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On the Annihilation of local homology modules

by
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Abstract

Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of commutative Noetherian ring R and A an Artinian R-module. For a non-negative integer n, we show that

$$\sqcap_{p+q=n} \operatorname{Ann}(\operatorname{Tor}_p^R(R/\mathfrak{b}, H_q^{\mathfrak{a}}(A))) \subseteq \operatorname{Ann}(\operatorname{Tor}_n^R(R/\mathfrak{b}, A)).$$

As an immediate consequence, if $H_i^{\mathfrak{a}}(A)$ is Artinian for all $i < n$ then $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}(H_i^{\mathfrak{b}}(A)))$ for all $i < n$. Moreover, we prove that if $\mathfrak{a} = (x_1, \ldots, x_n)$ and $\mathfrak{c} = \cap_{t\geq 1} \cap_{i=0}^{n} \operatorname{Ann}(\operatorname{Tor}_i^R(R/\mathfrak{a}^t, A))$, then $\mathfrak{c}^k \subseteq \cap_{i=0}^{n-1} \operatorname{Ann}(H_i^{\mathfrak{a}}(A))$ where $k = \binom{n}{[\frac{n}{2}]}$.

Key Words: Annihilator of local homology modules, local homology modules.
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1 Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of R, and A is an Artinian R-module. In [2], Cuong and Nam defined the i-th local homology module of A with respect to $\mathfrak{a}$ by

$$H_i^{\mathfrak{a}}(A) = \varprojlim_t \operatorname{Tor}_i^R(R/\mathfrak{a}^t, A).$$

This definition is in some sense dual to Grothendieck's definition of local cohomology modules. It is well known that the 0-th local homology module of A with respect to $\mathfrak{a}$, $H_0^{\mathfrak{a}}(A)$, is always Artinian, simply because there exists an integer t such that $H_0^{\mathfrak{a}}(A) \cong A/\mathfrak{a}^t A$. But what about the following question: what is the largest integer n such that all the modules $H_i^{\mathfrak{a}}(A)$ are Artinian for all $i < n$? This question is dual to the question of which ideals annihilate the local cohomology modules, and the classical theorem on local cohomology modules is Faltings' Annihilator Theorem [4]. There are not many results concerning the finiteness of local homology modules. In this regard, see [3], [6] and [7].

In this paper, for each ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R with $\mathfrak{a} \subseteq \mathfrak{b}$, we show a relationship between the annihilators of the modules $\mathrm{Tor}_i^R(R/\mathfrak{b}, A)$ and $\mathrm{Tor}_i^R(R/\mathfrak{b}, H_j^{\mathfrak{a}}(A))$. This provides a new characterization of the concept of A-coregular sequence of an arbitrary ideal of R. Also, we prove that if n is a non-negative integer such that $H_i^{\mathfrak{a}}(A)$ is Artinian for all $i < n$, then $\mathfrak{a} \subseteq \mathrm{Rad}(\mathrm{Ann}(H_i^{\mathfrak{b}}(A)))$ for all $i < n$. Moreover, we show that if $\mathfrak{a} = (x_1, \ldots, x_n)$ and $\mathfrak{c} = \cap_{t\geq 1} \cap_{i=0}^{n} \mathrm{Ann}(\mathrm{Tor}_i^R(R/\mathfrak{a}^t, A))$, then $\mathfrak{c}^k \subseteq \cap_{i=0}^{n-1} \mathrm{Ann}(H_i^{\mathfrak{a}}(A))$ where $k = \binom{n}{[\frac{n}{2}]}$.

2 The results

The following theorem is dual of [5, Theorem 2.2].

Theorem 2.1. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R, and n a non-negative integer. Then* $\sqcap_{p+q=n} \mathrm{Ann}(\mathrm{Tor}_p^R(R/\mathfrak{b}, H_q^{\mathfrak{a}}(A))) \subseteq \mathrm{Ann}(\mathrm{Tor}_n^R(R/\mathfrak{b}, A))$.

Proof: Let us consider functors $F(-) = R/\mathfrak{b} \otimes_R -$ and $G(-) = H_0^{\mathfrak{a}}(-)$. The functor F is obviously right exact and a projective module P implies $H_0^{\mathfrak{a}}(P)$ is flat by [1, 1.4.7] or [11, 2.4]. Combining [9, Theorem 11.39] with [11, Theorem 1.1] yields a Grothendieck spectral sequence

$$E_{p,q}^2 := \mathrm{Tor}_p^R(R/\mathfrak{b}, H_q^{\mathfrak{a}}(A)) \underset{p}{\Longrightarrow} \mathrm{Tor}_{p+q}^R(R/\mathfrak{b}, A).$$

Thus, for each $n \geq 0$, there is a finite filtration of the module $H^n = \mathrm{Tor}_n^R(R/\mathfrak{a}, A)$

$$0 = \phi^{-1}H^n \subseteq \phi^0 H^n \subseteq \ldots \subseteq \phi^{n-1}H^n \subseteq \phi^n H^n = H^n$$

such that $E_{i,n-i}^{\infty} \cong \phi^i H^n/\phi^{i-1}H^n$ for all $0 \leq i \leq n$ (see [9, §11]). Since $E_{i,n-i}^{\infty}$ is a subquotient of $E_{i,n-i}^2$ for all $0 \leq i \leq n$, it implies that $\phi^i H^n/\phi^{i-1}H^n$ is annihilated by $\mathrm{Ann}(\mathrm{Tor}_i^R(R/\mathfrak{b}, H_{n-i}^{\mathfrak{a}}(A)))$ for all $0 \leq i \leq n$. Thus, we get that $\sqcap_{p+q=n} \mathrm{Ann}(\mathrm{Tor}_p^q(R/\mathfrak{b}, H_q^{\mathfrak{a}}(A))$ annihilates the homology module $\mathrm{Tor}_n^R(R/\mathfrak{b}, A)$. This completes the proof. □

A sequence of elements $x_1, \dots, x_n$ in R is said to be an A-coregular sequence (see [8, Definition 3.1]) if $0 :_A (x_1, \dots, x_n) \neq 0$ and
$0 :_A (x_1, \dots, x_{i-1}) \xrightarrow{x_i} 0 :_A (x_1, \dots, x_{i-1})$ is surjective for $i = 1, 2, \dots, n$. We denote by $\operatorname{width}(\mathfrak{a}, A)$ the supremum of the lengths of all maximal A-coregular sequences in the ideal $\mathfrak{a}$.

Corollary 2.2. *Let* $\mathfrak{a}$ *be an ideal of* R *such that* $0 :_A \mathfrak{a} \neq 0$. *Then* $\operatorname{width}(\mathfrak{a}, A) = \inf\{n : \operatorname{Tor}_i^R(R/\mathfrak{a}, H_j^{\mathfrak{a}}(A)) \neq 0$ *for some non-negative integers* i, j *with* $i+j = n\}$.

Proof: We denote by B the set in the above equality. In view of [3, Theorem 4.11] it follows that $\operatorname{width}(\mathfrak{a}, A) \leq \inf B$. On the other hand, by Theorem 2.1 and [8, Theorem 3.9] we have $\inf B \leq \operatorname{width}(\mathfrak{a}, A)$. This finishes the proof. □

Corollary 2.3. *Let* $\mathfrak{a} \subseteq \mathfrak{b}$ *be two ideals of* R. *Then, for each non-negative integer* n, $\sqcap_{i=0}^{n} \operatorname{Ann}(H_i^{\mathfrak{a}}(A)) \subseteq \cap_{t \geq 1} \cap_{i=0}^{n} \operatorname{Ann}(\operatorname{Tor}_i^R(R/\mathfrak{b}^t, A))$.

Proof: Let t be a positive integer. Then $\mathfrak{a}^t \subseteq \mathfrak{b}^t$. Hence, for each non-negative integer m, it follows from Theorem 2.1 and [2, Remark 2.1(ii)] that
$\sqcap_{i+j=m} \operatorname{Ann} \operatorname{Tor}_i^R(R/\mathfrak{b}^t, H_j^{\mathfrak{a}}(A)) \subseteq \operatorname{Ann}(\operatorname{Tor}_m^R(R/\mathfrak{b}^t, A))$. Since the Tor functors are linear, the result now follows. □

Theorem 2.4. *Let* $\mathfrak{a} \subseteq \mathfrak{b}$ *be two ideals of* R, *and* n *a non-negative integer such that* $H_i^{\mathfrak{a}}(A)$ *is Artinian for all* $i < n$. *Then* $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}(H_i^{\mathfrak{b}}(A)))$ *for all* $i < n$.

Proof: By [2, Proposition 4.7], there exists a positive integer m such that $\mathfrak{a}^m H_i^{\mathfrak{a}}(A) = 0$ for all $i < n$. Put $l := mn$. Then $\mathfrak{a}^l \subseteq \sqcap_{i=0}^{n-1} \operatorname{Ann}(H_i^{\mathfrak{a}}(A))$. Thus, by Corollary 2.3 $\mathfrak{a}^l \subseteq \cap_{i=0}^{n-1} \operatorname{Ann}(\operatorname{Tor}_i^R(R/\mathfrak{b}^t, A))$ for all $t \in \mathbb{N}$. Therefore $\mathfrak{a}^l H_i^{\mathfrak{b}}(A) = 0$ for all $i < n$ and so $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}(H_i^{\mathfrak{b}}(A)))$ for all $i < n$, as required. □

We prove the following theorem by similar techniques that used in [10, Theorem 3].

Theorem 2.5. *Let* n *be a non-negative integer such that* $\mathfrak{a} = (x_1, \dots, x_n)$. *If* $\mathfrak{c} = \cap_{t \geq 1} \cap_{i=0}^{n} \operatorname{Ann}(\operatorname{Tor}_i^R(R/\mathfrak{a}^t, A))$, *then* $\mathfrak{c}^k \subseteq \cap_{i=0}^{n-1} \operatorname{Ann}(H_i^{\mathfrak{a}}(A))$ *where* $k = \binom{n}{[\frac{n}{2}]}$.

Proof: First we show, for $k \leq n$, that $(\mathfrak{c})^{\binom{k}{i}} H_i(x_1^t, \ldots, x_k^t; A) = 0$, for $0 \leq i < k$ and $t \geq 1$. To this end we make an induction on k. If $k = 1$ and $i = 0$, then $H_0(x_1^t; A) \cong A/x_1^t A$ is annihilated by $\mathfrak{c}$. Assume $k \geq 2$. We show the statement by induction on i. For $i = 0$ we have the exact sequence

$$A/x_1^t A \longrightarrow H_0(x_1^t, \ldots, x_k^t; A) \longrightarrow 0$$

and the assertion is true. For $i \geq 1$ there is a short exact sequence

$$0 \longrightarrow H_i(x_1^t, \ldots, x_{k-1}^t; A)/x_k^t H_i(x_1^t, \ldots, x_{k-1}^t; A) \longrightarrow H_i(x_1, \ldots, x_k^t; A)$$

$$\longrightarrow (0 :_{H_{i-1}(x_1^t, \ldots, x_{k-1}^t; A)} x_k^t) \longrightarrow 0,$$

$t \geq 1$. If $i < k - 1$, the induction hypothesis yields the statement. In the case $i = k - 1$ we get

$$H_i(x_1^t, \ldots, x_{k-1}^t; A)/x_k^t H_i(x_1^t, \ldots, x_{k-1}^t; A) \cong R/(x_1^t, \ldots, x_k^t) \otimes (0 :_A (x_1^t, \ldots, x_{k-1}^t)).$$

Hence the short exact sequence proves the statement on the annihilation. In particular, we have $(\mathfrak{c})^{\binom{k}{i}} H_i(x_1^t, \ldots, x_k^t; A) = 0$ $(0 \leq i < n,\ t \geq 1)$. By [2, Theorem 3.6], we have $H_i^{\mathfrak{a}}(A) = \varprojlim_t H_i(x_1^t, \ldots, x_n^t; A)$. Note that $(\mathfrak{c})^{\binom{n}{i}} \varprojlim_t H_i(x_1^t, \ldots, x_n^t; A) \subseteq \varprojlim_t (\mathfrak{c})^{\binom{n}{i}} H_i(x_1^t, \ldots, x_n^t; A) = 0$. Therefore it follows $(\mathfrak{c})^{\binom{n}{i}} H_i^{\mathfrak{a}}(A) = 0$, which proves the statement. □

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References

[1] J. Bartijn, *Flatness, completions, regular sequences, un ménage à trois.* Thesis, Utrecht, (1985).

[2] N.T. Cuong and T. T. Nam, The I-adic completion and local homology for Artinian modules, Math. Proc. Camb. Phil. Soc., **131**(2001), 61-72.

[3] N.T. Cuong and T.T. Nam, A local homology theory for lineary compact modules, J. Algebra, **319** (2008), 4712-4737.

[4] G. Faltings, Uber die annulatoren lokaler kohomologiegruppen, Arch. Math., **30**(5)(1978), 473-476.

[5] K. Khashyarmanesh, On the annihilators of local cohomology modules, Comm. Algebra, **37**(2009), 1787-1792.

[6] A. Mafi and H. Saremi, Coassociated primes of local homology and local cohomology modules, Rocky Mountain J. Math., **41**(5)(2011), 1631-1638.

[7] A. Mafi and H. Saremi, On the finiteness of local homology modules, Rend. Semin. Mat. Univ. Politec. Torino, **67**(2009), 115-122.

[8] A. Ooishi, Matlis duality and the width of a module, Hiroshima Math. J., **6** (1976), 573-587.

[9] J. Rotman, Introduction to homological algebra, (Academic Press, 1979).

[10] P. Schenzel, Cohomological annihilators, Math. Proc. Camb. Phil. Soc., **91**(1982), 345-350.

[11] A. M. Simon, Some homological properties of complete modules, Math. Proc. Camb. Phil. Soc., **108**(1990), 231-246.

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