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Some splitting criteria on Hirzebruch surfaces

by
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Abstract

We prove some criteria for splitting of rank-two vector bundles on Hirzebruch surfaces. As a main tool we use Beilinson's type spectral sequences.

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1 Introduction

The problem of the splitting of a vector bundle is not new and is far away from being completed. In the case of the projective space, there are known the results of Evans and Griffith or Horrocks, which give us necessary and sufficient conditions for an arbitrary vector bundle on $\mathbb{P}^n$ to be split. For example, Horrocks states in [13] that a vector bundle on a projective space decomposes into a direct sum of line bundles if, and only if, all of its twists have no intermediate cohomology. Related to the same topic, Ottaviani proved in [15] and [16] that a vector bundle on the hyperquadric $Q_n \subset \mathbb{P}^{n+1}$ (or on the Grassmannian $\mathrm{Gr}(k,n)$) is a direct sum of line bundles if it has no intermediate cohomology and satisfies other cohomological conditions involving Spinor bundles (or the tautological k-dimensional bundle respectively). Costa and Miró-Roig showed in [10] that Horrocks' criterion can be extended to vector bundles on multi-projective spaces and on smooth projective varieties with extra properties.

One idea that has gained momentum over the years in the field is to use Beilinson type spectral sequences. Originally developed on projective spaces (see [14]), it has since been generalized first to Hirzebruch surfaces ([9]), and then to arbitrary scrolls ([3]). The idea has also proved useful to the study of various moduli space problems (see [14] or [11]).

In this note, we use Buchdal's Beilinson type spectral sequence ([9]) to prove that a vector bundle on a Hirzebruch surface splits if, and only if, it fulfills some cohomological conditions. In section 2, we briefly review its construction following [9] (see also [3]).

We mention that general results on vector bundles can be found in [7] or [12]. We also notice that vector bundles on Hirzebruch surfaces, and more generally on ruled surfaces, have been the subject of intensive studies (see, for example, [1], [2], [5], [6], [8]).

Let $X = \Sigma_e \xrightarrow{\pi} \mathbb{P}^1$ be a Hirzebruch surface, $e \geq 0$. Denote generators of the Picard group of X by $C_0 = \mathcal{O}_X(1)$ for the negative section ($C_0^2 = -e$), and by $F = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ for a fiber of the ruling. The main result, proved in section 3, can be stated as follows:

Theorem. *Let $X = \Sigma_e$ be a Hirzebruch surface and $\mathcal{M}$ a rank-two vector bundle on X. Then*

(i) $\mathcal{M} \cong \mathcal{O}_X \oplus \mathcal{O}_X$ *if, and only if,* $c_1(\mathcal{M}) = 0, c_2(\mathcal{M}) = 0$ *and* $h^0(\mathcal{M}(-C_0)) = h^0(\mathcal{M}(-F)) = h^1(\mathcal{M}) = 0$.

(ii) $\mathcal{M} \cong \mathcal{O}_X(-F) \oplus \mathcal{O}_X(-C_0 - eF)$ *if, and only if,* $c_1(\mathcal{M}) = -C_0 - (e+1)F, c_2(\mathcal{M}) = 1$ *and* $h^0(\mathcal{M}) = 0$.

(iii) $\mathcal{M} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-F)$ *if, and only if,* $c_1(\mathcal{M}) = -F, c_2(\mathcal{M}) = 0$ *and* $h^0(\mathcal{M}(-C_0)) = h^0(\mathcal{M}(-F)) = h^1(\mathcal{M}) = 0$.

(iv) $\mathcal{M} \cong \mathcal{O}_X(-F) \oplus \mathcal{O}_X(-C_0 - (e+1)F)$ *if, and only if,* $c_1(\mathcal{M}) = -C_0 - (e+2)F, c_2(\mathcal{M}) = 1$ *and* $h^0(\mathcal{M}) = h^1(\mathcal{M}(-C_0 - F)) = h^2(\mathcal{M}(-F)) = 0$.

(v) $\mathcal{M} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-C_0 - (e+1)F)$ *if, and only if,* $c_1(\mathcal{M}) = -C_0 - (e+1)F, c_2(\mathcal{M}) = 0$ *and* $h^0(\mathcal{M}(-C_0)) = h^0(\mathcal{M}(-F)) = h^1(\mathcal{M}) = 0$.

We mention that the cases above are the only ones that can be obtained by analyzing the first page of the Beilinson spectral sequence. In order to prove other splitting criteria, one needs a finer analysis involving differentials of the spectral sequence.

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2 Preliminaries

Throughout, $X = \Sigma_e$ denotes the Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, where $\mathcal{E}$ is the rank two vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$. The canonical divisor is

$$K =_{\text{def}} K_X = -2C_0 - (e+2)F,$$

where C_0 is the class of the negative section satisfying $C_0^2 = -e$, and where F is the class of a fiber of π. The classes C_0 and F span freely $\text{Pic}(X)$ and we also have the intersection relations $F^2 = 0$ and $C_0 \cdot F = 1$.

Let $\Delta \subset X \times X$ be the diagonal. As we see in [9], Δ can be described scheme-theoretically as the zero locus of a global section in a rank-two vector bundle over $X \times X$. Such X is said to satisfy the *diagonal property* (see [17]). As the foundation of the Beilinson spectral sequence, this description can be achieved in two steps. We denote by $p_1, p_2 : X \times X \to X$ the two projections and let $Y = X \times_{\mathbb{P}^1} X \subset X \times X$. The first step is to consider the embedding $\Delta \subset Y$ and observe (see [9], also [3]) that

$$\mathcal{O}_Y(\Delta) = \left(p_1^* T_{X|\mathbb{P}^1}(-C_0) \otimes p_2^* \mathcal{O}_X(C_0)\right)|_Y.$$

Recall that $T_{X|\mathbb{P}^1}(-C_0) = \mathcal{O}_X(C_0 + eF)$ and $p_1^*\mathcal{O}_X(F)|_Y \cong p_2^*\mathcal{O}_X(F)|_Y$. For the second step, we use an extension lemma and proceed from the fibered product to the usual product. To be more precise, there exists a rank-two bundle G on $X \times X$, given by a non-trivial extension

$$0 \to p_1^* T_{X|\mathbb{P}^1}(-C_0) \otimes p_2^* \mathcal{O}_X(C_0) \to G \to \mathcal{O}_{X \times X}(Y) \to 0 \tag{1}$$

and a global section of G whose zero-scheme coincides with $\Delta \subset X \times X$ (see, for example, [3]). In particular, we get a truncated Koszul complex

$$0 \to \wedge^2 G^* \to G^* \to \mathcal{O}_{X \times X}. \tag{2}$$

If $\mathcal{M}$ is a vector bundle of arbitrary rank on X, then we can twist the complex (2) by $p_2^*(\mathcal{M})$ and take the hypercohomology. We obtain a spectral sequence abutting to $\mathcal{M}$ (see [9]):

$$E_1^{p,q} = R^q p_{1*}(\wedge^{-p} G^* \otimes p_2^*(\mathcal{M})) \Rightarrow \begin{cases} \mathcal{M} & \text{if } p+q=0 \\ 0 & \text{otherwise.} \end{cases}$$

We note that $E_1^{p,q} = 0$ if $p \notin \{-2,-1,0\}$ or if $q \notin \{0,1,2\}$ and the remaining terms of the spectral sequence are computed by twisting the dual of the extension (1) by $p_2^*(\mathcal{M})$ and applying p_{1*} (see [9], also [3]). For any q we have:

$$E_1^{0,q} \cong H^q(X,\mathcal{M}) \otimes \mathcal{O}_X, \; E_1^{-2,q} \cong H^q(X, \mathcal{M}(-C_0 - F)) \otimes \mathcal{O}_X(-C_0 - (e+1)F)), \tag{3}$$

and $E_1^{-1,q}$ can be determined from the exact sequence

$$H^q(X,\mathcal{M}(-F)) \otimes \mathcal{O}_X(-F) \to E_1^{-1,q} \to H^q(X,\mathcal{M}(-C_0)) \otimes \mathcal{O}_X(-C_0 - eF). \tag{4}$$

Morally, if we use the Beilinson spectral sequence, we should completely determine a vector bundle when we know the cohomology of suitable twists and vector bundle morphisms (the differentials of the spectral sequence). To illustrate this principle we'll see in the next section that in the rank-two case, with appropriate conditions, we can find the shape of the first sheet of the spectral sequence, and consequently, the vector bundle itself.

3 Splitting criteria

In this section we find some splitting criteria for a rank-two vector bundle $\mathcal{M}$ on a Hirzebruch surface X. We use the Beilinson spectral sequences and get equivalent conditions for $\mathcal{M}$ to be split in five different cases. We mention that one of this situations was treated in another context in [4]. We start with a useful lemma:

Lemma 1. *Let X a Hirzebruch surface and $a, b \in \mathbb{Z}$. Then*

$$H^1(\mathcal{O}_X(aC_0 + bF)) \cong \begin{cases} H^0(\mathbb{P}^1, \bigoplus_{k=1}^{-a-1} \mathcal{O}_{\mathbb{P}^1}(ke+b)), & \text{if } a \leq -2 \\ 0, & \text{if } a = -1 \\ H^0(\mathbb{P}^1, \bigoplus_{k=0}^{a} \mathcal{O}_{\mathbb{P}^1}(ke-b-2)), & \text{if } a \geq 0 \end{cases}.$$

Remark 1. The expression of $h^1(X, \mathcal{F})$ for some vector bundle $\mathcal{F}$ might be involved in problems such as the splitting of an extension or the computation of some particular variety dimension (see, for example, [5], [6]).

Proof: Consider the Leray spectral sequence for π and $\mathcal{F} = \mathcal{O}_X(aC_0 + bF)$:

$$E_2^{p,\,q} = H^p(\mathbb{P}^1, R^q\pi_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Hence, we have

$$\begin{aligned} E_2^{p,\,q} &= E_\infty^{p,\,q}, \text{ for } p+q=1, \\ E_\infty^{0,\,1} &= Gr^1(H^1(X,\mathcal{F})) = H^0(\mathbb{P}^1, R^1\pi_*\mathcal{F}), \\ E_\infty^{1,\,0} &= Gr^0(H^1(X,\mathcal{F})) = H^1(\mathbb{P}^1, \pi_*\mathcal{F}). \end{aligned}$$

We obtain the exact sequence

$$0 \to H^0(\mathbb{P}^1, R^1\pi_*\mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(\mathbb{P}^1, \pi_*\mathcal{F}) \to 0.$$

But

$$R^1\pi_*\mathcal{F} = R^1\pi_*\mathcal{O}_X(aC_0) \otimes \mathcal{O}_{\mathbb{P}^1}(b),$$

and since

$$R^1\pi_*\mathcal{O}_X(aC_0) = (\pi_*\mathcal{O}_X((-a-2)C_0))^* \otimes \det(\mathcal{E}),$$

we find

$$R^1\pi_*\mathcal{F} = (\pi_*\mathcal{O}_X((-a-2)C_0))^* \otimes \mathcal{O}_{\mathbb{P}^1}(e+b),$$

and

$$\pi_*\mathcal{F} = \pi_*\mathcal{O}_X(aC_0) \otimes \mathcal{O}_{\mathbb{P}^1}(b).$$

On the other hand,

$$\pi_*\mathcal{O}_X(\alpha C_0) = \begin{cases} \mathcal{S}^\alpha(\mathcal{E}), & \text{if } \alpha \geq 0 \\ 0, & \text{if } \alpha < 0 \end{cases}, \tag{5}$$

where $\mathcal{S}^\alpha(\mathcal{E}) = \mathcal{S}^\alpha(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) = \bigoplus_{k=0}^{\alpha} \mathcal{O}_{\mathbb{P}^1}(-ke)$.

Note that if $a \leq -2$, then $\pi_*\mathcal{F} = 0$, and we get

$$H^1(X, \mathcal{F}) \cong H^0(\mathbb{P}^1, (\pi_*\mathcal{O}_X((-a-2)C_0))^* \otimes \mathcal{O}_{\mathbb{P}^1}(e+b)).$$

For $\alpha = -a-2$ in (5) we find

$$\pi_*\mathcal{O}_X((-a-2)C_0) = \bigoplus_{k=0}^{-a-2} \mathcal{O}_{\mathbb{P}^1}(-ke),$$

and then

$$(\pi_*\mathcal{O}_X((-a-2)C_0))^* \otimes \mathcal{O}_{\mathbb{P}^1}(e+b) = \bigoplus_{k=0}^{-a-2} \mathcal{O}_{\mathbb{P}^1}((k+1)e+b).$$

So, if $a \leq -2$, then

$$H^1(X, \mathcal{F}) \cong H^0(\mathbb{P}^1, \bigoplus_{k=1}^{-a-1} \mathcal{O}_{\mathbb{P}^1}(ke+b)).$$

If $a \geq 0$, we get from (5) that $R^1\pi_*\mathcal{F} = 0$, and

$$H^1(X, \mathcal{F}) \cong H^1(\mathbb{P}^1, \pi_*\mathcal{F}) \cong H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1} \otimes (\pi_*\mathcal{F})^*).$$

From (5) for $\alpha = a$ we find that

$$\pi_*\mathcal{F} = \bigoplus_{k=0}^{a} \mathcal{O}_{\mathbb{P}^1}(-ke+b),$$

and

$$\omega_{\mathbb{P}^1} \otimes (\pi_*\mathcal{F})^* = \bigoplus_{k=0}^{a} \mathcal{O}_{\mathbb{P}^1}(ke-b-2).$$

This completes the case $a \geq 0$.

If $a = -1$ then, by (5), we get $\pi_*\mathcal{F} = 0$ and $R^1\pi_*\mathcal{F} = 0$, so $H^1(X, \mathcal{F}) = 0$. □

Before we state the main result of this paper, we mention that the cohomology on $X = \Sigma_e$ will be noted without specify the space X. Namely, $H^0(\mathcal{M})$ means $H^0(X, \mathcal{M})$, otherwise the space will be specified.

Theorem 1. *Let X be a Hirzebruch surface and $\mathcal{M}$ a rank-two vector bundle on X. Then* *(i)* $\mathcal{M} \cong \mathcal{O}_X \oplus \mathcal{O}_X$ *if and only if* $c_1(\mathcal{M}) = 0, c_2(\mathcal{M}) = 0$ *and* $h^0(\mathcal{M}(-C_0)) = h^0(\mathcal{M}(-F)) = h^1(\mathcal{M}) = 0$.

(ii) $\mathcal{M} \cong \mathcal{O}_X(-F) \oplus \mathcal{O}_X(-C_0 - eF)$ *if and only if* $c_1(\mathcal{M}) = -C_0 - (e+1)F, c_2(\mathcal{M}) = 1$ *and* $h^0(\mathcal{M}) = 0$.

(iii) $\mathcal{M} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-F)$ *if and only if* $c_1(\mathcal{M}) = -F, c_2(\mathcal{M}) = 0$ *and* $h^0(\mathcal{M}(-C_0)) = h^0(\mathcal{M}(-F)) = h^1(\mathcal{M}) = 0$.

(iv) $\mathcal{M} \cong \mathcal{O}_X(-F) \oplus \mathcal{O}_X(-C_0 - (e+1)F)$ *if and only if* $c_1(\mathcal{M}) = -C_0 - (e+2)F, c_2(\mathcal{M}) = 1$ *and* $h^0(\mathcal{M}) = h^1(\mathcal{M}(-C_0 - F)) = h^2(\mathcal{M}(-F)) = 0$.

(v) $\mathcal{M} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-C_0 - (e+1)F)$ *if and only if* $c_1(\mathcal{M}) = -C_0 - (e+1)F, c_2(\mathcal{M}) = 0$ *and* $h^0(\mathcal{M}(-C_0)) = h^0(\mathcal{M}(-F)) = h^1(\mathcal{M}) = 0$.

Proof: **(i)** Two different solutions (one in arbitrary rank) can be found in [4].
(ii) It is clear that $\mathcal{M} \cong \mathcal{O}_X(-F) \oplus \mathcal{O}_X(-C_0 - eF)$ satisfies the vanishing condition.
Conversely, we suppose that the bundle $\mathcal{M}$ has $c_1(\mathcal{M}) = -C_0 - (e+1)F, c_2(\mathcal{M}) = 1$ and $h^0(\mathcal{M}) = 0$ and we prove that $\mathcal{M} \cong \mathcal{O}_X(-F) \oplus \mathcal{O}_X(-C_0 - eF)$. First of all, we show that

$$\begin{array}{l} h^1(\mathcal{M}) = h^2(\mathcal{M}) = 0, \\ h^0(\mathcal{M}(-C_0)) = h^2(\mathcal{M}(-C_0)) = 0, h^1(\mathcal{M}(-C_0)) = 1, \\ h^0(\mathcal{M}(-F)) = h^2(\mathcal{M}(-F)) = 0, h^1(\mathcal{M}(-F)) = 1, \\ h^0(\mathcal{M}(-C_0 - F)) = h^1(\mathcal{M}(-C_0 - F)) = h^2(\mathcal{M}(-C_0 - F)) = 0. \end{array} \tag{6}$$

To see this, we compute the Chern classes for the bundles $\mathcal{M}(-C_0), \mathcal{M}(-F)$ and $\mathcal{M}(-C_0 - F)$. We use that $c_1(\mathcal{M}) = -C_0 - (e+1)F, c_2(\mathcal{M}) = 1$ and get

$$\begin{array}{lclllcr} c_1(\mathcal{M}(-C_0)) & = & -3C_0 - (e+1)F, & c_2(\mathcal{M}(-C_0)) & = & 2-e, \\ c_1(\mathcal{M}(-F)) & = & -C_0 - (e+3)F, & c_2(\mathcal{M}(-F)) & = & 2, \\ c_1(\mathcal{M}(-C_0 - F)) & = & -3C_0 - (e+3)F, & c_2(\mathcal{M}(-C_0 - F)) & = & 5-e. \end{array}$$

Applying the Riemann-Roch theorem we find that

$$\begin{array}{ll} \chi(\mathcal{M}) = 0, & \chi(\mathcal{M}(-C_0)) = -1, \\ \chi(\mathcal{M}(-C_0 - F)) = 0, & \chi(\mathcal{M}(-F) = -1. \end{array} \tag{7}$$

Since $h^0(\mathcal{M}) = 0$ and the divisor $D \in \{C_0, F, C_0 + F\}$ is effective,

$$h^0(\mathcal{M}(-C_0)) = h^0(\mathcal{M}(-F)) = h^0(\mathcal{M}(-C_0 - F)) = 0. \tag{8}$$

Since $c_1(\mathcal{M}) = -C_0 - (e+1)F$, it follows that $\det(\mathcal{M}) = \mathcal{O}_X(-C_0 - (e+1)F)$, hence $\mathcal{M}^* \cong \mathcal{M}(C_0 + (e+1)F)$. Then, by Serre Duality, we get

$$\begin{array}{l} h^2(\mathcal{M}) = h^0(\mathcal{M}^*(K)) = h^0(\mathcal{M}(-C_0 - F)) = 0, \\ h^2(\mathcal{M}(-C_0)) = h^0(\mathcal{M}^*(K + C_0)) = h^0(\mathcal{M}(-F)) = 0, \\ h^2(\mathcal{M}(-F)) = h^0(\mathcal{M}^*(K + F)) = h^0(\mathcal{M}(-C_0)) = 0, \\ h^2(\mathcal{M}(-C_0 - F)) = h^0(\mathcal{M}^*(K + C_0 + F)) = h^0(\mathcal{M}) = 0. \end{array} \tag{9}$$

The hypotheses, combined with (7), (8) and (9), imply the other conditions in (6).

Next, we compute $E_1^{p,q}$ for the bundle $\mathcal{M}$, with $p \in \{-2,-1,0\}$ and $q \in \{0,1,2\}$. From (3) we get

$$\begin{aligned} E_1^{0,q} &\cong H^q(\mathcal{M}) \otimes \mathcal{O}_X = 0, \\ E_1^{-2,q} &\cong H^q(\mathcal{M}(-C_0-F)) \otimes \mathcal{O}_X(-C_0-(e+1)F) = 0, \end{aligned}$$

for all $q \in \{0,1,2\}$, and the long exact sequence (4) gives us

$$\begin{aligned} 0 &\to H^0(\mathcal{M}(-F)) \otimes \mathcal{O}_X(-F) \to E_1^{-1,0} \to H^0(\mathcal{M}(-C_0)) \otimes \mathcal{O}_X(-C_0-eF) \to \\ &\to H^1(\mathcal{M}(-F)) \otimes \mathcal{O}_X(-F) \to E_1^{-1,1} \to H^1(\mathcal{M}(-C_0)) \otimes \mathcal{O}_X(-C_0-eF) \to \\ &\to H^2(\mathcal{M}(-F)) \otimes \mathcal{O}_X(-F) \to E_1^{-1,2} \to H^2(\mathcal{M}(-C_0)) \otimes \mathcal{O}_X(-C_0-eF) \to 0. \end{aligned}$$

Since we have $h^0(\mathcal{M}(-C_0)) = h^0(\mathcal{M}(-F)) = h^2(\mathcal{M}(-C_0)) = h^2(\mathcal{M}(-F)) = 0$ and $h^1(\mathcal{M}(-C_0)) = h^1(\mathcal{M}(-F)) = 1$ (from (6)), then we get $E_1^{-1,0} = E_1^{-1,2} = 0$ and the short exact sequence

$$0 \to \mathcal{O}_X(-F) \to E_1^{-1,1} \to \mathcal{O}_X(-C_0-eF) \to 0. \tag{10}$$

But the extensions of this type are parameterized by

$$\mathrm{Ext}^1(\mathcal{O}_X(-C_0-eF), \mathcal{O}_X(-F)) \cong H^1(\mathcal{O}_X(C_0+(e-1)F)).$$

Applying lemma 1 we have

$$H^1(\mathcal{O}_X(C_0+(e-1)F)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(-e-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = 0,$$

so, the exact sequence (10) splits, hence

$$E_1^{-1,1} \cong \mathcal{O}_X(-F) \oplus \mathcal{O}_X(-C_0-eF).$$

By what we know so far, the first page of the spectral sequence has the following shape:

0	0	0
0	$E_1^{-1,1}$	0
0	0	0

.

Since $E_\infty^{p,q} \Rightarrow \mathcal{M}$ for $p+q=0$ and $E_\infty^{p,q} = E_1^{p,q}$, it follows that

$$\mathcal{M} \cong E_1^{-1,1} \cong \mathcal{O}_X(-F) \oplus \mathcal{O}_X(-C_0-eF).$$

(iii) As in the previous case, we are more interested in the converse. First of all, we compute and find $\chi(\mathcal{M}) = 1, \chi(\mathcal{M}(-C_0)) = 0, \chi(\mathcal{M}(-F) = -1, \chi(\mathcal{M}(-C_0 - F)) = 0$. Then, by hypothesis, by Serre Duality, and from $\mathcal{M}^* \cong \mathcal{M}(F)$, we get

$$\begin{aligned} &h^0(\mathcal{M}) = 1, h^2(\mathcal{M}) = 0, \\ &h^1(\mathcal{M}(-C_0)) = h^2(\mathcal{M}(-C_0)) = 0, \\ &h^1(\mathcal{M}(-F)) = 1, h^2(\mathcal{M}(-F)) = 0, \\ &h^0(\mathcal{M}(-C_0-F)) = h^1(\mathcal{M}(-C_0-F)) = h^2(\mathcal{M}(-C_0-F)) = 0. \end{aligned}$$

It follows that $E_1^{p,q} = 0$, except $E_1^{0,0} \cong \mathcal{O}_X$ and $E_1^{-1,1} \cong \mathcal{O}_X(-F)$. The first page of the spectral sequence has the following shape:

0	0	0
0	$E_1^{-1,1}$	0
0	0	$E_1^{0,0}$

.

This means that we have the extension

$$0 \to \mathcal{O}_X \to \mathcal{M} \to \mathcal{O}_X(-F) \to 0$$

that arises from the Beilinson spectral sequence and which is split because

$$\mathrm{Ext}^1(\mathcal{O}_X(-F), \mathcal{O}_X) \cong H^1(\mathcal{O}_X(F)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(-3)) = 0$$

(last isomorphism follows by lemma 1). Hence $\mathcal{M} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-F)$.
(iv) In a similar manner we find $\chi(\mathcal{M}) = 0, \chi(\mathcal{M}(-C_0)) = 0, \chi(\mathcal{M}(-F) = -1$, $\chi(\mathcal{M}(-C_0 - F)) = 1$. Notice that $\mathcal{M}^* \cong \mathcal{M}(C_0 + (e+2)F)$ hence, by hypothesis and by Serre Duality, we find that

$$\begin{aligned}
&h^1(\mathcal{M}) = h^2(\mathcal{M}) = 0,\\
&h^0(\mathcal{M}(-C_0)) = h^1(\mathcal{M}(-C_0)) = h^2(\mathcal{M}(-C_0)) = 0,\\
&h^0(\mathcal{M}(-F)) = 0, h^1(\mathcal{M}(-F)) = 1,\\
&h^0(\mathcal{M}(-C_0 - F)) = 0, h^2(\mathcal{M}(-C_0 - F)) = 1.
\end{aligned}$$

Therefore $E_1^{p,q} = 0$, except $E_1^{-2,2} \cong \mathcal{O}_X(-C_0 - (e+1)F)$ and $E_1^{-1,1} \cong \mathcal{O}_X(-F)$. The first page of the spectral sequence has the following shape:

$E_1^{-2,2}$	0	0
0	$E_1^{-1,1}$	0
0	0	0

.

For this reason, the Beilinson spectral sequence gives rise to the extension

$$0 \to \mathcal{O}_X(-F) \to \mathcal{M} \to \mathcal{O}_X(-C_0 - (e+1)F) \to 0,$$

which is split because

$$\mathrm{Ext}^1(\mathcal{O}_X(-C_0 - (e+1)F), \mathcal{O}_X(-F)) \cong H^1(\mathcal{O}_X(C_0 + eF))$$

and

$$H^1(\mathcal{O}_X(C_0 + eF)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(-e-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$$

(by lemma 1). Hence $\mathcal{M} \cong \mathcal{O}_X(-F) \oplus \mathcal{O}_X(-C_0 - (e+1)F)$.
(v) In the last case we may follow the same path. We find $\chi(\mathcal{M}) = 1, \chi(\mathcal{M}(-C_0)) = 0, \chi(\mathcal{M}(-F) = 0, \chi(\mathcal{M}(-C_0 - F)) = 1$. If we note that $\mathcal{M}^* \cong \mathcal{M}(C_0 + (e+1)F)$ then, by hypothesis, Serre Duality we obtain that

$$\begin{aligned}
&h^0(\mathcal{M}) = 1, h^2(\mathcal{M}) = 0,\\
&h^1(\mathcal{M}(-C_0)) = h^2(\mathcal{M}(-C_0)) = 0,\\
&h^1(\mathcal{M}(-F)) = h^2(\mathcal{M}(-F)) = 0,\\
&h^0(\mathcal{M}(-C_0 - F)) = h^1(\mathcal{M}(-C_0 - F)) = 0, h^2(\mathcal{M}(-C_0 - F)) = 1.
\end{aligned}$$

It follows that $E_1^{p,q} = 0$, except $E_1^{0,0} \cong \mathcal{O}_X$ and $E_1^{-2,2} \cong \mathcal{O}_X(-C_0 - (e+1)F)$. The first page of the spectral sequence has the following shape:

$E_1^{-2,2}$	0	0
0	0	0
0	0	$E_1^{0,0}$

.

Consequently, we have the extension

$$0 \to \mathcal{O}_X \to \mathcal{M} \to \mathcal{O}_X(-C_0 - (e+1)F) \to 0$$

that arises from the Beilinson spectral sequence and which is split because

$$\mathrm{Ext}^1(\mathcal{O}_X(-C_0 - (e+1)F), \mathcal{O}_X) \cong H^1(\mathcal{O}_X(C_0 + (e+1)F))$$

and

$$H^1(\mathcal{O}_X(C_0 + (e+1)F)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(-e-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)) = 0$$

(by lemma 1). Hence $\mathcal{M} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-C_0 - (e+1)F)$. □

Remark 2. From the proof of the theorem, we see that these are the only cases that can occur at the first level of the Beilinson spectral sequence. Therefore, if we want other splitting criteria, we have to involve the second page of the spectral sequence, which means that we need more information about the differentials appearing at the first level.

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