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A new sum and its mean value*

by
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Abstract

The main purpose of this paper is to introduce a new sum analogous to Dedekind sum, then using the properties of Dirichlet L-functions to study the mean value of the new sum.

Key Words: Dedekind sum; Mean value; Identity; Asymptotic formula.
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1 Introduction

For a positive integer k and an arbitrary integer h, the classical Dedekind sum $S(h,k)$ is defined by

$$S(h,k) = \sum_{a=1}^{k} \left(\left(\frac{a}{k}\right)\right) \left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The various properties of $S(h,k)$ were investigated by many authors, see [2]-[4], [6]-[10]. For example, one of the most important properties of $S(h,q)$ is its reciprocity theorem. That is, for all positive integers h and q with $(h,q)=1$, we have the identity

$$S(h,q) + S(q,h) = \frac{h^2+q^2+1}{12hq} - \frac{1}{4}.$$

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In this paper, we introduce a new sum $C(h,k)$ as follows:

$$C(h,k)=\sum_{a=1}^{k}{}' \cot\left(\frac{\pi h a}{k}\right)\cot\left(\frac{\pi a}{k}\right),$$

where h be any integer with $(h,k)=1$, $\sum_{a=1}^{k}{}'$ denotes that the sum is taken only over the a satisfying $(a,k)=1$, and $\cot(x)=\cos(x)/\sin(x)$.

This sum is analogous to Dedekind sum, so we think that it must have some similar properties as the Dedekind sum. Based on this reason, we use the orthogonality relations for characters and the properties of Dirichlet L-functions to study the mean value distribution of $C(h,k)$, and obtain some interesting identities and asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any integer $q>1$, we have the asymptotic formula*

$$\sum_{h=1}^{q}{}' C^2(h,q)=\frac{5}{9}\cdot q^4\cdot\prod_{p|q}\frac{(p^2-1)^2}{p^2(p^2+1)}+O\left(q^3\cdot\exp\left(\frac{4\ln q}{\ln\ln q}\right)\right),$$

where $\prod_{p|q}$ denotes the product over all distinct prime divisors p of q, $\exp(y)=e^y$.

Theorem 2. *Let $q>2$ be a square-full number (That is, for any prime p, $p|q$ if and only if $p^2|q$). Then we have the identity*

$$\sum_{h=1}^{q}{}' C(h,q)R_q(h+1)=-\frac{1}{3}\cdot q\cdot\phi^2(q)\cdot\prod_{p|q}\left(1+\frac{1}{p}\right),$$

where $R_q(c)$ is the Ramanujan's sum, defined as (see Theorem 8.6 of [1])

$$R_q(c)=\sum_{\substack{k=1\\(k,q)=1}}^{q} e^{\frac{2\pi i k c}{q}}=\sum_{d|(c,q)} d\mu(q/d),$$

and $\mu(n)$ denotes the Möbius function defined as follows:

$$\mu(n)=\begin{cases}1, & \textit{if } n=1;\\ (-1)^r, & \textit{if } n=p_1p_2\cdots p_r,\ p_i\ (i=1,2,\cdots)\ \textit{denote distinct primes;}\\ 0, & \textit{otherwise.}\end{cases}$$

.

Theorem 3. *For any positive integer $q>1$, we have the identity*

$$C(1,q)=\frac{1}{3}\cdot\phi(q)\left[q\cdot\prod_{p|q}\left(1+\frac{1}{p}\right)-3\right].$$

2 Some Lemmas

In this section, we shall give some lemmas which are necessary for the proof of our theorems. First we have the following:

Lemma 1. *Let $q > 2$ be an integer, and let χ be any Dirichlet character* mod q *with $\chi(-1) = -1$. Then we have the identity*

$$L(1, \chi) = \frac{\pi}{2q} \sum_{r=1}^{q} \chi(r) \cot\left(\frac{\pi r}{q}\right),$$

where $L(1, \chi)$ denotes Dirichlet L-function corresponding to χ mod q.

Proof: See Lemma 1 in [7]. □

Lemma 2. *Let $q \geq 3$ be an integer. Then for any integer h with $(h, q) = 1$, we have the identity*

$$C(h, q) = \frac{4q^2}{\pi^2 \phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h) |L(1, \chi)|^2,$$

where χ runs through the Dirichlet characters mod q *with $\chi(-1) = -1$.*

Proof: For any integer a with $(a, q) = 1$, it is clear that $\sum_{b=1}^{q} \chi(b) \cot\left(\frac{\pi b}{q}\right) = 0$ if $\chi(-1) = 1$. In fact note that cot is an odd function and both χ and cot are periodic with period q, we have

$$\sum_{b=1}^{q} \chi(b) \cot\left(\frac{\pi b}{q}\right) = \sum_{b=1}^{q} \chi(-b) \cot\left(\frac{-\pi b}{q}\right) = -\sum_{b=1}^{q} \chi(b) \cot\left(\frac{\pi b}{q}\right),$$

or

$$\sum_{b=1}^{q} \chi(b) \cot\left(\frac{\pi b}{q}\right) = 0.$$

Using this identity, Lemma 1 and the orthogonality of characters mod q we have

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(a) L(1, \chi) = \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(a) \left(\frac{\pi}{2q} \sum_{r=1}^{q} \chi(r) \cot\left(\frac{\pi r}{q}\right)\right)$$

$$= \frac{\pi}{2q} \sum_{\chi \bmod q} \overline{\chi}(a) \left(\sum_{r=1}^{q} \chi(r) \cot\left(\frac{\pi r}{q}\right)\right) = \frac{\pi \phi(q)}{2q} \cdot \cot\left(\frac{\pi a}{q}\right),$$

or

$$\cot\left(\frac{\pi a}{q}\right) = \frac{2q}{\pi\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(a) L(1,\chi). \tag{1}$$

Then from the definition of $C(h,q)$, (1) and the orthogonality of characters mod q we have

$$\begin{aligned} C(h,q) &= \sum_{a=1}^{q}{}' \cot\left(\frac{\pi h a}{q}\right) \cot\left(\frac{\pi a}{q}\right) \\ &= \frac{4q^2}{\pi^2\phi^2(q)} \sum_{\substack{\chi_1 \bmod q \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2(-1)=-1}} \sum_{a=1}^{q} \overline{\chi}_1(ha)\overline{\chi}_2(a) L(1,\chi_1)L(1,\chi_2) \\ &= \frac{4q^2}{\pi^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(h) |L(1,\chi)|^2. \end{aligned}$$

This proves Lemma 2. □

Lemma 3. *Let $q \geq 3$ be an integer. Then we have the asymptotic formula and identity*

$$(A). \quad \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^4 = \frac{5\pi^4}{144}\phi(q) \prod_{p|q} \frac{(p^2-1)^2}{p^2(p^2+1)} + O\left(q \cdot \exp\left(\frac{4\ln q}{\ln\ln q}\right)\right);$$

$$(B). \quad \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12} \cdot \frac{\phi^2(q)}{q^2} \left[q \prod_{p|q} \left(1+\frac{1}{p}\right) - 3 \right].$$

Lemma 4. *Let $q \geq 3$ be a square-full number. Then we have the identity*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12} \frac{\phi^3(q)}{q^2} \prod_{p|q} \left(1+\frac{1}{p}\right),$$

where $\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}$ *denotes the summation over all primitive odd characters* mod q.

Proof: The proofs of Lemma 3 and Lemma 4 can be found in [9]. □

3 Proof of Theorems

In this section, we shall complete the proofs of our theorems. First we prove Theorem 1. For any integer $q \geq 3$, from Lemma 2, (A) of Lemma 3 and the orthogonality of characters mod q we have

$$\sum_{h=1}^{q}{}' C^2(h,q) = \frac{16q^4}{\pi^4\phi^2(q)} \sum_{\substack{\chi_1 \bmod q \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2(-1)=-1}} \sum_{h=1}^{q} \overline{\chi}_1(h)\overline{\chi}_2(h)|L(1,\chi_1)|^2 \cdot$$

$$\cdot |L(1,\chi_2)|^2 = \frac{16q^4}{\pi^4\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^4$$

$$= \frac{5}{9} \cdot q^4 \cdot \prod_{p|q} \frac{(p^2-1)^2}{p^2(p^2+1)} + O\left(q^3 \cdot \exp\left(\frac{4\ln q}{\ln\ln q}\right)\right),$$

where $\prod_{p|q}$ denotes the product over all distinct prime divisors p of q. This proves Theorem 1.

Now we prove Theorem 2. Note the identity

$$\sum_{\substack{c=1 \\ (c,q)=1}}^{q} \overline{\chi}(c) R_q(c+1) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{a}{q}\right) \sum_{\substack{c=1 \\ (c,q)=1}}^{q} \overline{\chi}(c) e\left(\frac{ac}{q}\right) = \overline{\chi}(-1)\left|\tau(\chi)\right|^2, \tag{2}$$

where $\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right)$ denotes the Gauss sum, and $e(z) = \exp(2\pi i z)$.

If q is a square-full number and χ is not a primitive character mod q, then from the properties of Gauss sums (see [1] and [5]) we know that $\tau(\chi) = 0$. If χ is a primitive character mod q, then $|\tau(\chi)|^2 = q$. Then from Lemma 2, Lemma 4 and (2) we have

$$\sum_{h=1}^{q}{}' C(h,q) R_q(h+1) = \frac{4q^2}{\pi^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \sum_{h=1}^{q}{}' \overline{\chi}(h) R_q(h+1) |L(1,\chi)|^2$$

$$= \frac{4q^2}{\pi^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(-1)\left|\tau(\chi)\right|^2 \cdot |L(1,\chi)|^2 = -\frac{4q^3}{\pi^2\phi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2$$

$$= -\frac{4q^3}{\pi^2\phi(q)} \cdot \frac{\pi^2}{12} \frac{\phi^3(q)}{q^2} \prod_{p|q}\left(1+\frac{1}{p}\right) = -\frac{1}{3} \cdot q \cdot \phi^2(q) \cdot \prod_{p|q}\left(1+\frac{1}{p}\right),$$

where $\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}$ denotes the summation over all primitive odd characters mod q.

This proves Theorem 2.

Theorem 3 follows from Lemma 2 with $h = 1$ and (B) of Lemma 3.

This completes the proof of our all theorems.

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