The Siegel norm of algebraic numbers

by

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Abstract

In this paper we investigate connections between the Siegel norm and the spectral norm on the algebraic closure \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \), and their extensions to the spectral completion \( \tilde{\mathbb{Q}} \) of \( \mathbb{Q} \).

Key Words: Galois groups, normed fields.

2010 Mathematics Subject Classification: Primary 11R80; Secondary 11R04.

1 Introduction

Let \( \overline{\mathbb{Q}} \) be the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \), and denote by \( \| \cdot \| \) the spectral norm on \( \overline{\mathbb{Q}} \), defined by

\[
\| \alpha \| = \max_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} |\sigma(\alpha)|,
\]

for any algebraic number \( \alpha \). We consider the map \( A: \overline{\mathbb{Q}} \to [0, \infty) \) given by

\[
A(\alpha) = \frac{1}{[K: \mathbb{Q}]} \sum_{\sigma} |\sigma(\alpha)|^2,
\]

where \( \sigma \) runs over all the embeddings of \( K \) into \( \mathbb{C} \). Here \( A(\alpha) \) depends only on \( \alpha \) and not on the field \( K \) containing \( \alpha \). Note that if \( \alpha = \beta^2 \), where \( \alpha \) is totally real and positive, then \( \beta \) is totally real and

\[
A(\beta) = \frac{\text{Tr}_\alpha}{\deg \alpha}.
\]

For a totally real and positive algebraic integer \( \alpha \in O_{\overline{\mathbb{Q}}} \), let \( n = \deg \alpha \) be the degree of \( \alpha \) over \( \mathbb{Q} \), and let \( \alpha_1 = \alpha, \alpha_2, \cdots, \alpha_n \) be the conjugates of \( \alpha \) over \( \mathbb{Q} \). Then

\[
\frac{\text{Tr}_\alpha}{\deg \alpha} = \frac{\alpha_1 + \cdots + \alpha_n}{n} \geq \sqrt[n]{\alpha_1 \cdots \alpha_n} \geq 1.
\]

*Research of the second author is supported by the NSF grant DMS-0901621
The well known trace problem of Siegel asks for the best possible constant \( \lambda_0 \) for which given \( \lambda < \lambda_0 \), the trace of a totally real and positive algebraic integer \( \gamma \) is at least \( \lambda \) times its degree, except for finitely many \( \gamma \)'s.

The best result to date is \( \lambda_0 \geq 1.78702 \). On the other hand, as Siegel pointed out, for every odd prime \( p \), the number \( 4 \cos^2 \frac{\pi}{p} \) is a totally real and positive algebraic integer of degree \( \frac{p-1}{2} \) and its trace is \( p-2 \). So the best possible constant \( \lambda_0 \) is at most 2.

In [3], and more recently in [9], the restriction of the map \( A(\cdot) \) to the ring of cyclotomic integers \( \mathcal{O}_{\mathbb{Q}^{ab}} \) is studied. It is shown in [9] that the set \( T = \{ A(\alpha) : \alpha \in \mathcal{O}_{\mathbb{Q}^{ab}} \} \) is closed under addition, that \( T \) is topologically closed in \( \mathbb{R} \), and that for any \( 0 \leq r < 1, r \in \mathbb{Q} \), there is a \( t_r \in \mathbb{Q} \) such that

\[
T \cap (r + \mathbb{Z}) = \{ t_r, t_r + 1, t_r + 2, \cdots \}.
\]

We also mention that from Siegel’s work [8] it follows that the intersection \( T \cap [0, \frac{3}{2}) \) consists of only two elements: 0 and 1, and they are attained when \( \alpha = 0 \), respectively when \( \alpha \) is a root of unity. A striking application of this result is provided in an unpublished theorem of Thompson. Recall that the values of a linear character of a finite group are roots of unity. A classical theorem of Burnside [1] says that a non-linear irreducible character of a finite group has at least one zero. Thompson’s theorem, whose proof also implies Burnside’s result, states that an irreducible character of a finite group is zero or a root of unity at more than a third of the group elements.

In the present paper we extend \( A(\alpha) \) to a larger set. The completion \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \) with respect to the spectral norm (see [4] and [5]), provides a natural setting for such an extension. The map \( A(\cdot) : \overline{\mathbb{Q}} \rightarrow [0, \infty) \) is continuous with respect to the spectral norm, and it naturally extends to a map, which we will still denote by \( A(\cdot) \), from \( \overline{\mathbb{Q}} \) to \( [0, \infty) \). For any algebraic number \( \alpha \), we define its Siegel norm \( \| \alpha \|_{S_i} \) by \( \| \alpha \|_{S_i} = \sqrt{A(\alpha)} \). In this paper we investigate connections between the Siegel norm and the spectral norm on \( \overline{\mathbb{Q}} \), and their extensions to the spectral completion \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \). In the last section we show how one can explicitly compute the Siegel norm for some classes of elements of \( \overline{\mathbb{Q}} \).

2 Construction of the Siegel norm

Let us first remark that the function \( \| \cdot \|_{S_i} : \overline{\mathbb{Q}} \rightarrow [0, \infty) \) has the following properties.

1. \( \| \alpha + \beta \|_{S_i} \leq \| \alpha \|_{S_i} + \| \beta \|_{S_i} \), for any \( \alpha, \beta \in \overline{\mathbb{Q}} \).
2. \( \| c \alpha \|_{S_i} = |c| \| \alpha \|_{S_i} \), for any \( c \in \mathbb{Q}, \alpha \in \overline{\mathbb{Q}} \).

Indeed, let \( \alpha, \beta \in \overline{\mathbb{Q}} \). Let \( K \) be a number field such that \( \alpha, \beta \in K \). Then

\[
\sum_{\sigma} |\sigma(\alpha) + \overline{\sigma(\beta)}|^2 \leq \sum_{\sigma} |\sigma(\alpha)|^2 + \sum_{\sigma} |\sigma(\beta)|^2 + | \sum_{\sigma} \sigma(\alpha) \overline{\sigma(\beta)} | + | \sum_{\sigma} \overline{\sigma(\alpha)} \sigma(\beta) |.
\]
Employing Cauchy’s inequality, we derive that

\[ |\sum_{\sigma} \sigma(\alpha)\overline{\sigma(\beta)}| = |\sum_{\sigma} \overline{\sigma(\alpha)}\sigma(\beta)| \leq \left( \sum_{\sigma} |\sigma(\alpha)|^2 \right)^{1/2} \left( \sum_{\sigma} |\sigma(\beta)|^2 \right)^{1/2}. \]

Hence \( A(\alpha + \beta) \leq A(\alpha) + A(\beta) + 2\sqrt{A(\alpha)A(\beta)}, \) so \( \sqrt{A(\alpha + \beta)} \leq \sqrt{A(\alpha)} + \sqrt{A(\beta)}, \) and the first remark follows.

The second part follows easily from the definition of the map \( A. \)

The above properties show that \( \|\cdot\|_{S_i} \) is a vector space norm on \( \overline{\mathbb{Q}} \). Note that \( A(\alpha) \leq \|\alpha\|_2, \) hence \( \|\alpha\|_{S_i} \leq \|\alpha\|, \) for any \( \alpha \in \overline{\mathbb{Q}}. \)

For \( x \in \tilde{\mathbb{Q}} \) and \( \delta > 0, \) consider the open ball \( B(x, \delta) = \{ y \in \overline{\mathbb{Q}} : \|y - x\| < \delta \}, \) and let \( n(x, \delta) = \min\{\deg \alpha : \alpha \in B(x, \delta)\}. \) We can now state the following theorem.

**Theorem 1.**

i) The map \( A : \overline{\mathbb{Q}} \to [0, \infty) \) is continuous with respect to the spectral norm and it extends canonically to a map, which we will still denote by \( A, \) from \( \tilde{\mathbb{Q}} \) to \([0, \infty)\).

ii) Let \( x \in \tilde{\mathbb{Q}}, x \neq 0. \) Then \( \|x\|_{S_i} \geq \frac{\|x\|}{4\sqrt{n(x, \frac{\|x\|}{2})}}. \)

iii) \( \|\cdot\|_{S_i} \) is a vector space norm on \( \tilde{\mathbb{Q}}. \)

**Proof:**

i) Let \( (x_n)_{n \geq 0} \) be a convergent sequence in \( \overline{\mathbb{Q}}. \) Then \( (x_n) \) is Cauchy and let \( M > 0 \) be such that \( |x_n| \leq M, \) for any \( n \geq 0. \) From the proof of the remarks at the beginning of this section it follows that for all \( m, n \geq 0 \) we have

\[ |\sqrt{A(x_n)} - \sqrt{A(x_m)}| \leq \sqrt{A(x_n - x_m)}. \quad (1) \]

On the other hand, since \( \sqrt{A(\alpha)} \leq \|\alpha\|, \) for any algebraic number \( \alpha, \) we derive that

\[ \sqrt{A(x_n - x_m)} \leq \|x_n - x_m\|. \quad (2) \]

Combining relations (1) and (2), we obtain

\[ |\sqrt{A(x_n)} - \sqrt{A(x_m)}| \leq \|x_n - x_m\|. \quad (3) \]

We have

\[ |A(x_n) - A(x_m)| = |\sqrt{A(x_n)} - \sqrt{A(x_m)}| \sqrt{A(x_n)} + \sqrt{A(x_m)}| \leq \sqrt{2M|\sqrt{A(x_n)} - \sqrt{A(x_m)}|} \leq 2M\|x_n - x_m\|, \]

and hence

\[ |A(x_n) - A(x_m)| \leq 2M\|x_n - x_m\|. \quad (4) \]
Since \((x_n)_n\) is Cauchy with respect to the spectral norm, it follows from inequality (4) that the sequence \((A(x_n))_n\) is Cauchy in \([0, \infty)\), hence it is convergent. Another consequence of relation (4) is that the map \(A\) can be extended to \(\overline{Q}\) as follows. Let \(\alpha \in \overline{Q}\). Then \(\alpha = \lim_{n \to \infty} \alpha_n\), with \(\alpha_n \in Q\), and \(A(\alpha) := \lim_{n \to \infty} A(\alpha_n) \in [0, \infty)\).

ii) Let \(x \in \overline{Q}, x \neq 0\), and let \(0 < \delta = \frac{\|x\|}{4}\). Also let \(\alpha, \alpha_0 \in \overline{Q} \cap B(x, \delta)\) and denote \(n = \deg \alpha, \alpha_0 = \deg \alpha_0\). Let \(K\) be a finite field extension of \(Q\) such that \(\alpha, \alpha_0 \in K\). Denote by \(m\) the degree of \(K\) over \(Q(\alpha_0)\). From the above choice of \(\alpha\) we have \(\|\alpha_0 - x\| < \frac{\|x\|}{4}\). It follows that

\[
\|\alpha_0\| \geq \|x\| - \|x - \alpha_0\| > \|x\| - \frac{\|x\|}{4} = \frac{3}{4}\|x\|. 
\]  

(5)

Let \(\sigma_1, \ldots, \sigma_{mn}\) be the embeddings of \(K\) in \(\mathbb{C}\). From relation (5) we deduce that

\[
\max_{1 \leq j \leq mn} |\sigma_j(\alpha_0)| = \|\alpha_0\| > \frac{3}{4}\|x\|. 
\]

There exist \(j_1, \ldots, j_m\) such that \(|\sigma_{j_1}(\alpha_0)| = |\sigma_{j_1}(0)| = \cdots = |\sigma_{j_m}(\alpha_0)| > \frac{3}{4}\|x\|\).

We have \(\|\alpha_0 - \alpha\| \leq \|\alpha_0 - x\| + \|x - \alpha\| < \frac{3}{4}\|x\| + \frac{3}{4}\|x\| = \frac{3}{2}\|x\|\).

It follows that \(\|\sigma_j(\alpha_0) - \sigma_j(\alpha)\| < \frac{3}{4}\|x\|\), for any \(j \in \{1, 2, \ldots, mn\}\). In particular, \(\|\sigma_j(\alpha_0) - \sigma_j(\alpha)\| < \frac{3}{4}\|x\|\). Since \(\|\sigma_j(\alpha_0)\| > \frac{3}{4}\|x\|\), we obtain

\[
|\sigma_{j_1}(\alpha)| \geq |\sigma_{j_1}(\alpha_0)| - |\sigma_{j_1}(\alpha_0) - \sigma_{j_1}(\alpha)| > \frac{3}{4}\|x\| - \frac{1}{2}\|x\| = \frac{1}{4}\|x\|. 
\]

We derive that

\[
A(\alpha) = \frac{1}{mn} \sum_{1 \leq j \leq mn} |\sigma_j(\alpha)|^2 \geq \frac{1}{mn} \left( |\sigma_{j_1}(\alpha)|^2 + \cdots + |\sigma_{j_m}(\alpha)|^2 \right) \geq \frac{1}{mn} \cdot \frac{\|x\|^2}{16} = \frac{\|x\|^2}{16mn}.
\]

Hence

\[
\|\alpha\|_{S_1} \geq \frac{\|x\|}{4\sqrt{mn}}, \text{ for any } \alpha \in B\left(x, \frac{\|x\|}{4}\right).
\]

iii) From the two remarks at the beginning of this section, by continuity it follows that \(\|\alpha + \beta\|_{S_1} \leq \|\alpha\|_{S_1} + \|\beta\|_{S_1}\), for any \(\alpha, \beta \in \overline{Q}\) and also that \(\|c\alpha\|_{S_1} = |c|\|\alpha\|_{S_1}\), for any \(c \in Q, \alpha \in \overline{Q}\). Moreover, part ii) shows that for \(x \in \overline{Q}\), \(\|x\|_{S_1} = 0\) if and only if \(x = 0\), which completes the proof of the theorem. \(\Box\)
Recall ([2]) that a Pisot-Vijayaraghavan number (or simply a Pisot number or a PV number) is a real algebraic integer \( \alpha > 1 \) such that all its conjugates are of absolute value < 1.

We remark that, apart from the constant \(1/4\) on its right hand side, the inequality from Theorem 1 part ii) is best possible. Indeed let us choose a PV number, \( \beta \) say, a positive integer \( m \), and put \( x = \beta^m \). Let \( d \) denote the degree of \( \beta \) over \( \mathbb{Q} \). Since all the conjugates \( \sigma(\beta) \) are in absolute value < 1, it is easy to see (using the fact that a natural power of a PV number \( \beta \) of degree \( d \) over \( \mathbb{Q} \) also has degree \( d \) over \( \mathbb{Q} \) that if we keep \( \beta \) fixed and let \( m \) tend to infinity, the ratio \(|x|_{S_i}/|x|\) will approach \(1/\sqrt{d} \). On the other hand \( n(x,||x||/4) \leq d \). Therefore, for any fixed \( \epsilon > 0 \), if \( m \) is large enough then \(|x|_{S_i} < (1+\epsilon)\sqrt{n(x,||x||/4)} \).

### 3 Explicit computations

In this section we obtain a formula that gives the value of the map \( A(\cdot) \) on a large class of elements of \( \overline{\mathbb{Q}} \). To proceed, we introduce a few notations and recall some of the results from [6].

Let \( G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) be the absolute Galois group of \( \mathbb{Q} \), endowed with the Krull topology, and let \( C(G_\mathbb{Q}) \) be the \( \mathbb{C} \)-Banach algebra of all continuous functions defined on \( G_\mathbb{Q} \) with values in \( \mathbb{C} \) (\( ||f|| = \sup \{|f(\sigma)|, \sigma \in G_\mathbb{Q} \} \) for any \( f \in C(G_\mathbb{Q}) \)). Denote by \( \mu \) the Haar measure on \( G_\mathbb{Q} \), normalized such that \( \mu(G_\mathbb{Q}) = 1 \), and let \( \int_{G_\mathbb{Q}} f(\sigma)d\sigma \) be the corresponding Haar integral of any continuous function \( f : G_\mathbb{Q} \rightarrow \mathbb{C} \).

Let \( x \) be an element of \( \overline{\mathbb{Q}} \) and let \( \{x_n\}_n \) a Cauchy sequence in \( \overline{\mathbb{Q}} \) (relative to the spectral norm \(|\cdot||\) in the class of \( x \), i.e. \( \lim_{n \to \infty} x_n \square x \). Since \( |\sigma(x_{n+p}) - \sigma(x_n)| \leq ||x_{n+p} - x_n|| \), for all \( \sigma \in G_\mathbb{Q}, \{\sigma(x_n)\}_n \) is also a Cauchy sequence in \( \mathbb{C} \). Let \( x(\sigma) \) be the limit of \( \{\sigma(x_n)\}_n \) in \( \mathbb{C} \). It can be shown that \( ||x|| = \sup \{|x(\sigma)|, \sigma \in G_\mathbb{Q} \} \) and that \( ||x|| = ||\varphi_x|| = \sup \{|\varphi_x(\sigma)|, \sigma \in G_K \} \), where \( \varphi_x : G_\mathbb{Q} \rightarrow \mathbb{C}, \varphi_x(\sigma) = x(\sigma) \). In [6] it is shown that for any \( x \in \overline{\mathbb{Q}} \) the function \( \varphi_x \) is continuous and that the mapping

\[
\Phi : \overline{\mathbb{Q}} \rightarrow C(G_\mathbb{Q}), \quad \Phi(x) = \varphi_x
\]  

(6)

is an isomorphism between the \( \mathbb{C} \)-Banach algebras \( \overline{\mathbb{Q}} \) and \( C(G_\mathbb{Q}) \).

Following [6], we introduce a continuous function \( H : G_\mathbb{Q} \rightarrow [0,1] \) with a special property: it is a measure preserving function, in the sense that it takes a Haar measurable subset of \( G_\mathbb{Q} \) to a Lebesgue measurable subset of \([0,1]\).

We begin by fixing a tower of subgroups of finite index \( G_\mathbb{Q} \supset G_1 \supset \cdots \supset G_n \supset \cdots \supset \{e\} \) for \( G_\mathbb{Q} \), where \( \bigcap_{i=0}^{\infty} G_i = \{e\} \) and \( e \) is the identity of \( G_\mathbb{Q} \), and for this tower we consider a complete set of left cosets \( \{\Delta G_i\}_{i \geq 1} \) of \( G_\mathbb{Q} \) relative to the subgroup \( G_i \), of the form \( \Delta G_i = \{G_i, \sigma_2^{(i)}G_i, \ldots, \sigma_k^{(i)}G_i\} \) (where \( k_i = [G_\mathbb{Q} : G_i] \),
Proposition 1. Let $H : G_{\mathbb{Q}} \to [0,1]$ be the above defined map, and let $f : [0,1] \to \mathbb{C}$ be a continuous function. Let $x_f \in \widetilde{G}$ be the element corresponding to the function $f \circ H$ under the isomorphism (6). Then

$$A(x_f) = \int_0^1 |f(t)|^2 dt.$$  

Proof: Let $x_f \in \widetilde{G}$ be such an element. Let $(\alpha_j)_j$ be a sequence of algebraic numbers that converges in the spectral norm to $x_f$. Let $M_j = \{ \sigma \in G_{\mathbb{Q}} : \sigma(\alpha_j) = \alpha_j \}$, and let $n_j = [G_{\mathbb{Q}} : M_j] = \deg_{\mathbb{Q}}(\alpha_j)$.

Consider a coset decomposition of $G_{\mathbb{Q}}$ with respect to $M_j$:

$$G_{\mathbb{Q}} = \bigcup_{s=1}^{n_j} \sigma_s M_j,$$

and let $\alpha_{1j}, \alpha_{2j}, \cdots, \alpha_{nj}$ be the conjugates of $\alpha_j$ over $\mathbb{Q}$, where $\sigma_s(\alpha_j) = \alpha_{sj}$.

For $\sigma \in \sigma_s M_j$, we have $\sigma = \sigma_s m_j$, with $m_j \in M_j$, and deduce that

$$\varphi_{\alpha_j}(\sigma) = \varphi_{\alpha_j}(\sigma_s m_j) = (\sigma_s m_j)(\alpha_j) = \sigma_s(\alpha_j) = \alpha_{sj}.$$ 

Hence

$$A(\alpha_j) = \frac{1}{n_j} |\alpha_{1j}|^2 + \frac{1}{n_j} |\alpha_{2j}|^2 + \cdots + \frac{1}{n_j} |\alpha_{nj}|^2 =$$

$$= \sum_{s=1}^{n_j} \int_{\sigma_s M_j} |\varphi_{\alpha_j}(\sigma)|^2 d\sigma = \int_{G_{\mathbb{Q}}} |\varphi_{\alpha_j}(\sigma)|^2 d\sigma.$$
Let $M > 0$ be such that $|\varphi_{\alpha_j}(\sigma)|, |\varphi_{x_f}(\sigma)| \leq M$ for any $\sigma \in G_\mathbb{Q}$. For $\sigma \in G_\mathbb{Q}$ one has:

$$\left| |\varphi_{\alpha_j}(\sigma)|^2 - |\varphi_{x_f}(\sigma)|^2 \right| = \left| |\varphi_{\alpha_j}(\sigma)| + |\varphi_{x_f}(\sigma)| \right| \cdot \left| |\varphi_{\alpha_j}(\sigma)| - |\varphi_{x_f}(\sigma)| \right|$$

$$\leq 2M \left| \varphi_{\alpha_j}(\sigma) - \varphi_{x_f}(\sigma) \right| \leq 2M \left\| \varphi_{\alpha_j} - \varphi_{x_f} \right\| = 2M \left\| \alpha_j - x_f \right\|.$$ 

This shows that the sequence $\left( |\varphi_{\alpha_j}(\cdot)|^2 \right)_j$ converges uniformly to $|\varphi_{\alpha_j}(\cdot)|^2$, and hence

$$\int_{G_\mathbb{Q}} |\varphi_{\alpha_j}(\sigma)|^2 d\sigma \rightarrow \int_{G_\mathbb{Q}} |\varphi_{x_f}(\sigma)|^2 d\sigma.$$ 

Since $A(x_f) = \lim_{j \to \infty} A(\alpha_j)$, and $\varphi_{x_f} = f \circ H$ we obtain that

$$A(x_f) = \int_{G_\mathbb{Q}} |(f \circ H)(\sigma)|^2 d\sigma. \quad (7)$$

In [6] it is proven that for any continuous function $g : [0,1] \to \mathbb{C}$ one has

$$\int_0^1 g(t) dt = \int_{G_\mathbb{Q}} (g \circ H)(\sigma) d\sigma.$$ 

Choosing $g(t) = |f(t)|^2$, we obtain

$$\int_0^1 |f(t)|^2 dt = \int_{G_\mathbb{Q}} |(f \circ H)(\sigma)|^2 d\sigma \quad (8)$$

Combining relations (7) and (8) we conclude that

$$A(x_f) = \int_0^1 |f(t)|^2 dt, \quad (9)$$

which completes the proof of the proposition.

We end this paper with a couple of examples. 

**Example 1.** Let $f : [0,1] \to \mathbb{C}$,

$$f(t) = \exp(-2\pi int) + \exp(2\pi int),$$

and let $x_f := \Phi^{-1}(f \circ H) \in \tilde{\mathbb{Q}}$. Then

$$A(x_f) = \int_0^1 |2\cos(2\pi nt)|^2 dt = 2.$$ 

Note that this example may be interpreted in some sense as a limiting case in $\tilde{\mathbb{Q}}$ of the sequence of examples in $\mathbb{Q}$ provided by Siegel, mentioned above in the
introduction, which showed that the best possible constant $\lambda_0$ in Siegel’s trace problem is at most 2.

**Example 2.** Let $f : [0, 1] \to \mathbb{C}$, $f(t) = 2t$ and let $x_f := \Phi^{-1}(f \circ H) \in \tilde{\mathbb{Q}}$. Then

$$A(x_f) = \int_0^1 4t^2 dt = \frac{4}{3}.$$

As an element of $\tilde{\mathbb{Q}}$, we know that $x_f$ is a limit of a Cauchy sequence $\{x_n\}_n$ in $\mathbb{Q}$, which must then satisfy $\lim_{n \to \infty} x_n = \frac{4}{3}$. Let us remark that $x_f$ cannot be a limit of algebraic integers, since by Siegel’s result, for any algebraic integer $x_n$ we must have $A(x_n) \in \{0, 1\}$ or $A(x_n) \geq \frac{3}{2}$.

**References**


Received: 29.10.2011,  
Accepted: 06.01.2012.

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