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A modified generalization of the Hermitian and skew-Hermitian splitting iteration*

by
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Abstract

In this paper, we further study the GHSS splitting method for non-Hermitian positive definite linear problems, which is introduced by Benzi [A generalization of the Hermitian and skew-Hermitian splitting iteration, *SIAM J. Matrix Anal. Appl.*, 31 (2009), pp. 360-374]. A modified generalization of the Hermitian and skew-Hermitian method (MGHSS) is given, and it still can be used as an effective preconditioner for saddle point systems. We will show the effectiveness of our preconditioner.

Key Words: Saddle point system; matrix splittings; HSS iteration
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1 Introduction

We consider the following Stokes type problem:

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u} + \nabla\mathbf{p} = f, & in\,\Omega \\ -div\mathbf{u} = 0, & in\,\Omega \\ \mathbf{u} = g, & on\,\Gamma \end{cases} \tag{1}$$

Here u denotes the velocity vector field, p is the pressure, Ω is a bounded domain in $\Re^d$. A stable finite element or finite difference method applied to discretize (1) leads to the solution of the following so-called saddle point linear system:

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{2}$$

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Where $A = \sigma M + \nu L + N$, L is symmetric positive definite (SPD) and consists of a direct sum of discrete Laplace operators. N is a skew-symmetric matrix, note that $N = 0$ for the generalized Stokes problem, Also, M is a mass matrix, possibly a scaled identity, C is a symmetric positive semidefinite pressure stabilization matrix and B has full rank.

The Hermitian and skew-Hermitian splitting (HSS) iteration was first introduced by Bai, Golub, and Ng in [1] and extended in [2] for the solution of a class of non-Hermitian linear systems $Ax = b$. In [3], the authors applied the standard HSS method to the system (2), which is based on the splitting:

$$\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} = \begin{pmatrix} H_A & 0 \\ 0 & C \end{pmatrix} + \begin{pmatrix} S_A & B^T \\ -B & 0 \end{pmatrix},$$

where $H_A = \frac{1}{2}(A + A^T)$, and in [6], Simoncini and Benzi studied the eigenvalue problem associated with the preconditioned matrix when is symmetric positive definite and . In [5], in order to have a "heavier" diagonal corresponding system which can be solved more easy than the standard HSS method, Benzi gave a generalization of the hermitian and skew-hermitian splitting iteration, which is based on the splittings:

$$\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} = \begin{pmatrix} G_A & 0 \\ 0 & C \end{pmatrix} + \begin{pmatrix} K_A + S_A & B^T \\ -B & 0 \end{pmatrix}$$

where

$$H_A = \frac{1}{2}(A + A^T) = G_A + K_A,$$

both G_A and K_A are Hermitian positive semidefinite matrices.

In this paper, in order to make the coefficient matrix more diagonally dominant, a modified generalization of the hermitian and skew-hermitian method (MGHSS) is given, and it still can be used as an effective preconditioner. In section 2, we will describe the MGHSS scheme, and the convergence theorem of stationary iteration will be given. We applied the MGHSS scheme as a preconditioner to to accelerate the convergence of the GMRES iteration in section 3, some numerical tests are discussed in section 4. Concluding remarks are given in section 5.

2 The MGHSS method

In this section, we give the MGHSS method based on the splitting:

$$\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} = \begin{pmatrix} G_A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} K_A + S_A & B^T \\ -B & C \end{pmatrix} = H_M + S_M, \tag{3}$$

First, let $\alpha > 0$, we will show the convergence of the alternating iteration:

$$\left\{ \begin{array}{l} (H_M + \alpha I)x^{k+\frac{1}{2}} = (\alpha I - S_M)x^k + b, \\ (S_M + \alpha I)x^{k+1} = (\alpha I - H_M)x^{k+\frac{1}{2}} + b. \end{array} \right. \tag{4}$$

We give the convergence theorem of stationary iteration as follows.

Theorem 2.1. *Assume that $A \in R^{n\times n}$ has positive definite symmetric part $H_A = (A + A^T)/2$, $C \in R^{m\times m}$ is symmetric positive semidefinite and has full rank. Assume further that H_A is split as $H_A = G_A + K_A$, with G_A SPD and K_A positive semidefinite. Then the MGHSS iteration converges unconditionally to the unique solution of problem (1) based on the splittings (4), that is, $\rho(T) < 1$ for all $\alpha > 0$.*

Proof: Consider the splitting (4). The iteration matrix T is similar to

$$T = (\alpha I - H_M)(\alpha I + H_M)^{-1}(\alpha I - S_M)(\alpha I + S_M)^{-1} = RU,$$

where

$$R = (\alpha I - H_M)(\alpha I + H_M)^{-1},\ \ U = (\alpha I - S_M)(\alpha I + S_M)^{-1}.$$

Now, R is orthogonally similar to the $(n + m) \times (n + m)$ diagonal matrix

$$D = \mathrm{diag}\left(\frac{\alpha - u_1}{\alpha + u_1}, \cdots, \frac{\alpha - u_n}{\alpha + u_n}, 1, \cdots, 1\right),$$

where $u_1, u_2, ..., u_n$ are the (positive) eigenvalues of G_A. That is, there is an orthogonal matrix V of order $n + m$ such that

$$V^T RV = D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},$$

with D_1 and D_2 diagonal matrices of order n and m, respectively.

Note that $\frac{\alpha - u_i}{\alpha + u_i} < 1$, by the famous Kellogg's lemma [7], we have

$$\rho(T) \leq 1.$$

Note that RU is orthogonally similar to

$$V^T RUV = (V^T RV)(V^T UV^T) = DQ,$$

where $Q = V^T UV^T$, therefore,

$$\rho(T) = \rho(DQ) = \rho(QD).$$

Let

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}.$$

Then,

$$QD = \begin{bmatrix} Q_{11}D_1 & Q_{12}D_2 \\ Q_{21}D_1 & Q_{22}D_2 \end{bmatrix}.$$

Now, let $\lambda \in C$ be an eigenvalue of QD and let $x \in C^{n+m}$ be a corresponding eigenvector with $\|x\|_2 = 1$. We assume $\lambda \neq 0$, then we have $QDx = \lambda x$, and taking norms on both sides, we have

$$|\lambda| = \|QDx\|_2 \leq \|Dx\|_2 = \sum_{i=1}^{n} \left(\frac{\alpha - u_i}{\alpha + u_i}\right) x_i \bar{x_i} + \sum_{i=n+1}^{n+m} x_i \bar{x_i}.$$

To prove that $\lambda < 1$, we show that there exists at least one i $(1 \leq i \leq n)$ such that $x_i \neq 0$. Using the assumption that B has full rank, we will show that $x_i = 0$ for all $1 \leq i \leq n$ implies $x = 0$, a contradiction. Indeed, if the eigenvector $x = [0; \hat{x}]$ (where $\hat{x} \in C^m$), the identity $DQx = \lambda x$ becomes

$$DQx = \begin{bmatrix} D_1Q_{11} & D_2Q_{12} \\ D_1Q_{21} & D_2Q_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix} = \begin{bmatrix} D_1Q_{12}\hat{x} \\ D_1Q_{12}\hat{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda\hat{x} \end{bmatrix}.$$

Therefore, in particular, it must be $D_1Q_{12} \neq 0$. We will show that Q_{12} has full column rank; hence, it must be $\hat{x} = 0$. Recall that $Q = V^TUV$ with

$$V = \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix},$$

where $V_{11} \in R^{n\times n}$ is the orthogonal matrix that diagonalizes $(\alpha I - H_A)(\alpha I + H_A)^{-1}$ and $V_{22} = I^{m\times m}$. Recall that the orthogonal matrix U is given by

$$(\alpha I - S_M)(\alpha I + S_M)^{-1} =$$

$$\begin{bmatrix} \alpha I_n - K_A - S_A & -B^T \\ B & \alpha I_m - C \end{bmatrix} \begin{bmatrix} \alpha I_n + K_A + S_A & -B^T \\ B & \alpha I_m + C \end{bmatrix}^{-1}$$

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.$$

An explicit calculation shows that

$$U_{12} = -[(\alpha I_n - S)(\alpha I_n + S)^{-1} + I_n]B^T[\alpha I_m + C + B(\alpha I_n + S)B^T]^{-1},$$

where $S = K_A + S_A$. And we can get $Q_{12} = V_{11}^T U_{12} V_{22}$, showing that Q_{12} has full column rank since V_{11} and V_{22} are orthogonal and B^T has full column rank. This completes the proof. □

Remark 2.1. In fact, if we split $C = G_C + K_C$, where G_C and K_C are Hermitian positive semidefinite matrices, we can get the different splitting:

$$\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} = \begin{pmatrix} G_A & 0 \\ 0 & G_C \end{pmatrix} + \begin{pmatrix} K_A + S_A & B^T \\ -B & K_C \end{pmatrix} = \hat{H}_M + \hat{S}_M. \tag{5}$$

Similarly, we can get the convergence theorem of the alternating iteration based on the splittings, we will not discuss it anymore.

Let

$$P_\alpha = \frac{1}{2\alpha}(\alpha I + H_M)(\alpha I + S_M), Q_\alpha = \frac{1}{2\alpha}(\alpha I - H_M)(\alpha I - S_M), T_\alpha = P_\alpha^{-1}Q_\alpha,$$

then, we have

$$A = P_\alpha - Q_\alpha, T_\alpha = P_\alpha^{-1}Q_\alpha = I - P_\alpha^{-1}A,$$

T and T_α have the same spectrum, $\Im(\lambda)$, $\Re(\lambda)$ denote the imaginary and real part of λ, which is the eigenvalue of the preconditioned matrix $P_\alpha^{-1}A$. From Theorem 2.1, we can know that,

$$|1 - \lambda| < 1$$

so,

$$\lambda \in D(1,1) := \{z \in C; |z - 1| \leq 1\} \setminus \{0\}.$$

3 Krylov subspace acceleration

The basic algorithm of the Krylov subspace methods is the conjugate gradient method (CG) which has the nice properties that it uses only three vectors in memory and minimizes the error in the A-norm. However, the algorithm mainly performs well if the matrix is symmetric, and positive definite. In cases where one of these two properties is violated, CG may break down. GMRES method has the advantage that theoretically the algorithm does not break down unless convergence has been reached. The main problem in GMRES method is that the amount of storage increases as the iteration number increases. Therefore, the application of GMRES method may be limited by the computer storage. To remedy this problem, a restarted version, GMRES(m). See [10] for more about the Krylov subspace methods.

Note that as a preconditioner we can use

$$P_\alpha = \frac{1}{2\alpha}(\alpha I + H_M)(\alpha I + S_M).$$

Application of the alternating preconditioner within GMRES method requires solving a linear system of the form $M_\alpha z = r$ at each iteration. This is done by first solving

$$(H_M + \alpha I)v = r$$

for v, followed by

$$(S_M + \alpha I)z = v.$$

Therefore, the main potential advantage of the MGHSS scheme over the standard HSS or the GHSS scheme is that the matrix $(S_M + \alpha I)$ have a "heavier" diagonal form and better conditioned. So it can be expected to be less expensive when we use the GMRES method.

4 Numerical examples

In this section we present a sample of numerical experiments conducted in order to assess the effectiveness of the alternating algorithm (4) as a preconditioner for the GMRES method. All experiments were performed in Matlab 7.0.

The generated test problems are leaky two-dimensional lid-driven cavity problems in square domain $(-1,1)\times(-1,1)$ with the lid flowing from the left to right. A Dirichlet no-flow condition is applied on the side and bottom boundaries. The nonzero horizontal velocity on the lid is chosen to be $\{y=1; -1 \leq x \leq 1 \,|u_x = 1\}$. Using the IFISS software written by Silvester, Elman and Ramage [8] to discretize (1), we take a finite element subdivision based on uniform grids of square elements. The mixed finite element used is the bilinear-constant velocity u pressure: $Q_1 - P_0$ pair. Thus, the (1, 1) block A of the coefficient matrix corresponding to the discretization of the conservative term is symmetric positive definite, and the (1, 2) block B corresponding to the discrete divergence operator is rank deficient. The example is derived by discretizing (1) on 16×16 meshes with stabilization (i.e., $C \neq 0$, $\beta = 0.25$ were used for the viscosity and stabilization parameters).

Grid	n	m	$nnz(A)$	$nnz(B)$
8×8	162	64	786	392
16×16	578	256	3806	1800
32×32	2078	1024	16818	7688
64×64	8450	4096	70450	31752

Table 1: Size and number of nonzeros of the relevant matrices.

Method	$m \times m$	8×8	16×16	32×32
MGHSS	IT	38	51	55
	CPU	0.4814	8.6060	44.3390
GHSS	IT	84	73	58
	CPU	1.0643	13.5072	47.3624
HSS	IT	103	95	70
	CPU	1.2587	17.0489	267.3157

Table 2: IT and CPU for the example with $\alpha = 0.02$.

In Figure 1, we display the eigenvalues of the preconditioned matrix $P^{-1}M$ in the case of 16×16 meshes when we use the MGHSS preconditioners. In Figure 2, we display the eigenvalues of the preconditioned matrix $P^{-1}M$ in the case of 16×16 meshes when we use the standard HSS preconditioners. We can find that they are all strongly clustered. All set $\alpha = 0.04$.

In Figure 3, we compare the performances of the GMRES iterations, with a convergence tolerance of 10^{-6}, and the inner GMRES iteration was preconditioned by an incomplete LU factorization, in both cases, the drop tolerance

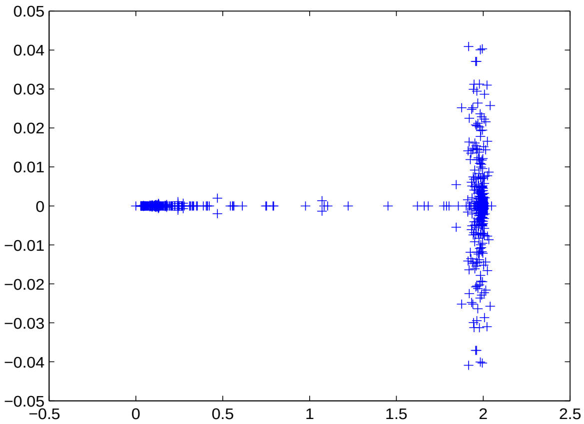

Figure 1: Eigenvalues of $P^{-1}M$(MGHSS) with $\alpha = 0.04$ on 16×16 grid .

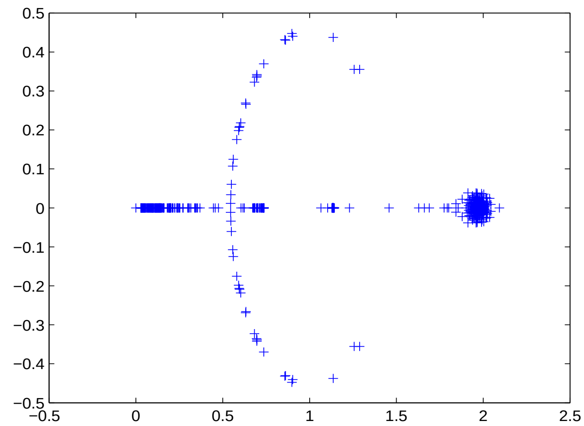

Figure 2: Eigenvalues of $P^{-1}M$(HSS) with $\alpha = 0.04$ on 16×16 grid

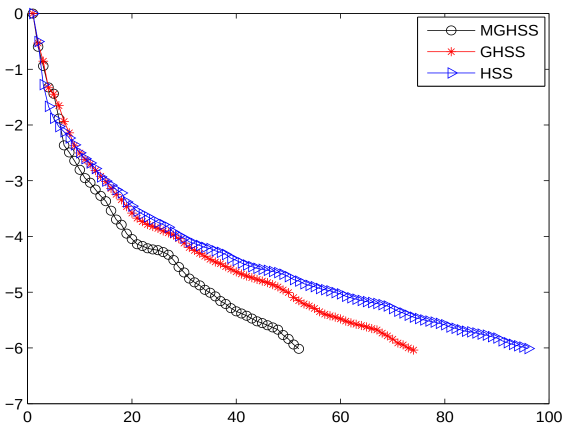

Figure 3: Convergence curve and total numbers of GMRES(10) iterations with $\alpha = 0.02$ on 16×16 grid

was set to 0.01. We can find that, with the same value $\alpha = 0.02$, the MGHSS preconditioner can outperform the optimally tuned HSS preconditioner, this is still true when we compare the results between the MGHSS preconditioner and GHSS preconditioner.

In Table 1, problem sizes and sparse information on the relevant matrices on different meshes are given.

In Table 2, we observe that as preconditioners for GMRES(10) with $\alpha = 0.02$, MGHSS performs much better than HSS and GHSS in both iteration steps and CPU times.

5 Conclusion

In this paper, a modified generalization of the HSS splitting method of Bai, Golub, and Ng has been described. The new scheme has been shown to be unconditionally convergent. Similar to the PHSS method and the PGHSS method, the new method (PMGHSS) can also be accelerated by a Krylov subspace method with inexact inner solves. Numerical tests have shown the effectiveness of the new approach, but we can not sure that the PMGHSS method always performs better than the PGHSS method and the method PHSS, in that case, we choose the PGHSS method or the PHSS method.

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