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Optimal Control Problem Governed by an Infinite Dimensional One-Nilpotent Bilinear Systems

by

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Abstract

The object of this work is to construct an explicit linear operators B which commute with a given linear operator A in infinite dimensional spaces. This construction can be applied to give exact optimal solution for a class of infinite dimensional bilinear systems.

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1 Introduction

Consider the bilinear system:

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & Ax(t) + u(t)Bx(t) \\ x(0) & = & x_0, \end{array} \right. \tag{1}$$

where A and B are bounded linear operators on a Hilbert space E, $x(t) \in E$ is the trajectory associated to the control $u(t) \in \mathbb{R}$ and $[0, T]$ is a fixed time interval, where $T > 0$. Then for all $t \in [0, T]$ we consider the following problem: under the constraint (1), find the control $u(t)$ which minimizes the functional

$$J(u) = \int_0^T u^2 dt + \langle x(T), Fx(T) \rangle , \tag{2}$$

where F is a coercive symmetric bounded operator. To determine the optimal control in finite dimension, we apply the Pontryagin's Maximum Principle [18]. This process is not always valid in infinite dimensional. In the context of our bilinear system (1) and (2), if u is a bounded control we use the results of [4] to obtain the optimal controls in terms of state vector and adjoint one. If the Lie algebra generated by operators A and B is nilpotent for the bracket $[A, B] = AB - BA$, the optimal control can be written in terms of time and initial conditions.

Our objective is to construct explicit operators B commuting with a given operator A for finite dimensional operators, finite rank operators and compact ones.

In the second section of this work, we obtain an explicit expression of the extremal control, where the degree of nilpotency is equal to one (Theorem 1 [19]). In the third section, we look to the existence of nilpotent bilinear systems of degree one in finite dimension. The idea is to write one matrix in Jordan normal form, this allows to define all matrices commuting with it. These results are generalized to the infinite dimensional for an operator of finite rank in a Hilbert space. In the fourth and fifth sections, the study is extended to a compact operator in a Banach space with a positive quasinilpotent part which is the limit of the normed nilpotent operators. We use in this context the properties of the spectrum of this operator to define the set of commuting matrices. We conclude this work with examples and application.

2 Optimal control for a bilinear system and a quadratic cost

2.1 The finite dimensional case

To find the optimal control when E is a finite dimensional vector space, we can apply the **Pontryagin's Maximum Principle** [18], it follows that for optimal control $u(t)$, there exists an adjoint vector $\lambda(t)$ solution of:

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \frac{\partial}{\partial\lambda}H(x,u,\lambda) \\ \dot{\lambda}(t) & = & -\frac{\partial}{\partial x}H(x,u,\lambda) \\ 0 & = & \frac{\partial}{\partial u}H(x,u,\lambda), \end{array}\right. \tag{3}$$

where

$$H(x,u,\lambda) \quad = \quad \langle \lambda(t), (A+uB)x(t)\rangle - u^2(t),$$

with terminal condition: $\lambda(T) = Fx(T)$.

Since the extremal are solutions of (3), then:

$$\frac{\partial}{\partial u}H(x,u,\lambda) = 0 \Longrightarrow \langle \lambda(t), Bx(t)\rangle - 2u(t) = 0,$$

we obtain the **optimal control** in terms of the state vector and adjoint one

$$\hat{u}(x, \lambda) = \frac{1}{2} \langle \lambda(t), Bx(t) \rangle . \tag{4}$$

2.2 The infinite dimensional case

In this paragraph, we propose a generalization of the previous characterization of optimal control, when E is an **infinite Hilbert space.** The proof is obtained by applying the results of [4].

Proposition 2.1. [19] *If $\bar{u}$ is a bounded optimal control which belongs to $L^2([0,T], \mathbb{R})$ then*

$$\bar{u}(t) = \frac{1}{2} \left\langle \overline{\lambda}(t), B\overline{x}(t) \right\rangle , \tag{5}$$

where $\overline{x}(t)$ is the trajectory associated to $\bar{u}(t)$ and $\overline{\lambda}(t)$ is the solution of adjoint system:

$$\begin{cases} \dot{\overline{\lambda}}(t) = -\left\langle \overline{\lambda}(t), (A + \bar{u}B) \right\rangle \\ \overline{\lambda}(T) = F\ \overline{x}(T). \end{cases} \tag{6}$$

Proof: Let $\bar{u}$ be a bounded optimal control, we set

$$a = \max \left\{ |\bar{u}(t)| , \quad t \in [0,T] \right\} ,$$

and $K_n = [-n, n]$ a compact of $\mathbb{R}$, with $n \in \mathbb{N}$. Then for $n \geq a$, $\bar{u}(t)$ is an optimal control among all controls $u(.) \in L^2([0,T], K_n)$.

From ([4] Theorem 3.1), $\bar{u}(t)$ satisfies the following relation for almost all $t \in [0,T]$:

$$\left\langle \overline{\lambda}(t), (A + \bar{u}(t)B)\overline{x}(t) \right\rangle - \bar{u}(t)^2 = \max_{v \in K_n} \left\{ \left\langle \overline{\lambda}(t), (A + vB)\overline{x}(t) \right\rangle - v^2 \right\} ,$$

where $\overline{x}(t)$ is the trajectory associated to $\bar{u}(t)$ and $\overline{\lambda}(t)$ is the solution of adjoint system (6).

According to the Gronwall's lemma, there exists a constant M such that $\|\overline{x}(t)\|$ and $\left\|\overline{\lambda}(t)\right\|$ are uniformly bounded by M for all $t \in [0,T]$.

Then for t fixed the $\max_{v \in K_n} \left\{ \left\langle \overline{\lambda}(t), (A + vB)\overline{x}(t) \right\rangle - v^2 \right\}$ is reached in

$$\bar{u}(t) = \frac{1}{2} \left\langle \overline{\lambda}(t), B\overline{x}(t) \right\rangle .$$

$\square$

2.3 Optimal control for a nilpotent bilinear system

We denote by $\mathcal{L}(E)$ the space of bounded operators in E, which has a structure of **Lie algebra** for the bracket $[A, B] = AB - BA$.

Let be Ad_u the endomorphism defined by:

$$\begin{array}{cccc} Ad_u & \mathcal{L}(E) & \longrightarrow & \mathcal{L}(E) \\ & v & \longrightarrow & [u,v] = uv - vu, \end{array}$$

and

$$Ad_M(N) = \{[u,v] \diagup u \in M \quad et \quad v \in N\},$$

where M and N parts of $\mathcal{L}(E)$. We define by recurrence for $k \geq 1$

$$\begin{array}{rcl} (Ad_M)^k(N) & = & Ad_M\left\{(Ad_M)^{k-1}(N)\right\}, \\ (Ad_M)^0(N) & = & N. \end{array}$$

Recall that a Lie sub-algebra M of $\mathcal{L}(E)$ is **nilpotent** of k degree if there exists an integer l where $(Ad_M)^l(M) = \{0\}$, and k is the smallest element of all these integers l.

We end this section by the following theorem that gives the expression of optimal control independent of $\lambda(t)$ and $x(t)$:

Theorem 1. *(See* [19] *Theorem 3.1)Consider the minimization problem* (1) *and* (2) *with* $k = 1$ *(i.e.* $[A, B] = 0$*) then the optimal control* $\bar{u}$ *is **constant** and this constant is characterized by the equation:*

$$2\bar{u} + \left\langle e^{(A+\bar{u}B)T}x_0, [FB + B^\star F]\, e^{(A+\bar{u}B)T}x_0 \right\rangle = 0,$$

where $B^\star$ *denotes the adjoint of* B.

3 Set of all operators commuting with a given operator A

3.1 Finite dimensional case

In this section, each matrix A of $M_n(\mathbb{C})$ will be assimilate with the canonical endomorphism of $\mathbb{C}^n$ whose matrix is A in the canonical basis of $\mathbb{C}^n$.

We consider a matrix A of $M_n(\mathbb{C})$ which is characterized by its elementary divisors $(\lambda-\lambda_1)^{n_1}, (\lambda-\lambda_2)^{n_2}, ..., (\lambda-\lambda_\mu)^{n_\mu}$ where $\{\lambda_1, \lambda_2, ..., \lambda_\mu\}$ are the eigenvalues of A and we look for all matrices B that $Ad_A(B) = [A, B] = 0.$ This leads us to the *Frobenius problem* (See [10] page 218) which determine the set of all matrices B that commuting with $A.$ This amounts to solve a matrix equation of the form:

$$AB = BA.$$

We reduce A to Jordan normal form, in the basis:
$e_1 = (A-\lambda_1 I)^{n_1-1}v_1,\ \ e_2 = (A-\lambda_1 I)^{n_1-2}v_1, ..., e_{n_1} = v_1,$
$e_{n_1+1} = (A-\lambda_2 I)^{n_2-1}v_2,\ \ e_{n_1+2} = (A-\lambda_2 I)^{n_2-2}v_2, ..., e_{n_1+n_2} = v_2,$
$\vdots$
$e_{n_1+...+n_{\mu-1}+1} = (A-\lambda_\mu I)^{n_\mu-1}v_\mu, ..., e_n = v_\mu,$

where v_i are called generalized eigenvectors (See [21]). In this basis $\{e_1, e_2, ..., e_n\}$, A has the form:

$$\left(\begin{array}{c|c|c}
\begin{matrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_1 \end{matrix} & 0 & 0 \\ \hline
0 & \ddots & 0 \\ \hline
0 & 0 & \begin{matrix} \lambda_\mu & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_\mu \end{matrix}
\end{array}\right)$$

Then there exists an invertible matrix U in $M_n(\mathbb{C})$ where A has a Jordan normal form associated to the elementary divisors:

$$A = U\widetilde{A}U^{-1}.$$

We obtain all matrices commuting with A under the following form

$$B = U\widetilde{B}U^{-1},$$

where $\widetilde{B}$ denotes an arbitrary matrix which commutes with $\widetilde{A}$, and $\widetilde{B}$ is decomposed into B_{ij} blocks where $i, j = 1, 2, ..., \mu$ (μ is the number of elementary divisors of A)

$$\widetilde{B} = (B_{ij}) \qquad i, j = 1, 2, ..., \mu$$

where B_{ij} is the **zero matrix** if $\lambda_i \neq \lambda_j$, or a regular **upper triangular** matrix if $\lambda_i = \lambda_j$. The diagrams below show the structure of these matrices when $\lambda_i = \lambda_j$:

$$B_{ij} = \left.\left(\underbrace{\begin{matrix} a_1 & a_2 & \cdots & a_{n_i} \\ 0 & a_1 & \ddots & \vdots \\ \vdots & & \ddots & a_2 \\ 0 & \cdots & 0 & a_1 \end{matrix}}_{n_i} \right)\right\} n_i \qquad \text{if} \quad n_i = n_j \tag{7}$$

$$B_{ij} = \left.\left(\underbrace{\begin{array}{|cccc|cccc|} \hline 0 & \cdots & \cdots & 0 & a_1 & a_2 & \cdots & a_{n_i} \\ \vdots & & & \vdots & 0 & a_1 & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & a_2 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & a_1 \\ \hline \end{array}}_{n_j} \right)\right\} n_i \qquad \text{if} \quad n_i < n_j \tag{8}$$

$$B_{ij} = \left.\left(\underbrace{\begin{array}{|cccc|} \hline a_1 & a_2 & \cdots & a_{n_j} \\ 0 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \cdots & 0 & a_1 \\ \hline 0 & \cdots\cdots & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots\cdots & \cdots & 0 \\ \hline \end{array}}_{n_j} \right)\right\} n_i \qquad \text{if} \quad n_j < n_i \tag{9}$$

Where $a_1, a_2, a_3, ...$ are arbitrary parameters and the elements of any direction parallel to the principal diagonal are equal.

Definition 3.1. *We say that a matrix $B \in M_n(\mathbb{C})$ is $A-$**similar** to a matrix $A \in M_n(\mathbb{C})$ if there exists an invertible U in $M_n(\mathbb{C})$ such that:*

$$B = U\widetilde{B}U^{-1}, \qquad A = U\widetilde{A}U^{-1}$$

where $\widetilde{A}$ is the canonical Jordan form of A.

So we get the following proposition:

Proposition 3.1. *(Frobenius Problem) Any matrix $B \in M_n(\mathbb{C})$ which commutes with a given matrix $A \in M_n(\mathbb{C})$ characterized by its elementary divisors $(\lambda - \lambda_1)^{n_1}$, $(\lambda - \lambda_2)^{n_2}, ..., (\lambda - \lambda_\mu)^{n_\mu}$, is $A-$similar to a matrix decomposed into μ^2 blocks of type* (7), (8), (9) *or zero blocks.*

(See Example 1, Section 6)

3.2 Case where A is of finite rank in a Hilbert space

Let E be infinite complex Hilbert space, and $\mathcal{L}(E)$ the set of bounded linear operators on E. We define:

$$\mathcal{L}'(E) = \{A \in \mathcal{L}(E) : \ A \text{ of finite rank } \}.$$

For any operator A of $\mathcal{L}(E)$ we denote by $R(A) = A(E)$ the range of A. An operator A is of **finite rank** (of rank n) if dimension of $R(A)$ is finite (n dimensional).

A finite rank operator A can be written (See [22] Theorem 6.1 p 129):

$$Ax = \sum_{i=1}^{n} \alpha_i \langle u_i, x \rangle v_i \quad for\ all \quad x \in E,$$

where $\{u_i\}$ and $\{v_i\}$ are finite orthonormal families. The previous decomposition is called the canonical form of finite rank operators. Moreover, the rank of an operator A is n if and only if its adjoint operator $A^\star$ is also of finite rank n. Indeed we have:

$$A^\star x = \sum_{i=1}^{n} \alpha_i \langle v_i, x \rangle u_i \quad for\ all \quad x \in E.$$

We will say that an endomorphism A of E is similar to an endomorphism $\widetilde{A}$ if there exists an automorphism U of E such that $A = U\widetilde{A}U^{-1}$. In this case, we say that an endomorphism B is $A-$similar to $\widetilde{B}$ if for the same automorphism U we have $B = U\widetilde{B}U^{-1}$.

Proposition 3.2. *Let $A \in \mathcal{L}'(E)$ and denote by E_1 the Hilbert subspace generated by $R(A)$ and $R(A^\star)$. Then A is similar to the quasi-diagonal forms $\left(\begin{array}{|c|c|} \hline A_1 & 0 \\ \hline 0 & 0 \\ \hline \end{array}\right)$, where A_1 is an endomorphism of E_1. In particular, there exists a basis $\{e_1, e_2, .., e_n\}$ of E_1 so that the matrix of A_1 has a Jordan normal form associated to the elementary divisor of A_1 (See Section 3.1).*

Proof: Using the previous canonical forms for A and $A^\star$, we see that $E_1 = R(A)+R(A^\star)$ is generated by $\{u_i\}$ and $\{v_i\}$. So E_1 is well defined and is invariant by A ($AE_1 \subset E_1$).

Then $E = E_1 \oplus E_2$, where $E_2 = E_1^\perp$ is in fact the intersection of kernel of A and $A^\star$.

Where $A_{|E_1}$ is characterized by its elementary divisors $(\lambda - \lambda_1)^{m_1}, ..., (\lambda - \lambda_\mu)^{m_\mu}$, $m_1 + m_2 + ... + m_\mu = n$, according to Section 3.1, we have a basis $\{e_1, e_2, .., e_n\}$ of E_1 so the matrix A_1 in this basis has a Jordan normal form associated to the elementary divisors of A_1. So, there exists an invertible matrix V, such that $VA_{|E_1}V^{-1}$ is the Jordan normal form of $A_{|E_1}$.

Denote by U the automorphism of E defined by $U_{|E_1} = V$ and $U_{|E_2} = Id_{E_2}$ then, A is similar to the quasi-diagonal form: $\left(\begin{array}{|c|c|}\hline A_1 & 0 \\ \hline 0 & 0 \\ \hline\end{array}\right)$. □

Theorem 2. *Let A be an operator of $\mathcal{L}'(E)$ and $E_1 = R(A) + R(A^\star)$, the invariant subspace associated to A. Then, the set of all $B \in \mathcal{L}(E)$ which leaves E_1 invariant and such that $[A, B] = 0$, is $A-$similar to $\left(\begin{array}{|c|c|}\hline B_1 & 0 \\ \hline 0 & B_2 \\ \hline\end{array}\right)$ where B_1 is an endomorphism of E_1 which commutes with A_1 and B_2 is an endomorphism of E_2. In particular, there exists a basis $\{e_1, e_2, .., e_n\}$ of E_1 so that the matrix A_1 has a Jordan normal form associated to the elementary divisors of A_1, and the matrix of B_1 is decomposed into μ^2 blocks of type* (7), (8), (9) *or zero blocks (See Proposition 3.1).*

Proof: Let be $A \in \mathcal{L}'(E)$, according to Proposition 3.2, there exists an automorphism U such that:

$$A = U \left(\begin{array}{|c|c|}\hline A_1 & 0 \\ \hline 0 & 0 \\ \hline\end{array}\right) U^{-1}.$$

We consider the set of all endomorphisms B_1 of E_1 which commute with A_1. According to Section 3.1, there exists a basis $\{e_1, e_2, .., e_n\}$ of E_1 so that the matrix A_1 has a Jordan normal form and the matrix of B_1 is decomposed into μ^2 blocks of type (7), (8), (9) or zero blocks.

For any endomorphism B_2 of E_2, let be B defined by the quasi-diagonal form:

$$B = U \left(\begin{array}{|c|c|}\hline B_1 & 0 \\ \hline 0 & B_2 \\ \hline\end{array}\right) U^{-1},$$

where U is the automorphism previously associated to A.

It is clear that:

$$[A, B] = 0.$$

□

Remark 3.1. *In the previous Theorem then $B \in \mathcal{L}'(E)$ if and only if B_2 is a finite rank.*

4 Case where A is a compact operator in Banach space

In this section we will look for the operators which commute with a given compact operator on an infinite dimensional **Banach space which has countably many non-zero eigenvalues**. We will apply the **Riesz** functional calculus to disjoint circles surrounding the nonzero isolated eigenvalues of this operator to obtain projections. Then the space is the direct sum of the ranges of these projections together with the space on which **the operator is quasinilpotent** (i.e. complement of the direct sum of the nonzero eigenspaces). For each direct sum corresponding to nonzero eigenvalues the operator has a **Jordan matrix representation**. If the quasinilpotent part of the operator is **positive**, then the corresponding subspace is decomposed into maximal chains of bands which are invariant subspaces, hence the Jordan matrix representation. Such that each quasinilpotent operator is the norm limit of nilpotent ones.

More precisely, according to standard book [5] and [14] we recall in this context:

Let E be a Banach space, A a compact operator on E (i.e. the image of a subset of E by A is a relatively compact subset), and $\sigma(A)$ the spectrum of A. The spectral properties of A are:

- Every nonzero $\lambda \in \sigma(A)$ is an eigenvalue of A.
- For all nonzero $\lambda \in \sigma(A)$, there exists m such that:

$$Ker(\lambda I - A)^m = Ker(\lambda I - A)^{m+1}.$$

- The eigenvalues can only accumulate at 0. If the dimension of E is not finite, then $\sigma(A)$ must contain 0.
- $\sigma(A)$ is countable.
- Every nonzero $\lambda \in \sigma(A)$ is a pole of the resolvent function:

$$\zeta \to (\zeta I - A)^{-1}.$$

4.1 Invariant subspaces of nonzero eigenvalues

As in finite matrix case, the above spectral properties lead to a decomposition of E into invariant subspaces of a compact operator A. Let $\lambda \neq 0$ be an eigenvalue of A, so λ is an **isolated point** of $\sigma(A)$. Using the holomorphic functional calculus, define the **Riesz projection** $P(\lambda)$ by:

$$P(\lambda) = \frac{1}{2\pi i} \int_{\gamma} (\zeta I - A)^{-1} d\zeta,$$

where γ is the **Jordan contour** containing the single point λ of $\sigma(A)$, and $P(\lambda)$ satisfy $P(\lambda)^2 = P(\lambda)$, so they are **spectral projections**, by definition they commute with A. Moreover, $P(\lambda)P(\mu) = 0$ if $\lambda \neq \mu$.

Let $E(\lambda)$ be the subspace $E(\lambda) = P(\lambda)E$, the restriction of A to $E(\lambda)$ is a compact invertible operator with spectrum $\{\lambda\}$.

And let be m an integer such that:

$$Ker(\lambda I - A)^m = Ker(\lambda I - A)^{m+1},$$

the Laurent series of the resolvent mapping centered at λ shows that

$$P(\lambda)(\lambda I - A)^m = (\lambda I - A)^m P(\lambda) = 0,$$

by inspecting the Jordan form, there exists n that $(\lambda I - A)^n = 0$ while $(\lambda I - A)^{n-1} \neq 0$, where

$$E(\lambda) = Ker(\lambda I - A)^m.$$

Then $E(\lambda)$ is a **finite-dimensional invariant subspace**. Hence the restriction of A to $E(\lambda)$ admits a **Jordan matrix representation** with one eigenvalue λ.

If we set $E(0)$ the intersection of the kernels of $P(\lambda)$, then $E(0)$ is a closed subspace invariant under A and the restriction of A to $E(0)$ is a **compact operator with spectrum** $\{0\}$.

So we get the following proposition:

Proposition 4.1. *Let A be a compact operator on Banach space E, and $\sigma(A)$ the spectrum of A. Then E is a direct sum of the ranges of the invariant subspaces projections together $E(\lambda_i)$ ($\lambda_i \neq 0$) with $E(0)$ where*

$$E = \underset{i}{\oplus} E(\lambda_i) \oplus E(0).$$

4.2 The set of operators commuting with a compact operator with a nonzero eigenvalues

Theorem 3. *Let E be a complex Banach space, A_c a compact operator in E with a nonzero eigenvalues. Any compact operators B_c which have the same invariant subspaces as A_c (i.e $E = \underset{i}{\oplus} E(\lambda_i) \oplus (\underset{i}{\oplus} E(\lambda_i))^{\perp}$) and commutes with A_c is A_c-similar to the following quasi-diagonal form:*

$$\begin{pmatrix} B_{\lambda_1} & & & & \\ & \ddots & & & \\ & & B_{\lambda_n} & & \\ & & & \ddots & \\ & & & & B_0 \end{pmatrix}$$

where $B_{\lambda_i} E(\lambda_i) \subset E(\lambda_i)$ and $B_0 = B_{c|(\underset{i}{\oplus} E(\lambda_i))^{\perp}}$ is the zero operator.

Proof: If λ_i is an eigenvalue such that $\lambda_i \neq 0$, hence the restriction $A_{c|E(\lambda_i)} = A_{\lambda_i}$ admits a **Jordan matrix representation** of one eigenvalue λ_i (See Proposition 4.1) and $A_{c|(\underset{i}{\oplus} E(\lambda_i))^{\perp}} = A_0$ is the zero operator. In the sequence, for simplicity, any operator on quasi-diagonal form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ will be written $A_1 \oplus A_2$.

Then we can construct a chain of operators as follows:
$A_1 = A_{\lambda_1} \oplus A_0$
$A_2 = A_{\lambda_1} \oplus A_{\lambda_2} \oplus A_0$
$A_3 = A_{\lambda_1} \oplus A_{\lambda_2} \oplus A_{\lambda_3} \oplus A_0$
$\vdots$
$A_n = A_{\lambda_1} \oplus A_{\lambda_2} \oplus A_{\lambda_3} \oplus \cdots \oplus A_{\lambda_n} \oplus A_0$
$A_i : E \longrightarrow E$ is a finite rank operators where its limit is the compact operator $A_c : E \longrightarrow E$, that $\underset{n\to\infty}{lim} \|A_c - A_n\| = 0$ and $A_c = \underset{i\in\mathbb{N}}{\oplus} (A_{c|E(\lambda_i)}) \oplus A_0$.

To define all operators commuting with A_c, let be $B_0 = B_{c|(\underset{i}{\oplus} E(\lambda_i))^{\perp}}$ the null endomorphism of $(\underset{i}{\oplus} E(\lambda_i))^{\perp}$ and we apply the Proposition 3.1 to define the set of all endomorphisms B_{λ_i} of $E(\lambda_i)$ and that satisfy $[A_{\lambda_i}, B_{\lambda_i}] = 0$ then we construct a chain of operators B_i as follows:
$B_1 = B_{\lambda_1} \oplus B_0$
$B_2 = B_{\lambda_1} \oplus B_{\lambda_2} \oplus B_0$

$B_3 = B_{\lambda_1} \oplus B_{\lambda_2} \oplus B_{\lambda_3} \oplus B_0$
$\vdots$

$B_n = B_{\lambda_1} \oplus B_{\lambda_2} \oplus B_{\lambda_3} \oplus \cdots \oplus B_{\lambda_n} \oplus B_0$
For $i := 1, ..., n \quad B_i : E \longrightarrow E$ and satisfy: $[A_i, B_i] = 0$.

The operator B_n is of finite rank where its limit is the compact operator $B_c : E \longrightarrow E$, that $\underset{n\to\infty}{lim} \|B_c - B_n\| = 0$.

To complete the proof we will show that: $[A_c, B_c] = 0$.

First we prove that:

$$[A_c, B_c] = \lim_{n\to\infty} [A_n, B_n],$$

that is equivalent to:

$$A_cB_c - B_cA_c = \lim_{n\to\infty}(A_nB_n - B_nA_n),$$

we have:

$$\begin{aligned}\lim_{n\to\infty}\|A_nB_n - A_cB_c\| &= \lim_{n\to\infty}\|A_nB_n - A_cB_n + A_cB_n - A_cB_c\|\\ &= \lim_{n\to\infty}\|(A_n - A_c)B_n + A_c(B_n - B_c)\|\\ &\le \lim_{n\to\infty}\|A_n - A_c\|\,\|B_n\| + \|A_c\|\,\|B_n - B_c\| = 0,\end{aligned}$$

with the same method we prove:

$$\begin{aligned}\lim_{n\to\infty}\|B_nA_n - B_cA_c\| &= \lim_{n\to\infty}\|B_nA_n - B_cA_n + B_cA_n - B_cA_c\|\\ &= \lim_{n\to\infty}\|(B_n - B_c)A_n + B_c(A_n - A_c)\|\\ &\le \lim_{n\to\infty}\|B_n - B\|\,\|A_n\| + \|B\|\,\|A_n - A\| = 0,\end{aligned}$$

since A_n and B_n are bounded then:

$$\begin{aligned}\lim_{n\to\infty}[A_n, B_n] &= \lim_{n\to\infty}(A_nB_n - B_nA_n)\\ &= \lim_{n\to\infty}A_nB_n - \lim_{n\to\infty}B_nA_n\\ &= A_cB_c - B_cA_c = [A_c, B_c].\end{aligned}$$

Finally, since $[A_n, B_n] = 0$ for each $n \in \mathbb{N}^*$ then

$$[A_c, B_c] = 0.$$

□

5 The spectral decomposition of a positive compact quasinilpotent operator

For this part we refer the reader to ([1], [2] , [3] , [8] , [13], [20]). We also, need to recall the following definitions:

- A continuous operator A in Banach space is said to be **quasinilpotent** if its spectral radius is zero. It is well known that A is quasinilpotent if and only if for each $x \in E$:
$$\lim_{n\to\infty} \left\| A^n x \right\|^{\frac{1}{n}} = 0.$$
- A real vector space E which is ordered by some order relation $\leq$ is called a **vector lattice** if any two elements $x, y \in E$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$ and the following properties are satisfied:

  1) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in E$,

  2) $0 \leq x$ implies $0 \leq tx$ for all $x \in E$ and $t \in \mathbb{R}_+$.
- A norm on vector lattice E is called a **lattice norm** if:
$$|x| \leq |y| \Longrightarrow \|x\| \leq \|y\| \quad \text{for } x, y \in E$$
where $|x|$ be the absolute value of x: $|x| := x \vee (-x)$.
- A **Banach lattice** is a real Banach space E endowed with an ordering $\leq$ such that $(E, \leq)$ is a vector lattice and the norm on E is a lattice norm.
- A linear operator A is said to be **positive** operator (in symbol $A \geq 0$) if $Ax \geq 0$ for each $x \geq 0$.
- A **chain of bands** is a family of subspaces that are ordered by inclusion.
- Let be L the lattice of all closed chains of subspaces in Banach lattice E, we said that a chain C is **maximal** in L, if C included in an other chain C' of L, then $C = C'$.
- Afterwards we set $E' = E(0)$ and A' the restriction of A to $E(0)$.

5.1 Maximal chains of bands

Let L be the lattice of all closed chains of closed subspaces of E'. Recall that a chain C in L is said to be **simple** whenever it satisfies the following three conditions:

1) $\{0\} \in C, \;\; E' \in C$.

2) If C_0 is subfamily of C, then the closed subspaces: $\wedge C_0 = \cap \{c : c \in C_0\}$, $\vee C_0 = \overline{\cup \{c : c \in C_0\}}$ are in C.

3) For each M in C, the quotient space M/M_- where $M_- = \vee \{c \in C : c \subsetneqq M\}$ is at most one-dimensional.

By condition (2) $M_- \in C$ for all $M \in C$.

Result 1: A chain C is **maximal** if and only if it is **simple**.

Result 2: If $(E', \| \quad \|)$ is a Banach lattice ($\dim E' \geq 2$), and if A' is a positive compact quasinilpotent operator in E', then there exists a non trivial A'-invariant closed ideal in E'.

Proposition 5.1. *Every positive compact quasinilpotent operator A' in E' possesses a maximal chain of bands which are $A'-$invariant (invariant subspaces), is called a $A'-$**invariant maximal chain of bands.***

Proof: By Result 2, every operator such A' has a non-trivial A'-invariant band. An application of Zorn's lemma guarantees the existence of maximal chain C of A'-invariant bands. □

If we denote by B the **lattice of closed ideals bands**, then C is a maximal chain in B. Where $\dim B/B_- \leq 1$ for all $B \in C$.

Let (P) be a partition of E' depending on the maximal chain C then

$$(P) \qquad \{0\} = E'_0 \subset E'_1 \subset ....E'_n \subset ...E' \qquad (E'_i \in C),$$

where $\dim E'_i/E'_{i-1} \leq 1$.

And A'_i is the restriction of A' to E'_i then:

$$A'_i = A'_{|E'_i} \qquad E'_i \longrightarrow E'_i$$

It is evident that $A'_n = A'_{|E'_n}$ is nilpotent of rank at most n, and if E'_1 is generated by the nonzero vector $\{v\}$, then for every $n \in \mathbb{N}^\star$ we have:

$$\begin{aligned}
&E'_1 = span\,\{v\}\\
&E'_2 = span\,\{v, A'v\}\\
&E'_3 = span\,\left\{v, A'v, A'^2v\right\}\\
&\vdots\\
&E'_n = span\,\left\{v, A'v, ..., A'^{n-1}v\right\}.
\end{aligned}$$

If we denote by $U_n = \left\{v, A'v, ..., A'^{n-1}v\right\}$ then A'_n admits a **Jordan matrix representation** of one eigenvalue equal to zero as the form:

$$A'_n = U_n \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & 0 & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} U_n^{-1}$$

where

$$(A'_n)^n = 0.$$

By recurrence we prove for $n+1$:

$$A'_{n+1}{=}U_{n+1}\left(\begin{array}{c|c} \begin{matrix} & & & \\ & & & \\ & A'_n & & \\ & & & \\ & & & \end{matrix} & \begin{matrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{matrix} \\ \hline 0 \quad \cdots \quad \cdots \quad 0 \quad 0 & 0 \end{array}\right)U_{n+1}^{-1} =$$

$$\mathrm{U}_{n+1}\left(\begin{array}{ccccc|c} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 \\ \hline 0 & \cdots & \cdots & 0 & 0 & 0 \end{array}\right)U_{n+1}^{-1}$$

where

$$(A'_{n+1})^{n+1} = 0,$$

then for every $n \in \mathbb{N}^\star$, $(A'_n)^n = 0$.

Where the quasinilpotent operator A' admits a **Jordan matrix representation** depending on the increasing partition (P) of E' under the form:

$$A' = U \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots & \\ \vdots & \vdots & 0 & 1 & \ddots & \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots \\ & & & & \ddots & \ddots \\ & & & & & \ddots \end{pmatrix} U^{-1}$$

5.2 The set of operators commuting with a positive compact quasinilpotent operator

Theorem 4. *Let E' be a Banach lattice space, A' a positive compact quasinilpotent operator in E', then any operator B' which is $A'-$similar and satisfies*

$[A', B'] = 0$, *admits an infinite upper triangular matrix representation (depending on the same increasing partition* (P) *of* E' *previously considered for* A*) of the following type:*

$$U \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_n & \cdots \\ 0 & a_1 & \cdots & \cdots & a_{n-1} & \cdots \\ \vdots & \ddots & \ddots & \cdots & \vdots & \\ \vdots & & \ddots & \ddots & a_2 & \\ 0 & \cdots & \cdots & 0 & a_1 & \ddots \\ \vdots & & & & \ddots & \ddots \end{pmatrix} U^{-1}$$

where $a_1, a_2, a_3, ...$ *are arbitrary parameters.*

Proof: Let A' be the quasinilpotent part of A and $E(0) = E'$ which is Banach lattice that $A' = A'_{|E'}$ is a positive compact quasinilpotent operator. From the Proposition 5.1, A' admits a spectral decomposition which allows us to define for each finite submatrix A'_n a set of all commuting matrices B'_n that are defined in the same invariant subspace as A'_n and B'_n have an upper triangular matrix representation where the elements of all directions parallel to the principal diagonal are equal (See Proposition 3.1).

$$A'_n = U_n \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} U_n^{-1},$$

$$B'_n = U_n \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_n \\ 0 & a_1 & \cdots & \cdots & a_{n-1} \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ \vdots & & \ddots & \ddots & a_2 \\ 0 & \cdots & \cdots & 0 & a_1 \end{pmatrix} U_n^{-1}$$

we can prove by recurrence that $\forall n \in \mathbb{N}^\star$

$$[A'_n, B'_n] = 0,$$

it's clear that for an integer $n \in \mathbb{N}^\star$ the relation $[A'_n, B'_n] = 0$ is satisfied, and for $n + 1$ we write A'_{n+1}, B'_{n+1} under the form:

$$A'_{n+1} = U_{n+1} \left(\begin{array}{|cccc|c|} \hline 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ \hline 0 & \cdots & \cdots & 0 & 0 \\ \hline \end{array} \right) U_{n+1}^{-1},$$

$$B'_{n+1} = U_{n+1} \left(\begin{array}{|cccc|c|} \hline a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & \ddots & \cdots & \vdots & a_n \\ \vdots & & \ddots & a_2 & \vdots \\ 0 & \cdots & 0 & a_1 & a_2 \\ \hline 0 & \cdots & \cdots & 0 & a_1 \\ \hline \end{array} \right) U_{n+1}^{-1}$$

that satisfies

$$\left[A'_{n+1}, B'_{n+1}\right] = 0,$$

then we can say $\forall n \in \mathbb{N}^\star$

$$[A'_n, B'_n] = 0.$$

If we write

$$B' = \lim_{n \to \infty} B'_n = B'_\infty,$$

then

$$[A', B'] = 0.$$

$\square$

Conclusion. *From Section 4 and Section 5 we can conclude that: If A is a compact operator with positive quasinilpotent part on complex Banach space E then A can be writhen as the form $A = \underset{i}{\oplus} A_i \oplus A'$ and $E = \underset{i}{\oplus} E_i \oplus E_0$ (See Proposition 4.1) where $A_i = A_{|E_i}$ and $A' = A_{|E_0}$ is a quasinilpotent part of A then we define in E all operators B where $B = \underset{i}{\oplus} B_i \oplus B'$ and $BE_i \subset E_i$ (See Theorem 3 and Theorem 4) that satisfy $[A, B] = 0$.*

Remark 5.1. *The operator B is compact if and only if B' is compact.*

6 Examples and application on the calculation of optimal control

Example 1. Let A be a matrix characterized by its elementary divisors $(\lambda -$

$\lambda_1)^3,\ (\lambda-\lambda_1)^2,\ (\lambda-\lambda_2)^3,\ (\lambda-\lambda_2)^1,\quad \lambda_1 \neq \lambda_2$. In this case all matrices which commutes with A are as follows:

$$A = U\left(\begin{array}{ccc|cc|ccc|c}
\lambda_1 & 1 & 0 & & & & & & \\
0 & \lambda_1 & 1 & 0 & & & 0 & & 0 \\
0 & 0 & \lambda_1 & & & & & & \\
\hline
 & 0 & & \lambda_1 & 1 & & 0 & & 0 \\
 & & & 0 & \lambda_1 & & & & \\
\hline
 & & & & & \lambda_2 & 1 & 0 & \\
 & 0 & & 0 & & 0 & \lambda_2 & 1 & \\
 & & & & & 0 & 0 & \lambda_2 & \\
\hline
 & 0 & & 0 & & & 0 & & \lambda_2
\end{array}\right)U^{-1}$$

$$B = U\left(\begin{array}{ccc|cc|ccc|c}
a & b & c & f & j & & & & \\
0 & a & b & 0 & f & & 0 & & 0 \\
0 & 0 & a & 0 & 0 & & & & \\
\hline
0 & h & k & d & e & & 0 & & 0 \\
0 & 0 & h & 0 & d & & & & \\
\hline
 & & & & & x & y & z & p \\
 & 0 & & 0 & & 0 & x & y & 0 \\
 & & & & & 0 & 0 & x & 0 \\
\hline
 & 0 & & 0 & & 0 & 0 & n & m
\end{array}\right)U^{-1}$$

The parameters $a, b, c, ...$ are arbitrary. Where $[A, B] = 0$.

Example 2. Optimal control of error associated to an observer:
We consider the observed bilinear system:

$$(\textstyle\sum)\ \begin{cases} \dot{x}(t) &= \quad \Delta x(t) + A(t)u(t) + B(u(t), x(t)) \\ x(0) &= x_0 \\ y(t) &= Cx(t), \end{cases}$$

where $u \in L^2([0,T], \mathbb{R})$, E a Hilbert space and C a bounded linear operator in E.

We assume that Δ is a generator of strongly continuous semigroup $G(t)$, that:

$$G(0) = Id \qquad \text{and} \qquad \|G(t)\| \leq M.$$

The observer problem finds many applications in fields of robotics, mechanics, heat transfer and bio-chemical processes (See [7], [9], [15], [16]).

For our type of systems $(\sum)$, the classical observer, which is a generalization of finite dimensional linear systems is the following (See [11], [12]):

$$\begin{cases} \dot{\widehat{x}}(t) &= \quad \Delta\widehat{x}(t) + A(t)u(t) + B(u(t),\widehat{x}(t)) - \widehat{K}(C\widehat{x}(t) - y(t)) \\ \widehat{x}(0) &= \widehat{x}_0 \\ \widehat{y}(t) &= C\widehat{x}(t). \end{cases}$$

The objective is, by intermediate of this auxiliary system, we give an estimate $\widehat{x}(t)$ to the state $x(t)$ of initial system, where the "error":

$$\varepsilon(t) = \widehat{x}(t) - x(t),$$

which tends to zero as t tends to infinity, the error equation is:

$$(I) \qquad \begin{cases} \dot{\varepsilon} &= \Delta\varepsilon + B(u,\varepsilon) - \widehat{K}C\varepsilon \\ \varepsilon(0) &= \varepsilon_0 \\ \widehat{y} - y &= C\varepsilon. \end{cases}$$

Let $\Delta' = \Delta - \widehat{K}C$, then the system (I) is written:

$$(II) \qquad \begin{cases} \dot{\varepsilon}(t) &= \Delta'\varepsilon(t) + B(u(t),\varepsilon(t)) \\ \varepsilon(0) &= \varepsilon_0 \\ \widehat{y}(t) - y(t) &= C\varepsilon(t), \end{cases}$$

with these notations, $C^\star C \in \mathcal{L}(E)$ and $\Delta' \in L(D,E)$ the space of bounded linear operators from D to E, where D is the domain of Δ.

It follows from [6] and [17] that $\Delta' = \Delta - C^\star C$ is the infinitesimal generator of a continuous semigroup $S(t)$ at E that:

$$\|S(t)\| \leq Me^{M\|C\|^2 t} = M_1.$$

Application.

Consider the case where:

$$J(u) = \int_0^1 \|u\|^2 \, ds + \|\varepsilon(1)\|^2,$$

and we suppose that $B(u(t),\varepsilon(t)) = u(t)B\varepsilon(t)$, where B is bounded linear operator of finite rank, which defined by

$$B = B_1 \oplus B_0 = \left(\begin{array}{|c|c|} \hline B_1 & 0 \\ \hline 0 & B_0 \\ \hline \end{array}\right)$$

where $B_1 \in M_{5,5}(\mathbb{C})$, and B_0 is an infinite dimensional block of zero:

$$B_1 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

If Δ' is defined as a Lie bracket $[\Delta', B]$ is zero, then according to Theorem 1, there is a constant optimal control that minimizes the functional J. By Theorem 2, all matrices which commute with B are of the form:

$$\left(\begin{array}{ccccc|c} a_1 & a_2 & a_3 & 0 & 0 & \\ 0 & a_1 & a_2 & 0 & 0 & \\ 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 & b_2 & \\ 0 & 0 & 0 & 0 & b_1 & \\ \hline & & 0 & & & 0 \end{array}\right)$$

we choose Δ' as the form :

$$\Delta' = \Delta'_1 \oplus \Delta'_0 = \left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \\ \hline & & 0 & & & 0 \end{array}\right)$$

where Δ'_0 is infinity zero block.

We determine the optimal control $\bar{u}$ of problem (II) which is constant and this constant is solution of the equation:

$$(III) \qquad 2\bar{u} + \varepsilon_0^t[\exp(\Delta' + \bar{u}B)]^t \, [B + B^*] \, [\exp(\Delta' + \bar{u}B)]\varepsilon_0 = 0,$$

for $\varepsilon_0 = (1, 1, 1, 1, 1, 0, 0, 0, ...)^t$, where

$$\exp(\Delta' + \bar{u}B) =$$

$$\left(\begin{array}{ccccc|c} e^{-2} & e^{-2}\left(\bar{u}+1\right) & \frac{e^{-2}}{2}\left(\bar{u}+1\right)^2 & 0 & 0 & \\ 0 & e^{-2} & e^{-2}\left(\bar{u}+1\right) & 0 & 0 & \\ 0 & 0 & e^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-3} & e^{-3}\left(\bar{u}+1\right) & \\ 0 & 0 & 0 & 0 & e^{-3} & \\ \hline & & 0 & & & I_\infty \end{array}\right)$$

and (III) is equivalent to:

$$\bar{u}^3 + 6\bar{u}^2 + (2e^4 + 2e^{-2} + 15)\bar{u} + 4e^{-2} + 14 = 0,$$

the solution is :

$$\bar{u} = -0.117\,48,$$

and

$\varepsilon(t) = e^{(\Delta + \bar{u}B)t}\varepsilon_0$

$\varepsilon(t)=e^{-2t}\left(1 + 0.882\,52t + 0.389\,42t^2 \ ,1 + 0.882\,52t \ ,1 \ ,(1 + 0.882\,52t)e^{-t} \ ,e^{-t}\right.$

,0 ,0 , . . .)
such that $\varepsilon(t)$ tends to zero as t tends to infinity.

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