Bull. Math. Soc. Sci. Math. Roumanie
Tome 55(103) No. 1, 2012, 51–61

Some Generalized Fibonomial Sums related with the Gaussian $q-$Binomial sums

by
Emrah Kilic, Ilker Akkus and Hideyuki Ohtsuka

Abstract

In this paper, we consider some generalized Fibonomial sums formulae and then prove them by using the Cauchy binomial theorem and $q-$Zeilberger algorithm in Mathematica session.

Key Words: Gaussian binomial coefficient, fibonomial coefficient, $q-$binomial theorem, sums.
2010 Mathematics Subject Classification: Primary 11B65; Secondary 05A10, 11B37.

1 Introduction

For all real n and integer k with $k \geq 0$, the Gaussian q–binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

and as zero otherwise, where

$$(x;q)_n := (1-x)(1-xq)\dots(1-xq^{n-1}).$$

Thus far, the $q-$identities have taken the interest of many authors. For a detailed information about the $q-$identities, we may refer to [1, 6, 7] and its to the list of references.

We recall some well known identities related to the $q-$identities: firstly *Gauss' identity* is as follows :

$$\sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q = \prod_{k=1}^{n} (1-q^{2k-1})$$

and the Cauchy binomial theorem is given by

$$\sum_{k=0}^{n} y^k q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{k=1}^{n} \left(1 + yq^k\right). \tag{1.1}$$

Define the non-degenerate second order linear sequences $\{U_n\}$ and $\{V_n\}$ by, for $n > 1$

$$\begin{aligned} U_n &= pU_{n-1} + U_{n-2}\ ,\ \ U_0 = 0\ ,\ \ U_1 = 1, \\ V_n &= pV_{n-1} + V_{n-2}\ ,\ \ V_0 = 2\ ,\ \ V_1 = p. \end{aligned}$$

The Binet formulas of $\{U_n\}$ and $\{V_n\}$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n\ ,$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$.

For $n \geq k \geq 1$ and any positive integer m, define the generalized Fibonomial coefficient by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} := \frac{U_m U_{2m} \ldots U_{nm}}{(U_m U_{2m} \ldots U_{km}) \left(U_m U_{2m} \ldots U_{m(n-k)}\right)} = \prod_{i=1}^{k} \frac{U_{m(n-i+1)}}{U_{mi}}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_{U;m} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_{U;m} = 1$.

When $m = 1$, we obtain the usual Fibonomial coefficient, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{U}$. For more details about the Fibonomial and generalized Fibonomial coefficients, see [2, 3, 4, 9].

The link between the generalized Fibonomial and Gaussian q–binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

By taking $q = \beta/\alpha$, the Binet formulae are reduced to the following forms:

$$U_n = \alpha^{n-1} \frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n (1 + q^n),$$

where $\mathbf{i} = \sqrt{-1} = \alpha\sqrt{q}$. All the identities we will derive hold for general q, and results about generalized Fibonacci and Lucas numbers come out as corollaries for the special choice of q.

In the present study, we shall consider new generalized Fibonomial sums formulae and prove them by using the Cauchy binomial theorem and the $q-$Zeilberger algorithm (for such computer algorithms, we may refer to [5]).

2 Some Fibonomial Sums

In this section, we consider some generalized Fibonomial sums and compute them by means of Gaussian q-binomial sums formulae.

Lemma 1. *For $n \geq 0$,*

(i) $\sum_{k=0}^{2n+1} \mathbf{i}^{-k^2} a(k) = \frac{1-\mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k),$

(ii) $\sum_{k=0}^{2n} \mathbf{i}^{mk^2 \pm k} b(k) = \mathbf{i}^{mn^2} \sum_{k=0}^{2n} \mathbf{i}^{\pm k} b(k),$

(iii) $\sum_{k=0}^{2n+1} \mathbf{i}^{k(2n+1-k) \pm k} a(k) = \frac{1 \pm (-1)^n \mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k),$

where the sequences $a(k)$ and $b(k)$ satisfy $a(k) = a(2n+1-k)$ and $b(k) = b(2n-k)$, respectively.

Proof: Consider

(i) $\sum_{k=0}^{2n+1} \mathbf{i}^{-k^2} a(k) = \frac{1}{2} \sum_{k=0}^{2n+1} (\mathbf{i}^{-k^2} + \mathbf{i}^{-(2n+1-k)^2}) a(k) = \frac{1-\mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k).$

(ii) Let $C_{n;m}(k) := \mathbf{i}^{mn^2 \pm k} - \mathbf{i}^{mk^2 \pm k}$. Then we have $C_{n;m}(k) + C_{n;m}(2n-k) = 0$. Therefore,

$$\sum_{k=0}^{2n} C_{n;m}(k) b(k) = \frac{1}{2} \sum_{k=0}^{2n} \left(C_{n;m}(k) + C_{n;m}(2n-k)\right) b(k) = 0.$$

This result together with some manipulations proves (ii).

(iii)
$$\begin{aligned} &\sum_{k=0}^{2n+1} \mathbf{i}^{k(2n+1-k)+k} a(k) \\ &= \frac{1}{2} \sum_{k=0}^{2n+1} \left((-1)^{k(n+1)} \mathbf{i}^{-k^2} + (-1)^{(2n+1-k)(n+1)} \mathbf{i}^{-(2n+1-k)^2} \right) a(k) \\ &= \frac{1+(-1)^n \mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k). \end{aligned}$$

By a similar think, it is seen that $\sum_{k=0}^{2n+1} \mathbf{i}^{k(2n+1-k)-k} a(k) = \frac{1-(-1)^n \mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k)$, which completes the proof. □

Lemma 2. *For positive integers n and m,*

$$\sum_{k=0}^{n}(-1)^{\frac{mk(2n+1-k)}{2}}q^{\frac{mk(k-2n-1)}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_{q^m} = \begin{cases}(-1)^{\binom{n+1}{2}}q^{-m\binom{n+1}{2}}(-q^{2m};q^{2m})_n & \text{if } m \text{ is odd,}\\ q^{-m\binom{n+1}{2}}(-q^m;q^m)_n^2 & \text{if } m \text{ is even.}\end{cases}$$

Proof: If m is odd,

$$\begin{aligned} & 2\sum_{k=0}^{n}(-q)^{\frac{mk(k-2n-1)}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_{q^m} \\ = & \sum_{k=0}^{2n+1}(-q^m)^{\frac{k(k-2n-1)}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_{q^m} \\ = & \sum_{k=0}^{2n+1}\mathbf{i}^{k^2}q^{m\binom{k+1}{2}}(\mathbf{i}^{-(2n+1)}q^{-m(n+1)})^k\begin{bmatrix}2n+1\\k\end{bmatrix}_{q^m}\end{aligned}$$

which, by (i) in Lemma 1, gives us

$$\begin{aligned} = & \frac{2}{1-\mathbf{i}}\sum_{k=0}^{2n+1}q^{m\binom{k+1}{2}}(\mathbf{i}^{-(2n+1)}q^{-m(n+1)})^k\begin{bmatrix}2n+1\\k\end{bmatrix}_{q^m} \\ = & (1+\mathbf{i})\prod_{k=1}^{2n+1}(1+\mathbf{i}^{-(2n+1)}q^{m(k-n-1)}) \\ = & (1+\mathbf{i})(1+\mathbf{i}^{-(2n+1)})\prod_{j=1}^{n}(1+\mathbf{i}^{-(2n+1)}q^{mj})(1+\mathbf{i}^{-(2n+1)}q^{-mj}) \\ = & 2\mathbf{i}^{n^2}\prod_{j=1}^{n}\mathbf{i}^{-(2n+1)}(q^{mj}+q^{-mj}) \\ = & 2(-q^{-m})^{\binom{n+1}{2}}(-q^{2m};q^{2m})_n.\end{aligned}$$

Thus the proof is complete for the case m is odd.

If m is even,

$$
\begin{aligned}
& 2\sum_{k=0}^{n}(-q)^{\frac{mk(k-2n-1)}{2}}\begin{bmatrix}2n+1\\ k\end{bmatrix}_{q^m} \\
= & \sum_{k=0}^{2n+1} q^{\frac{mk(k-2n-1)}{2}}\begin{bmatrix}2n+1\\ k\end{bmatrix}_{q^m} \\
= & \sum_{k=0}^{2n+1} q^{m\binom{k+1}{2}}q^{-mk(n+1)}\begin{bmatrix}2n+1\\ k\end{bmatrix}_{q^m} \\
= & \prod_{k=1}^{2n+1}(1+q^{m(k-n-1)}) = 2\prod_{j=1}^{n}(1+q^{mj})(1+q^{-mj}) \\
= & 2\prod_{j=1}^{n}(2+q^{mj}+q^{-mj}) = 2\prod_{j=1}^{n}q^{-mj}(1+q^{mj})^2 \\
= & 2q^{-m\binom{n+1}{2}}(-q^m;q^m)_n^2.
\end{aligned}
$$

So we have the conclusion. ∎

Theorem 1. *For positive integers n and m,*

$$
\sum_{k=0}^{n}\begin{Bmatrix}2n+1\\ k\end{Bmatrix}_{U;m} = \begin{cases} \prod_{k=1}^{n} V_{2mk} & \text{if } m \text{ is odd,} \\ \prod_{k=1}^{n} V_{mk}^2 & \text{if } m \text{ is even.} \end{cases}
$$

Proof: The proof could be obtained from Lemma 2 by taking $q=\beta/\alpha$. ∎

Lemma 3. *For even m and any integer t,*

$$
\sum_{k=0}^{2n}(-1)^{(\frac{m}{2}+t)k}q^{\frac{mk(k-2n)}{2}}\begin{bmatrix}2n\\ k\end{bmatrix}_{q^m} = \prod_{k=1}^{n}\left((-1)^{\frac{m}{2}+t}q^{-\frac{m(2k-1)}{2}}(1+q^{m(2k-1)})+2\right).
$$

Proof: Consider the RHS of the claimed identity, we write

$$\begin{aligned} &\sum_{k=0}^{2n}(-1)^{(\frac{m}{2}+t)k}q^{\frac{mk(k-2n)}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_{q^m} \\ =\;& \sum_{k=0}^{2n} q^{m\binom{k+1}{2}}((-1)^{\frac{m}{2}+t}q^{-\frac{m(2n+1)}{2}})^k\begin{bmatrix}2n\\k\end{bmatrix}_{q^m} \\ =\;& \prod_{k=1}^{2n}(1+(-1)^{\frac{m}{2}+t}q^{\frac{m(2k-2n-1)}{2}}) \\ =\;& \prod_{j=1}^{n}(1+(-1)^{\frac{m}{2}+t}q^{\frac{m(2j-1)}{2}})(1+(-1)^{\frac{m}{2}+t}q^{-\frac{m(2j-1)}{2}}) \\ =\;& \prod_{j=1}^{n}((-1)^{\frac{m}{2}+t}q^{-\frac{m(2j-1)}{2}}(q^{m(2j-1)}+1)+2). \end{aligned}$$

Thus the proof is complete. □

Theorem 2. *For even m and any integer t,*

$$\sum_{k=0}^{2n}(-1)^{tk}\begin{Bmatrix}2n\\k\end{Bmatrix}_{U;m} = \prod_{k=1}^{n}\left((-1)^t V_{m(2k-1)}+2\right).$$

As a consequence of Theorem 2, we have the following Corollary:

Corollary 1. *For even m,*

(i) $\displaystyle\sum_{k=0}^{n}\begin{Bmatrix}2n\\2k\end{Bmatrix}_{U;m} = \frac{1}{2}\left\{\prod_{k=1}^{n}\left(V_{m(2k-1)}+2\right)+\prod_{k=1}^{n}\left(-V_{m(2k-1)}+2\right)\right\},$

(ii) $\displaystyle\sum_{k=1}^{n}\begin{Bmatrix}2n\\2k-1\end{Bmatrix}_{U;m} = \frac{1}{2}\left\{\prod_{k=1}^{n}\left(V_{m(2k-1)}+2\right)-\prod_{k=1}^{n}\left(-V_{m(2k-1)}+2\right)\right\}.$

Lemma 4. *For any positive integers n and m,*

$$\sum_{k=0}^{2n}\mathbf{i}^{k(2mn-mk\pm1)}q^{\frac{mk(k-2n)}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_{q^m} = \mathbf{i}^{n(mn\pm1)}q^{-\frac{mn^2}{2}}(-q^m;q^{2m})_n.$$

Proof: Consider the RHS of the claimed identity and by using (ii) in Lemma 1, we get

$$
\begin{aligned}
&\sum_{k=0}^{2n} \mathbf{i}^{k(2mn-mk\pm1)} q^{\frac{mk(k-2n)}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^m} \\
=\ & \sum_{k=0}^{2n} \mathbf{i}^{-mk^2} q^{m\binom{k+1}{2}} \left(\mathbf{i}^{2mn\pm1} q^{-\frac{m(2n+1)}{2}}\right)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^m} \\
=\ & \mathbf{i}^{-mn^2} \sum_{k=0}^{2n} q^{m\binom{k+1}{2}} \left(\mathbf{i}^{2mn\pm1} q^{-\frac{m(2n+1)}{2}}\right)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^m} \\
=\ & \mathbf{i}^{-mn^2} \prod_{k=1}^{2n} \left(1+\mathbf{i}^{2mn\pm1} q^{\frac{m(2k-2n-1)}{2}}\right) \\
=\ & \mathbf{i}^{-mn^2} \prod_{j=1}^{n} \left(1+\mathbf{i}^{2mn\pm1} q^{\frac{m(2j-1)}{2}}\right)\left(1+\mathbf{i}^{2mn\pm1} q^{-\frac{m(2j-1)}{2}}\right) \\
=\ & \mathbf{i}^{-mn^2} \prod_{j=1}^{n} \mathbf{i}^{2mn\pm1} q^{-\frac{m(2j-1)}{2}} (1+q^{m(2j-1)}) \\
=\ & \mathbf{i}^{mn^2\pm n} q^{-\frac{mn^2}{2}} (-q^m;q^{2m})_n.
\end{aligned}
$$

So we have the claimed identity. □

Theorem 3. *For any positive integers n and m,*

$$\sum_{k=0}^{2n} \mathbf{i}^{\pm k} \left\{ \begin{matrix} 2n \\ k \end{matrix} \right\}_{U;m} = \mathbf{i}^{\pm n} \prod_{k=1}^{n} V_{m(2k-1)}.$$

Using Theorem 3, the following corollary can be obtained:

Corollary 2. *For any positive integers n and m,*

(i) $\displaystyle\sum_{k=0}^{n} (-1)^k \left\{ \begin{matrix} 2n \\ 2k \end{matrix} \right\}_{U;m} = \cos\frac{n\pi}{2} \prod_{k=1}^{n} V_{m(2k-1)},$

(ii) $\displaystyle\sum_{k=0}^{n} (-1)^k \left\{ \begin{matrix} 2n \\ 2k-1 \end{matrix} \right\}_{U;m} = -\sin\frac{n\pi}{2} \prod_{k=1}^{n} V_{m(2k-1)}.$

Lemma 5. *For any positive integers n and m,*

$$
\begin{aligned}
&\sum_{k=0}^{2n+1} \mathbf{i}^{k(m(k-2n-1)\pm1)} q^{\frac{mk(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\
&\qquad = (1\pm\mathbf{i})\mathbf{i}^{\pm n} \begin{cases} (-1)^{\binom{n+1}{2}} q^{-m\binom{n+1}{2}} (-q^m;q^m)_n^2 & \textit{if } m \textit{ is odd,} \\ q^{-m\binom{n+1}{2}} (-q^{2m};q^{2m})_n & \textit{if } m \textit{ is even.} \end{cases}
\end{aligned}
$$

Proof: If m is odd, then we write

$$\begin{aligned} & \sum_{k=0}^{2n+1} \mathbf{i}^{mk(2n+1-k)\pm k} q^{\frac{mk(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\ = & \sum_{k=0}^{2n+1} \mathbf{i}^{k(2n+1-k)\pm k} q^{\frac{mk(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \end{aligned}$$

which, by (iii) in Lemma 1, gives us

$$\begin{aligned} = & \frac{1 \pm (-1)^n \mathbf{i}}{2} \sum_{k=0}^{2n+1} q^{m\binom{k+1}{2}} q^{-m(n+1)k} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\ = & \frac{1 \pm (-1)^n \mathbf{i}}{2} \prod_{k=1}^{2n+1} (1 + q^{m(k-n-1)}) \\ = & \frac{1 \pm (-1)^n \mathbf{i}}{2} 2 \prod_{j=1}^{n} (1 + q^{mj})(1 + q^{-mj}) \\ = & (1 \pm (-1)^n \mathbf{i}) \prod_{j=1}^{n} (2 + q^{mj} + q^{-mj}) \\ = & (1 \pm (-1)^n \mathbf{i}) \prod_{j=1}^{n} q^{-mj} (1 + q^{mj})^2 \\ = & (1 \pm (-1)^n \mathbf{i}) q^{-m\binom{n+1}{2}} (-q^m; q^m)_n^2 \\ = & (1 \pm \mathbf{i}) \mathbf{i}^{\pm n} (-q^{-m})^{\binom{n+1}{2}} (-q^m; q^m)_n^2. \end{aligned}$$

Thus the claim is seen for odd m.

If m is even, then we similarly obtain by (1.1),

$$\begin{aligned}
&\sum_{k=0}^{2n+1} \mathbf{i}^{mk(2n+1-k)\pm k} q^{\frac{mk(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\
= & \sum_{k=0}^{2n+1} \mathbf{i}^{\pm k} q^{\frac{mk(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\
= & \sum_{k=0}^{2n+1} q^{m\binom{k+1}{2}} \left(\mathbf{i}^{\pm 1} q^{-m(n+1)}\right)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\
= & \prod_{k=1}^{2n+1} (1+\mathbf{i}^{\pm 1} q^{m(k-n-1)}) = (1+\mathbf{i}^{\pm 1}) \prod_{j=1}^{n} (1+\mathbf{i}^{\pm 1} q^{mj})(1+\mathbf{i}^{\pm 1} q^{-mj}) \\
= & (1 \pm \mathbf{i}) \prod_{j=1}^{n} \mathbf{i}^{\pm 1} (q^{mj} + q^{-mj}) = (1 \pm \mathbf{i}) \mathbf{i}^{\pm n} \prod_{j=1}^{n} q^{-mj}(1+q^{2mj}) \\
= & (1 \pm \mathbf{i}) \mathbf{i}^{\pm n} q^{-m\binom{n+1}{2}} (-q^{2m}; q^{2m})_n.
\end{aligned}$$

∎

Theorem 4. *For any positive integers n and m,*

$$\sum_{k=0}^{2n+1} \mathbf{i}^{\pm k} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_{U;m} = (1 \pm \mathbf{i}) \mathbf{i}^{\pm n} \begin{cases} \prod_{k=1}^{n} V_{mk}^2 & \text{if } m \text{ is odd,} \\ \prod_{k=1}^{n} V_{2mk} & \text{if } m \text{ is even.} \end{cases}$$

Using Theorem 4, we have the following Corollary:

Corollary 3. *For any positive integers n and m,*

(i) $$\sum_{k=0}^{n} (-1)^k \begin{Bmatrix} 2n+1 \\ 2k \end{Bmatrix}_{U;m} = (-1)^{\binom{n+1}{2}} \begin{cases} \prod_{k=1}^{n} V_{mk}^2 & \text{if } m \text{ is odd,} \\ \prod_{k=1}^{n} V_{2mk} & \text{if } m \text{ is even.} \end{cases}$$

(ii) $$\sum_{k=0}^{n} (-1)^k \begin{Bmatrix} 2n+1 \\ 2k+1 \end{Bmatrix}_{U;m} = (-1)^{\binom{n}{2}} \begin{cases} \prod_{k=1}^{n} V_{mk}^2 & \text{if } m \text{ is odd,} \\ \prod_{k=1}^{n} V_{2mk} & \text{if } m \text{ is even.} \end{cases}$$

Lemma 6. *For positive integers n and m,*

$$\sum_{k=0}^{2n} (-1)^k q^{mk(k-2n)} \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^m}^2 = (-1)^n q^{-mn^2} \frac{(-q^{2m}; q^{2m})_n (q^{2m}; q^{4m})_n}{(q^{2m}; q^{2m})_n}. \qquad (2.1)$$

Proof: We will show how to evaluate and prove such sums entirely mechanically. We used Zeilberger's own version [8], which is a Mathematica program. After loading the package `qZeil.m` in Mathematica, define

$$\mathtt{SUM[n]} := \sum_{\mathtt{k=0}}^{\mathtt{2n}} (-1)^{\mathtt{k}} \mathtt{q}^{\mathtt{mk(k-2n)}} \begin{bmatrix} \mathtt{2n} \\ \mathtt{k} \end{bmatrix}_{\mathtt{q^m}}^2 . \tag{2.2}$$

If we run the RHS of (2.2) in the program, then we obtain following first order recurrence relation:

$$\mathtt{SUM[n]} = -\frac{\mathtt{q}^{\mathtt{m-2mn}}(1+\mathtt{q}^{\mathtt{2mn}})(1-\mathtt{q}^{\mathtt{2m(2n-1)}})}{1-\mathtt{q}^{\mathtt{2mn}}} \mathtt{SUM[n-1]}.$$

Clearly the right hand side of equation (2.1) satisfies the same recurrence relation (and the both sides of (2.1) have the same initial conditions). Also, if we solve this recurrence relation, we obtain

$$\mathtt{SUM[n]} = \frac{(-1)^{\mathtt{n-1}} \mathtt{q}^{\mathtt{m-mn^2}} (-\mathtt{q}^{\mathtt{3m}}; \mathtt{q}^{\mathtt{2m}})_{\mathtt{n-1}} (\mathtt{q}^{\mathtt{3m}}; \mathtt{q}^{\mathtt{2m}})_{\mathtt{n-1}} (-\mathtt{q}^{\mathtt{4m}}; \mathtt{q}^{\mathtt{2m}})_{\mathtt{n-1}}}{(\mathtt{q}^{\mathtt{4m}}; \mathtt{q}^{\mathtt{2m}})_{\mathtt{n-1}}} . \tag{2.3}$$

Thus, the right hand side of (2.1) and (2.3) are the same; it is no more necessary to distinguish the parity of m. Consequently, the proof is complete. □

Theorem 5. *For positive integers n and m,*

$$\sum_{k=0}^{2n} (-1)^{(m+1)k} \left\{ \begin{matrix} 2n \\ k \end{matrix} \right\}_{U;m}^2 = (-1)^{(m+1)n} \prod_{k=1}^{n} \frac{V_{2mk} U_{2m(2k-1)}}{U_{2mk}} .$$

References

[1] G. E. Andrews, The Theory of Partitions, Encyclopedia of mathematics and its applications, Vol. 2, Addison-Wesley, Reading, MA, 1976.

[2] V.E. Hoggatt Jr., Fibonacci numbers and generalized binomial coefficients, Fibonacci Quart. 5 (1967) 383–400.

[3] E. Kilic, The generalized Fibonomial matrix, European J. Combin. 31 (1) (2010) 193-209.

[4] E. Kilic and P. Stanica, Generating matrices of C-nomial coefficients and their spectra, Proc. International Conf. Fibonacci Numbers & Applic. 2010, accepted.

[5] M. Petkovsek, H.Wilf, and D.Zeilberger, "$A = B$", A K Peters, Wellesley, MA, 1996.

[6] H. Prodinger, Some applications of the q-Rice formula, Random Structures and Algorithms 19 (3-4) (2001) 552–557.

[7] A. Riese, A Mathematica q-Analogue of Zeilberger's Algorithm for Proving $q-$Hypergeometric Identities, Diploma Thesis, RISC, J. Kepler University, Linz, Austria, 1995.

[8] A. Riese, http://www.risc.uni-linz.ac.at/research/combinat/software/qZeil/index.php.

[9] P. Trojovsky, Pavel On some identities for the Fibonomial coefficients via generating function, Discrete Appl. Math. 155 (15) (2007) 2017–2024.

Received: 09.09.2010,
Accepted: 07.01.2012.

TOBB
University of Economics and Technology Mathematics Department
06560 Sögütözü
Ankara Turkey
E-mail: `ekilic@etu.edu.tr`

Kırıkkale University
Department of Mathematics
71450 Yahşihan Kırıkkale Turkey
E-mail: `ilkerakkus@kku.edu.tr`

Bunkyo University High School,
1191-7, Kami, Ageo-city,
Saitama Pref.,
362-0001, Japan
E-mail: `otsukahideyuki@gmail.com`