tau'| > 2r_j$ and so $\tau \notin B(a_j;r_j)$. Then for any k, we have $\tau \notin B(a_k;r_k)$ i.e. $\tau \in \overline{B}(0;R) \setminus \cup_{j=1}^N B(a_j;r_j)$. Thus we have

$$\ln|P(\xi)| > -h(\varepsilon)(C + (b^{-1}\rho_\nu + \delta)K) + (h(\varepsilon)+1)\ln|P(0)|.$$

Therefore setting $K_\varepsilon := b^{-2}h(\varepsilon)K$ and $C_\varepsilon := \mathrm{e}^{Ch(\varepsilon)+\delta K} + (h(\varepsilon)-1)\ln|P(0)|$, by (6.3), we have on each C_ν,

$$\frac{1}{|P(\xi)|} \leqslant C_\varepsilon \mathrm{e}^{K_\varepsilon|\xi|}. \tag{6.5}$$

Now for any entire function $f(x) = \sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha x^\alpha \in \mathcal{O}(\mathbb{C}^n)$, we define

$$u(x) := \sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha \oint_{C_{|\alpha|}} \frac{\mathrm{e}^{x\cdot\xi}}{P(\xi)} \Phi_\alpha(-\xi)d\xi \tag{6.6}$$

(where the function $\Phi_\alpha(\xi)$ is defined in [18]; for example, we refere to [6], (5.8)). For any $\delta > 0$, there exists $C' > 0$ such that $|c_\alpha| \leqslant C'\delta^{|\alpha|}$ for any α. By (II), setting $d := c/b$ (b, c being in (6.2) and (6.3)), we have that $u(x)$ converges and then is a holomorphic function for any $x \in \mathbb{C}^n$. We can easily verify that $u(x)$ is a solution of the equation $Pu(x) = f(x)$. We have thus proved the following:

Theorem 6.1. *Let $P(\xi)$ be a constant coefficient symbol of a non-local differential operator P. Suppose that $P(\xi)$ satisfies the conditions* (I) *and* (II). *Then for any $f(x) \in \mathcal{O}(\mathbb{C}^n)$, the function $u(x)$ defined by the above formula* (6.6) *is an entire holomorphic function on $\mathbb{C}^n$ and is a special solution of the non-local differential equation $Pu(x) = f(x)$.*

7 Example: An operational calculus for differential-difference equations with constant coefficients

As the following Example shows, Theorem 6.1 leads to develop an operational calculus for differential-difference equations with constant coefficients.

Example 7.1. $u(x) + u(x+1) = \mathrm{e}^{\lambda x}$.

Recall $Z_{(1+e^D)} = \mathbb{C} \times \{(2k+1)\pi\sqrt{-1} \mid k \in \mathbb{Z}\}$; $\exists\delta > 0$ small enough s.t, $|\lambda| - \delta \notin 2\mathbb{Z}\pi$. Take integral paths $C_k := \{\xi \mid |\xi| = 2k\pi + \delta\}$ $(k \in \mathbb{Z}_+)$. Then $\{\xi \in \mathbb{C} \mid \mathbb{C} \times \{\xi\infty\} \subset Car_\infty(P),\ \xi \text{ is inside of } C_k\} = \{(2j+1)\pi\sqrt{-1} \mid -k \leqslant j \leqslant k-1\}$. Setting $F_k(x) := \dfrac{1}{2\pi\sqrt{-1}} \oint_{C_k} \dfrac{\mathrm{e}^{x\xi}}{1+\mathrm{e}^{\xi}} \dfrac{k!}{\xi^{k+1}} d\xi$, we have a solution $u(x) = \displaystyle\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} F_k(x)$:

$$\begin{aligned} F_k(x) &= \mathrm{Res}_{\xi=0} \frac{k!\mathrm{e}^{x\xi}}{\xi^{k+1}(1+e^{\xi})} + \sum_{j=-k}^{k-1} \mathrm{Res}_{\xi=(2j+1)\pi\sqrt{-1}} \frac{k!\mathrm{e}^{x\xi}}{\xi^{k+1}(1+e^{\xi})} \\ &= \partial_\xi^k \left(\frac{\mathrm{e}^{x\xi}}{1+\mathrm{e}^{\xi}}\right)_{|\xi=0} - \sum_{j=-k}^{k-1} \frac{\mathrm{e}^{(2j+1)\pi\sqrt{-1}x}}{((2j+1)\pi\sqrt{-1})^{k+1}}, \end{aligned}$$

$$F_k(x+1) = \partial_\xi^k \left(\frac{\mathrm{e}^{(x+1)\xi}}{1+\mathrm{e}^{\xi}}\right)_{|\xi=0} - \sum_{j=-k}^{k-1} \frac{\mathrm{e}^{(2j+1)\pi\sqrt{-1}(x+1)}}{((2j+1)\pi\sqrt{-1})^{k+1}} = x^k - F_k(x).$$

Thus we have

$$(1+\mathrm{e}^D)u(x) = u(x) + u(x+1) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(F_k(x) + F_k(x+1)) = \sum_{k=0}^{\infty} \frac{\lambda^k x^k}{k!} = \mathrm{e}^{\lambda x}.$$

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