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On the periodicity of some systems of nonlinear difference equations

by
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Abstract

In this paper we deal with the form of the solutions of some difference systems on a rational form of second order with nonzero real number initial conditions.

Key Words: periodic solutions, system of difference equations.
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1 Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so on. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solution see [1]–[16] and the references cited therein. There are many papers related to the difference equations system, see for example [5], [6], [17]–[18].

Our aim in this paper is to get the form of the solutions of the following rational difference systems

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)},\ y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)},$$

with nonzero real number initial conditions.

2 The System: $x_{n+1} = \frac{y_n}{x_{n-1}(1+y_n)}$, $y_{n+1} = \frac{x_n}{y_{n-1}(1+x_n)}$

In this section, we study the solutions of the system of the difference equations

$$x_{n+1} = \frac{y_n}{x_{n-1}(1 + y_n)}, \quad y_{n+1} = \frac{x_n}{y_{n-1}(1 + x_n)}, \quad n = 0, 1, ..., \tag{1}$$

with nonzero real initials conditions x_0, x_{-1}, y_0, y_{-1} such that x_0, x_{-1}, y_0, $y_{-1} \neq -1$, $(x_0 + y_{-1}x_0 + y_{-1})(y_0 + x_{-1}y_0 + x_{-1}) \neq 0$.

Theorem 1. *Let $\{x_n, y_n\}_{n=-1}^{+\infty}$ be solutions of system (1). Then*

1. $x_{n+5} = y_n$, $y_{n+5} = x_n$ *for* $n \geq -1$.
2. $\{x_n\}_{n=-1}^{+\infty}$ *and* $\{y_n\}_{n=-1}^{+\infty}$ *are periodic with period ten i.e.,*

$$x_{n+10} = x_n,\ y_{n+10} = y_n,$$

for $n \geq -1$.

3. *We have*

$$\begin{aligned} x_{10n-1} = x_{-1}, \quad x_{10n} = x_0, \quad x_{10n+1} &= \frac{y_0}{x_{-1}(1+y_0)}, \\ x_{10n+2} &= \frac{1}{x_0 + y_{-1}x_0 + y_{-1}}, \\ x_{10n+3} = \frac{x_{-1}}{y_0(1+x_{-1})}, \quad x_{10n+4} &= y_{-1}, \quad x_{10n+5} = y_0, \\ x_{10n+6} = \frac{x_0}{y_{-1}(1+x_0)}, x_{10n+7} = \frac{1}{y_0 + x_{-1}y_0 + x_{-1}}, \quad x_{10n+8} &= \frac{y_{-1}}{x_0(1+y_{-1})}, \end{aligned}$$

and

$$\begin{aligned} y_{10n-1} = y_{-1}, \quad y_{10n} = y_0, \quad y_{10n+1} &= \frac{x_0}{y_{-1}(x_0+1)}, \\ y_{10n+2} &= \frac{1}{y_0 + x_{-1}y_0 + x_{-1}}, \\ y_{10n+3} = \frac{y_{-1}}{x_0(1+y_{-1})}, \quad y_{10n+4} &= x_{-1}, \quad y_{10n+5} = x_0, \\ y_{10n+6} = \frac{y_0}{x_{-1}(y_0+1)}, y_{10n+7} = \frac{1}{x_0 + y_{-1}x_0 + y_{-1}}, \quad y_{10n+8} &= \frac{x_{-1}}{y_0(1+x_{-1})}. \end{aligned}$$

Or equivalently

$$\begin{aligned} \{x_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} x_{-1},\ x_0,\ \frac{y_0}{x_{-1}(1+y_0)},\ \frac{1}{x_0+y_{-1}x_0+y_{-1}},\ \frac{x_{-1}}{y_0(1+x_{-1})},\ y_{-1}, y_0, \\ \frac{x_0}{y_{-1}(1+x_0)},\ \frac{1}{y_0+x_{-1}y_0+x_{-1}},\ \frac{y_{-1}}{x_0(1+y_{-1})},\ x_{-1},\ x_0, \ldots \end{array} \right\} \\ \{y_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} y_{-1},\ y_0,\ \frac{x_0}{y_{-1}(1+x_0)},\ \frac{1}{y_0+x_{-1}y_0+x_{-1}},\ \frac{y_{-1}}{x_0(1+y_{-1})},\ x_{-1}, x_0, \\ \frac{y_0}{x_{-1}(1+y_0)},\ \frac{1}{x_0+y_{-1}x_0+y_{-1}},\ \frac{x_{-1}}{y_0(1+x_{-1})},\ y_{-1},\ y_0, \ldots \end{array} \right\}. \end{aligned}$$

Proof: 1. From Eq.(1) we see that

$$\begin{aligned}
x_{n+5} &= \frac{y_{n+4}}{x_{n+3}(1+y_{n+4})}, \quad y_{n+5}=\frac{x_{n+4}}{y_{n+3}(1+x_{n+4})},\\
x_{n+5} &= \frac{\frac{x_{n+3}}{y_{n+2}(1+x_{n+3})}}{x_{n+3}(1+\frac{x_{n+3}}{y_{n+2}(1+x_{n+3})})}, \quad y_{n+5}=\frac{\frac{y_{n+3}}{x_{n+2}(1+y_{n+3})}}{y_{n+3}(1+\frac{y_{n+3}}{x_{n+2}(1+y_{n+3})})},\\
x_{n+5} &= \frac{1}{x_{n+3}(1+y_{n+2})+y_{n+2}}, \quad y_{n+5}=\frac{1}{y_{n+3}(1+x_{n+2})+x_{n+2}},\\
x_{n+5} &= \frac{1}{\frac{y_{n+2}(1+y_{n+2})}{x_{n+1}(1+y_{n+2})}+y_{n+2}}, \quad y_{n+5}=\frac{1}{\frac{x_{n+2}(1+x_{n+2})}{y_{n+1}(1+x_{n+2})}+x_{n+2}},\\
x_{n+5} &= \frac{x_{n+1}}{y_{n+2}(1+x_{n+1})}, \quad y_{n+5}=\frac{y_{n+1}}{x_{n+2}(1+y_{n+1})},\\
x_{n+5} &= \frac{x_{n+1}}{\frac{x_{n+1}(1+x_{n+1})}{y_n(1+x_{n+1})}}, \quad y_{n+5}=\frac{y_{n+1}}{\frac{y_{n+1}(1+y_{n+1})}{x_n(1+y_{n+1})}}.
\end{aligned}$$

Therefore

$$x_{n+5}=y_n, \quad y_{n+5}=x_n.$$

2. From 1), we get

$$x_{n+10}=y_{n+5}=x_n,$$

and

$$y_{n+10}=x_{n+5}=y_n.$$

3. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$x_{10n-11}=x_{-1},\ x_{10n-10}=x_0,\ x_{10n-9}=\frac{y_0}{x_{-1}(1+y_0)},$$
$$x_{10n-8}=\frac{1}{x_0+y_{-1}x_0+y_{-1}},$$
$$x_{10n-7}=\frac{x_{-1}}{y_0(1+x_{-1})}, \quad x_{10n-6}=y_{-1},\ x_{10n-5}=y_0,$$
$$x_{10n-4}=\frac{x_0}{y_{-1}(1+x_0)}, x_{10n-3}=\frac{1}{y_0+x_{-1}y_0+x_{-1}}, \quad x_{10n-2}=\frac{y_{-1}}{x_0(1+y_{-1})},$$

and

$$y_{10n-11}=y_{-1},\ y_{10n-10}=y_0,\ y_{10n-9}=\frac{x_0}{y_{-1}(x_0+1)},$$
$$y_{10n-8}=\frac{1}{y_0+x_{-1}+y_0+x_{-1}},$$
$$y_{10n-7}=\frac{y_{-1}}{x_0(1+y_{-1})},\ y_{10n-6}=x_{-1},\ y_{10n-5}=x_0,$$
$$y_{10n-4}=\frac{y_0}{x_{-1}(y_0+1)}, y_{10n-3}=\frac{1}{x_0+y_{-1}x_0+y_{-1}}, y_{10n-2}=\frac{x_{-1}}{y_0(1+x_{-1})}.$$

Now, it follows from Eq.(1) that

$$
\begin{aligned}
x_{10n-1} &= \frac{y_{10n-2}}{x_{10n-3}(1+y_{10n-2})} = \frac{\frac{x_{-1}}{y_0(1+x_{-1})}}{\left(\frac{1}{y_0+x_{-1}y_0+x_{-1}}\right)\left(1+\frac{x_{-1}}{y_0(1+x_{-1})}\right)} = x_{-1}, \\
y_{10n-1} &= \frac{x_{10n-2}}{y_{10n-3}(1+x_{10n-2})} = \frac{\frac{y_{-1}}{x_0(1+y_{-1})}}{\left(\frac{1}{x_0+y_{-1}x_0+y_{-1}}\right)\left(1+\frac{y_{-1}}{x_0(1+y_{-1})}\right)} = y_{-1},
\end{aligned}
$$

$$
\begin{aligned}
x_{10n} &= \frac{y_{10n-1}}{x_{10n-2}(1+y_{10n-1})} = \frac{y_{-1}}{\left(\frac{y_{-1}}{(1+y_{-1})x_0}\right)(1+y_{-1})} = x_0, \\
y_{10n} &= \frac{x_{10n-1}}{y_{10n-2}(1+x_{10n-1})} = \frac{x_{-1}}{\left(\frac{x_{-1}}{(1+x_{-1})y_0}\right)(1+x_{-1})} = y_0,
\end{aligned}
$$

and

$$
\begin{aligned}
x_{10n+1} &= \frac{y_{10n}}{x_{10n-1}(1+y_{10n})} = \frac{y_0}{x_{-1}(1+y_0)}, \\
y_{10n+1} &= \frac{x_{10n}}{y_{10n-1}(1+x_{10n})} = \frac{x_0}{y_{-1}(1+x_0)}.
\end{aligned}
$$

Similarly one can prove the others relations. The proof is complete. □

Example 1. *In order to illustrate the results of this section and to support our theoretical discussions, we consider interesting numerical example for the difference system (1) with the initial conditions* $x_{-1} = -2,\ x_0 = 0.5,\ y_{-1} = 0.7$ *and* $y_0 = 0.6$.*(See Fig. 1).*

The following cases can be proved similarly.

3 The System: $x_{n+1} = \frac{y_n}{x_{n-1}(-1+y_n)}$, $y_{n+1} = \frac{x_n}{y_{n-1}(-1+x_n)}$

In this section, we obtain the solutions of the following system of difference equations

$$x_{n+1} = \frac{y_n}{x_{n-1}(-1+y_n)},\ y_{n+1} = \frac{x_n}{y_{n-1}(-1+x_n)},\ n = 0, 1, ..., \tag{2}$$

with nonzero real initials conditions $x_0,\ x_{-1},\ y_0,\ y_{-1}$ such that $x_0,\ x_{-1},\ y_0,\ y_{-1} \neq 1,\ (x_0 - y_{-1}x_0 + y_{-1})(y_0 - x_{-1}y_0 + x_{-1}) \neq 0$.

Theorem 2. *Let* $\{x_n, y_n\}_{n=-1}^{+\infty}$ *be solutions of system (2). Then*

1. $x_{n+5} = y_n,\ y_{n+5} = x_n$ *for* $n \geq -1$.

2. $\{x_n\}_{n=-1}^{+\infty}$ *and* $\{y_n\}_{n=-1}^{+\infty}$ *are periodic with period ten i.e.,*

$$x_{n+10} = x_n,\ y_{n+10} = y_n,$$

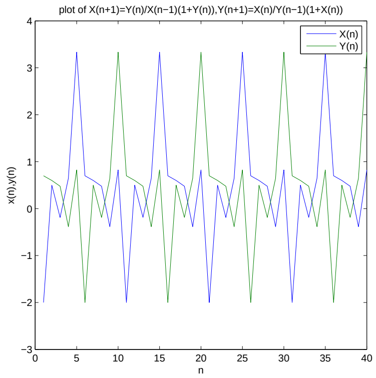

Figure 1: This figure shows the periodicity of the solutions of system (1) with the initial conditions $x_{-1} = -2$, $x_0 = 0.5$, $y_{-1} = 0.7$ and $y_0 = 0.6$.

for $n \geq -1$.

3. We have

$$\begin{aligned}
x_{10n-1} &= x_{-1},\ x_{10n} = x_0,\ x_{10n+1} = \frac{y_0}{x_{-1}(-1+y_0)},\ x_{10n+2} = \frac{1}{x_0 - y_{-1}x_0 + y_{-1}},\\
x_{10n+3} &= \frac{x_{-1}}{y_0(-1+x_{-1})},\ x_{10n+4} = y_{-1},\ x_{10n+5} = y_0,\ x_{10n+6} = \frac{x_0}{y_{-1}(-1+x_0)},\\
x_{10n+7} &= \frac{1}{y_0 - x_{-1}y_0 + x_{-1}},\ x_{10n+8} = \frac{y_{-1}}{x_0(-1+y_{-1})},
\end{aligned}$$

and

$$\begin{aligned}
y_{10n-1} &= y_{-1},\ y_{10n} = y_0,\ y_{10n+1} = \frac{x_0}{y_{-1}(-1+x_0)},\ y_{10n+2} = \frac{1}{y_0 - x_{-1}y_0 + x_{-1}},\\
y_{10n+3} &= \frac{y_{-1}}{x_0(-1+y_{-1})},\ y_{10n+4} = x_{-1},\ y_{10n+5} = x_0,\ y_{10n+6} = \frac{y_0}{x_{-1}(-1+y_0)},\\
y_{10n+7} &= \frac{1}{x_0 - y_{-1}x_0 + y_{-1}},\ y_{10n+8} = \frac{x_{-1}}{y_0(-1+x_{-1})}.
\end{aligned}$$

Or equivalently

$$\begin{aligned}
\{x_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} x_{-1},\ x_0,\ \frac{y_0}{x_{-1}(-1+y_0)},\ \frac{1}{x_0-y_{-1}x_0+y_{-1}},\ \frac{x_{-1}}{y_0(-1+x_{-1})},\\ y_{-1}, y_0, \frac{x_0}{y_{-1}(-1+x_0)},\ \frac{1}{y_0-x_{-1}y_0+x_{-1}},\ \frac{y_{-1}}{x_0(-1+y_{-1})},\ x_{-1},\ x_0, \ldots \end{array} \right\},\\
\{y_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} y_{-1},\ y_0,\ \frac{x_0}{y_{-1}(-1+x_0)},\ \frac{1}{y_0-x_{-1}y_0+x_{-1}},\ \frac{y_{-1}}{x_0(-1+y_{-1})},\\ x_{-1}, x_0, \frac{y_0}{x_{-1}(-1+y_0)},\ \frac{1}{x_0-y_{-1}x_0+y_{-1}},\ \frac{x_{-1}}{y_0(-1+x_{-1})},\ y_{-1},\ y_0, \ldots \end{array} \right\}.
\end{aligned}$$

Example 2. *For the initial conditions* $x_{-1} = 0.1$, $x_0 = 0.6$, $y_{-1} = 0.5$ *and* $y_0 = -0.3$ *when we take the system (2). (See Fig. 2).*

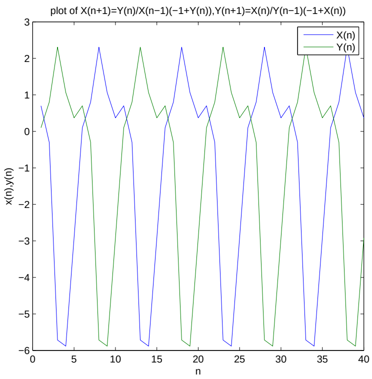

Figure 2: This figure shows the periodicity of the solutions of system (2) with the initial values $x_{-1} = 0.1$, $x_0 = 0.6$, $y_{-1} = 0.5$ and $y_0 = -0.3$.

Remark 1. *Consider the systems*

$$\begin{aligned} x_{n+1} &= \frac{y_n}{x_{n-1}(1+y_n)},\ y_{n+1} = \frac{x_n}{y_{n-1}(-1+x_n)},\ n = 0,1,..., &(3)\\ x_{n+1} &= \frac{y_n}{x_{n-1}(-1+y_n)},\ y_{n+1} = \frac{x_n}{y_{n-1}(1+x_n)},\ n = 0,1,..., &(4)\end{aligned}$$

with nonzero real initials conditions $x_0,\ x_{-1},\ y_0,\ y_{-1}$.

- *Let* $\{x_n, y_n\}_{n=-1}^{+\infty}$ *be solutions of system (3) such that* $x_0,\ x_{-1} \neq 1,\ \ y_0,\ y_{-1} \neq -1,\ (-x_0 - y_{-1}x_0 + y_{-1})(y_0 - x_{-1}y_0 - x_{-1}) \neq 0$. *Then*

$$\begin{aligned} \{x_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} x_{-1},\ x_0,\ \frac{y_0}{x_{-1}(1+y_0)},\ \frac{1}{x_0+y_{-1}x_0-y_{-1}},\ \frac{x_{-1}}{y_0(1-x_{-1})},\ -y_{-1},\ -y_0, \\ \frac{x_0}{y_{-1}(1-x_0)},\ \frac{1}{-y_0+x_{-1}y_0+x_{-1}},\ \frac{y_{-1}}{x_0(1+y_{-1})},\ x_{-1},\ x_0, ... \end{array} \right\} \\ \{y_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} y_{-1},\ y_0,\ \frac{x_0}{y_{-1}(-1+x_0)},\ \frac{1}{y_0-x_{-1}y_0-x_{-1}},\ \frac{-y_{-1}}{x_0(1+y_{-1})},\ -x_{-1},\ -x_0, \\ \frac{-y_0}{x_{-1}(1+y_0)},\ \frac{1}{-x_0-y_{-1}x_0+y_{-1}},\ \frac{x_{-1}}{y_0(-1+x_{-1})},\ y_{-1},\ y_0, ... \end{array} \right\}. \end{aligned}$$

- *Let* $\{x_n, y_n\}_{n=-1}^{+\infty}$ *be solutions of system (4) such that* $x_0,\ x_{-1} \neq -1,\ \ y_0,\ y_{-1} \neq 1,\ (x_0 - y_{-1}x_0 - y_{-1})(-y_0 - x_{-1}y_0 + x_{-1}) \neq 0$. *Then*

$$\begin{aligned} \{x_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} x_{-1},\ x_0,\ \frac{y_0}{x_{-1}(-1+y_0)},\ \frac{1}{x_0-y_{-1}x_0-y_{-1}},\ \frac{-x_{-1}}{y_0(1+x_{-1})},\ -y_{-1},\ -y_0, \\ \frac{-x_0}{y_{-1}(1+x_0)},\ \frac{1}{-y_0-x_{-1}y_0+x_{-1}},\ \frac{y_{-1}}{x_0(-1+y_{-1})},\ x_{-1},\ x_0, ... \end{array} \right\} \\ \{y_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} y_{-1},\ y_0,\ \frac{x_0}{y_{-1}(1+x_0)},\ \frac{1}{y_0+x_{-1}y_0-x_{-1}},\ \frac{y_{-1}}{x_0(1-y_{-1})},\ -x_{-1},\ -x_0, \\ \frac{y_0}{x_{-1}(1-y_0)},\ \frac{1}{-x_0+y_{-1}x_0+y_{-1}},\ \frac{x_{-1}}{y_0(1+x_{-1})},\ y_{-1},\ y_0, ... \end{array} \right\}. \end{aligned}$$

We see that the solutions of the systems (3), (4) are periodic with period ten.

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