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Primal, completely irreducible, and primary meet decompositions in modules

by
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Abstract

This paper was inspired by the work of Fuchs, Heinzer, and Olberding concerning primal and completely irreducible ideals. It is proved that if R is a commutative Noetherian ring then every primal submodule of an R-module M is a primary submodule of M if and only if for all prime ideals $\mathfrak{p} \subset \mathfrak{q}$ of R, every $\mathfrak{p}$-primary submodule of M is contained in every $\mathfrak{q}$-primary submodule of M. Moreover, for a commutative Noetherian ring R, every primal ideal of R is primary if and only if R is a finite direct product of Artinian rings and one-dimensional domains. Given a general ring R, a right R-module M has the property that every submodule contains a completely coirreducible submodule if and only if the Jacobson radical of any non-zero submodule N of M is zero and an irredundant intersection of maximal submodules of N. The paper closes with seven open problems.

Key Words: Primal submodule, irreducible submodule, completely irreducible submodule, completely coirreducible module, primary submodule, primary decomposition, completely irreducible decomposition, Noetherian ring.
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Introduction

This paper is a continuation of Albu and Smith [2] which was inspired by Fuchs, Heinzer, and Olberding [6], [7]. In Section 1 we consider primary submodules of a module over a commutative ring and their relation (if any) to completely

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irreducible submodules. We analyze in Section 2 the connections between primal submodules and primary submodules of a module over a commutative ring. We show that for any module M, every primary submodule is also primal. Then we study when the converse holds and characterize those modules M over a Noetherian ring R for which every primal submodule is primary (Theorem 2.7). In particular, a commutative Noetherian ring R has the property that every primal ideal is primary if and only if R is a finite direct product of Artinian rings and one-dimensional domains (Theorem 2.9). In Section 3 we examine irredundant decompositions of a submodule of a module over an arbitrary ring as an intersection of irreducible, completely irreducible, or primal submodules. Similar to the characterization, due to Fort [4], of modules rich in coirreducibles via irredundant irreducible decompositions, we characterize modules rich in completely coirreducibles via irredundant completely irreducible decompositions. The final section contains a list of seven questions.

0 Preliminaries

We first present the basic terminology and notation that will be used in this paper and then briefly consider completely irreducible submodules of a general module.

Throughout, R will denote an associative ring with non-zero identity element and all modules considered will be unital right modules over R. The notation M_R will be used to designate a (unital right) R-module M, and the lattice of all submodules of M_R will be denoted by $\mathcal{L}(M)$. The notation $N \leqslant M$ (resp. $N < M$) means that N is a submodule (resp. proper submodule) of M. Whenever we want to indicate that X is merely a subset (resp. proper subset) of M, then we shall use $X \subseteq M$ (resp. $X \subset M$). We denote by $\mathbb{N}$ the set $\{1, 2, \ldots\}$ of all positive integers, by $\mathbb{Z}$ the ring of rational integers, and by $\mathbb{Q}$ the field of rational numbers. For all undefined terms and notation the reader is referred to Albu and Smith [2].

Let R be any ring. Recall that a submodule N of a module M is called *meet irreducible* or, simply, *irreducible* provided $N \neq M$ and whenever $N = K \cap L$, for some submodules K, L of M, then $N = K$ or $N = L$. On the other hand, a submodule N of M is called *completely irreducible* or, more briefly, *CI* if $N \neq M$ and N is not the intersection of any collection of submodules of M each properly containing N. Clearly CI submodules of M are irreducible submodules of M, but not conversely: the zero ideal of the ring $\mathbb{Z}$ is irreducible but not CI. A module M is called *coirreducible* (resp. *completely coirreducible*) if 0 is an irreducible (resp. CI) submodule of M. Note that the coirreducible modules are often known as *uniform* modules. We use here the term of “coirreducible” in accordance with the following more general terminology: if $\mathbb{P}$ is a property of submodules of a module, then a module M is called “co$\mathbb{P}$” if the submodule 0 of

M has $\mathbb{P}$ (e.g., *primary submodule* and *coprimary module*, *primal submodule* and *coprimal module*).

Recall the following known elementary facts (see, e.g., Albu [1] and/or Albu and Smith [2]).

Lemma 0.1. *Let R be any ring and let M be a non-zero right R-module. Then every proper submodule of M is an intersection of* CI *submodules of M.*

Lemma 0.2. *Let R be any ring. Then the following statements are equivalent for a submodule N of a right R-module M.*

(i) *N is a* CI *submodule of M.*

(ii) *$N \neq M$ and $N \neq \bigcap L$, where the intersection is taken over all submodules L of M with $N \subset L$.*

(iii) *N is an irreducible submodule of M such that the module M/N has non-zero socle.*

(iv) *The module M/N has a simple essential socle.*

For any module M, the collection of CI submodules will be denoted by $\mathcal{I}^c(M)$ and the collection of all irreducible submodules by $\mathcal{I}(M)$.

1 Completely irreducible submodules and primary submodules

In this section R will always denote a commutative ring (with a non-zero identity) and M an arbitrary (unital) R-module. We analyze in this section the connections between CI submodules and primary submodules of a module. We show that an R-module M is a module with primary decomposition if and only if every CI submodule of M is $\mathfrak{m}$-primary for some maximal ideal $\mathfrak{m}$ of R. This implies that if R is a Noetherian ring then a submodule of M is CI if and only if it is an irreducible $\mathfrak{m}$-primary submodule for some maximal ideal $\mathfrak{m}$ of R.

Recall from Bourbaki [3] some definitions concerning primary submodules. If N is a submodule of M, then the *radical* of N in M is the ideal of R defined by

$$\mathrm{Rad}_M(N) := \{\, r \in R \,|\, \forall\, m \in M,\ \exists\, k_m \in \mathbb{N} \text{ such that } r^{k_m} m \in N \,\}.$$

If $M = R$ and N is an ideal $\mathfrak{a}$ of R, then $\mathrm{Rad}_R(\mathfrak{a})$ is precisely the usual radical $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$. Note that if M is finitely generated, then

$$\mathrm{Rad}_M(N) = \{\, r \in R \,|\, \exists\, k \in \mathbb{N} \text{ such that } r^k M \subseteq N \,\}.$$

A *primary submodule* of M is a proper submodule N of M satisfying the following condition: whenever $r \in R$ and $m \in M$ are such that $rm \in N$, then

$r \in \mathrm{Rad}_M(N)$ or $m \in N$. Equivalently, N is a primary submodule of M if $\mathrm{Ass}_f(M/N)$ has exactly one element, say $\mathfrak{p} \in \mathrm{Spec}(R)$, and in that case we call N a $\mathfrak{p}$-*primary submodule* of M. Note that if N is a $\mathfrak{p}$-primary submodule of M, then $\mathfrak{p} = \mathrm{Rad}_M(N)$. A module M is said to be *coprimary* if 0 is a primary submodule of M; so, N is a primary submodule of M if and only if the quotient module M/N is coprimary.

A submodule N of M is said to be *strongly primary* if it is a $\mathfrak{p}$-primary submodule of M such that $\mathfrak{p}^n M \subseteq N$ for some positive integer n. A module M is said to be *with primary decomposition* (resp. *Laskerian*) if each of its proper submodules is an intersection, possibly infinite, (resp. a finite intersection) of primary submodules of M.

For all other undefined terms and notation the reader is referred to Albu and Smith [2].

Lemma 1.1. (Radu [9, Propoziţia 4.3]). *An R-module M is a module with primary decomposition if and only if every* CI *submodule of M is primary.*

Proof: If any CI submodule of M is primary, then any proper submodule of M is an intersection of primary submodules, i.e., the module M is with primary decomposition, because every proper submodule of M is an intersection of CI submodules by Lemma 0.1.

Conversely, if M is with primary decomposition, since any CI submodule N of M cannot be an intersection of primary submodules that properly contain it, it follows that N must be primary. □

Proposition 1.2. *An R-module M is a module with primary decomposition if and only if every* CI *submodule of M is $\mathfrak{m}$-primary for some $\mathfrak{m} \in \mathrm{Max}(R)$. In particular, if M is a Laskerian module, then every* CI *submodule of M is $\mathfrak{m}$-primary for some $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof: In view of Lemma 1.1, it is sufficient to show that if M is a module with primary decomposition then every CI submodule of M is $\mathfrak{m}$-primary for some $\mathfrak{m} \in \mathrm{Max}(R)$. Let N be any CI submodule of M. Then, by Lemma 0.2, there exists $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\mathrm{Soc}(M/N) = N^*/N \simeq R/\mathfrak{m}$ and N^*/N is an essential submodule of M/N. By Lemma 1.1, N is a primary submodule of M, say $\mathfrak{p}$-primary. Thus $\{\mathfrak{p}\} = \mathrm{Ass}_f(M/N) \supseteq \mathrm{Ass}(M/N) = \mathrm{Ass}(R/\mathfrak{m}) = \{\mathfrak{m}\}$, and then, $\mathfrak{p} = \mathfrak{m}$, which implies that N is $\mathfrak{m}$-primary, as desired. □

Corollary 1.3. *If* $\mathrm{Spec}(R) = \mathrm{Max}(R)$, *i.e., if the ring R has classical Krull dimension* 0, *then for every R-module M, every* CI *submodule of M is $\mathfrak{m}$-primary for some $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof: By Radu [9, Corolar 4.10], every module over any ring with classical Krull dimension 0 is with primary decomposition, so we can apply Proposition 1.2. □

Corollary 1.4. *Let M be an R-module such that $V(\mathrm{Ann}_R(M)) \subseteq \mathrm{Max}(R)$. Then every* CI *submodule of M is $\mathfrak{m}$-primary for some $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof: By Radu [9, Corolar 4.11], M is a module with primary decomposition. The result now follows from Proposition 1.2. □

Corollary 1.5. *Let $\mathfrak{m} \in \mathrm{Max}(R)$ and let M be any R-module. Then every irreducible strongly $\mathfrak{m}$-primary submodule of M is* CI*. Moreover, if $\mathfrak{m}$ is finitely generated then every irreducible $\mathfrak{m}$-primary submodule of M is* CI*. In particular, if R is a Noetherian ring then a submodule of M is* CI *if and only if it is an irreducible $\mathfrak{p}$-primary submodule for some $\mathfrak{p} \in \mathrm{Max}(R)$.*

Proof: Let N be an irreducible strongly $\mathfrak{m}$-primary submodule of M. Then $\mathfrak{m}^n M \subseteq N$ for some positive integer n. Without loss of generality we may assume that $n \geqslant 1$ is minimum with $\mathfrak{m}^n M \subseteq N$. If $n = 1$, then $\mathfrak{m}M \subseteq N$, so the non-zero R-module M/N is semisimple. If $n \geqslant 2$, then $\mathfrak{m}^n M \subseteq N$ and $\mathfrak{m}^{n-1} M \not\subseteq N$, so there exists $x \in \mathfrak{m}^{n-1} M \setminus N$. Since $\mathfrak{m}(\mathfrak{m}^{n-1} M) \subseteq N$, we deduce that $\mathfrak{m}x \subseteq N$, and then M/N has non-zero socle. Thus, in any case, M/N has non-zero socle. Now apply Lemma 0.2 to conclude that N is CI.

Now assume that $\mathfrak{m}$ is a finitely generated, and let Q be an irreducible $\mathfrak{m}$-primary submodule of M. Then $\mathfrak{m} = \{\, a \in R \mid \forall\, x \in M, \exists\, n(x) \in \mathbb{N}, a^{n(x)} x \in Q \,\}$ $(= \mathrm{Rad}_M(Q))$. Without loss of generality we may assume that $Q = 0$. Suppose that $\mathfrak{m} = Ra_1 + \ldots + Ra_n$. Let $0 \neq x \in M$. For all i, $1 \leqslant i \leqslant n$, there exists $k(i) \geqslant 1$ such that $a_i^{k(i)} x = 0$. Let $k = k(1) \cdots k(n)$. Then $\mathfrak{m}^k x = 0$. There exists $t \geqslant 1$ such that $\mathfrak{m}^{t-1} x \neq 0$ but $\mathfrak{m}^t x = 0$. Therefore $\mathrm{Soc}(Rx) \neq 0$. Since $Q\,(= 0)$ is an irreducible submodule of M, it follows that Q is CI by Lemma 0.2, as desired.

If R is a Noetherian ring then every R-module is with primary decomposition by Radu [9, Corolar 4.8], so that, for any R-module M, every CI submodule of M is $\mathfrak{p}$-primary for some $\mathfrak{p} \in \mathrm{Max}(R)$ by Proposition 1.2. Conversely, if $\mathfrak{p} \in \mathrm{Max}(R)$, then $\mathfrak{p}$ is finitely generated, hence every irreducible $\mathfrak{p}$-primary submodule of M is CI, as we have already proved. □

Remarks 1.6. (1) If a module M is not with primary decomposition, then, by Lemma 1.1, it has CI submodules that are not primary. According to Radu [9, Exemple 4.15 (ii)], the trivial extension $\mathbb{Z} \rtimes \mathbb{Q}$ is not a ring with primary decomposition, so it has CI ideals which are not primary; more precisely, every CI ideal of $\mathbb{Z} \rtimes \mathbb{Q}$ strictly contained in $0 \rtimes \mathbb{Q}$ is not primary, since the primary ideals of $\mathbb{Z} \rtimes \mathbb{Q}$ have one of the following forms: $\mathbb{Z}p^n \rtimes \mathbb{Q}$, where $p > 0$ is prime in $\mathbb{Z}$ and $n \in \mathbb{N}$, $0 \rtimes \mathbb{Q}$, and $0 \rtimes 0$ by Radu [9, Exemple 2.9 (v)]. For example, $0 \rtimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at any non-zero prime ideal $p\mathbb{Z}$, is not a primary ideal, but it is a CI ideal of $\mathbb{Z} \rtimes \mathbb{Q}$.

(2) An example of an $\mathfrak{m}$-primary submodule that is not CI is the following one. Let R be a rank-one nondiscrete valuation domain with maximal ideal $\mathfrak{m}$, and let $0 \neq x \in R$. If we set $\mathfrak{q} := Rx$, then $\sqrt{\mathfrak{q}}$ is a non-zero prime ideal of R, so necessarily $\sqrt{\mathfrak{q}} = \mathfrak{m}$, and hence $\mathfrak{q}$ is a principal $\mathfrak{m}$-primary ideal. Then $\mathfrak{q}$ is irreducible but not CI (see Fuchs, Heinzer, and Olberding [7, Remark 1.8]).

Another example is the following one. Let F be a field of non-zero characteristic p, let G be the Prüfer p-group $C(p^{\infty})$, and let R be the group algebra $F[G]$. Let $\mathfrak{m}$ denote the augmentation ideal of R. Then $\mathfrak{m}$ is a nil idempotent ideal of R. For any two elements a, b in R there exists a finite cyclic subgroup H of G such that $a, b \in F[H]$. But H is a finite p-group so that $F[H]$ is a valuation ring. Thus $F[H]a \subseteq F[H]b$ or $F[H]b \subseteq F[H]a$. It follows that $Ra \subseteq Rb$ or $Rb \subseteq Ra$ for all a and b in R. Thus R is a valuation ring. It follows that 0 is an irreducible $\mathfrak{m}$-primary submodule of $_RR$. Suppose that 0 is CI. Then there exists $0 \neq r \in R$ such that $r\mathfrak{m} = 0$. This is impossible because G is infinite. Thus 0 is not a CI submodule of $_RR$.

2 Primal submodules and primary submodules

Throughout this section R will again be a commutative ring with non-zero identity element. We analyze the connections between primal submodules and primary submodules of a module. We show that for any module M, each primary submodule is also primal. Then we study when the converse holds. We characterize those modules M over a Noetherian ring R for which every primal submodule is primary. In particular, we describe the structure of Noetherian rings R for which all primal ideals are primary.

If M is an R-module, then we denote by

$$Z(M) := \{\, a \in R \,|\, \exists\, x \in M,\, x \neq 0, \text{ with } ax = 0 \,\}$$

the set of all *zero divisors* on M. Recall that a submodule N of M is said to *primal* if $N \neq M$ and $Z(M/N)$ is an ideal of R, which is necessarily prime; in this case, if $Z(M/N) = \mathfrak{p}$, then N is called $\mathfrak{p}$-*primal* and $\mathfrak{p}$ is called the *adjoint ideal* of N and is denoted by adj N. The module M is called *coprimal* if 0 is a primal submodule of M. By $\mathcal{P}(M)$ (resp. $\mathcal{Q}(M)$) we denote the set of all primal (resp. primary) submodules of M.

For basic properties of primal submodules of a module, see Albu and Smith [2].

Proposition 2.1. *Any coprimary module is coprimal, so $\mathcal{Q}(M) \subseteq \mathcal{P}(M)$ for any module M, in other words, any primary submodule of a module is also primal.*

Proof: Let M be a coprimary module; this means that whenever $c \in R$ and $z \in M$ are such that $cz = 0$, then $c \in \mathrm{Rad}_M(0)$ or $z = 0$. Let $a, b \in Z(M)$. Then, there exist non-zero elements $x, y \in M$ such that $ax = 0$ and $by = 0$. Since M is a coprimary module, it follows that $a, b \in \mathrm{Rad}_M(0)$, so also $a - b \in \mathrm{Rad}_M(0) \subseteq Z(M)$. Thus M is coprimal. □

Remarks 2.2. (1) A primal submodule of a module M is not necessarily primary, as an example from Fuchs [5] shows: the ideal (X^2, XY) of the ring $F[X,Y]$ of polynomials over a field F is primal but is not primary.

(2) It would be interesting to characterize those modules M such that $\mathcal{Q}(M) = \mathcal{P}(M)$; see Section 4, Problem 6. A partial answer is given below for modules over Noetherian rings. □

Lemma 2.3. *Let M be a module such that the zero submodule $0 = \bigcap_{i \in I} N_i$ is an intersection of a family $(N_i)_{i \in I}$ of submodules of M. Then*

$$Z(M) \subseteq \bigcup_{i \in I} Z(M/N_i).$$

Proof: Let $r \in Z(M)$. Then $rm = 0$ for some $0 \neq m \in M$. There exists $i \in I$ such that $m \notin N_i$ but $rm \in N_i$ so that $r \in Z(M/N_i)$. □

Lemma 2.4. *Let M be a module such that the zero submodule $0 = \bigcap_{i \in I} N_i$ is an irredundant intersection of a family $(N_i)_{i \in I}$ of primary submodules of M. Then*

$$Z(M) = \bigcup_{i \in I} Z(M/N_i).$$

Proof: Let $r \in Z(M/N_i)$ for some $i \in I$. Because $\bigcap_{j \neq i} N_j \nsubseteq N_i$, there exists $m \in \bigcap_{j \neq i} N_j$ with $m \notin N_i$. There exists $k \in \mathbb{N}$ such that $r^k m \in N_i$, and so $r^k m \in \bigcap_{j \in I} N_j = 0$. Then, there exists $t \in \mathbb{N}$, $1 \leqslant t \leqslant k$ such that $r^{t-1} m \neq 0$ but $r^t m = 0$, hence $r \in Z(M)$. The result follows by Lemma 2.3. □

Lemma 2.5. *Let R be a Noetherian ring, and let M be an arbitrary non-zero R-module. Then every irreducible submodule N of M is a primary submodule of M, and moreover, $Z(M/N) = \mathfrak{p}$, where $\mathrm{Ass}(M/N) = \mathrm{Ass}_f(M/N) = \{\mathfrak{p}\}$.*

Proof: Let N be an irreducible submodule of M. Then M/N is a coirreducible module. Since R is a Noetherian ring, $\mathrm{Ass}(M/N) = \mathrm{Ass}_f(M/N) \neq \varnothing$. Let $\mathfrak{p} \in \mathrm{Ass}(M/N)$. Then $\mathfrak{p} = \mathrm{Ann}_R(y)$ for some $y \in M/N$, so $R/\mathfrak{p} \simeq Ry \hookrightarrow M/N$. Since Ry is an essential submodule of M/N, we have $\{\mathfrak{p}\} = \mathrm{Ass}(R/\mathfrak{p}) = \mathrm{Ass}(M/N) = \mathrm{Ass}_f(M/N)$, and so, $\mathrm{Ass}_f(M/N) = \{\mathfrak{p}\}$, which means precisely that N is a $\mathfrak{p}$-primary submodule of M.

Since $Z(M/N) = \bigcup_{\mathfrak{q} \in \mathrm{Ass}_f(M/N)} \mathfrak{q}$, we deduce that $Z(M/N) = \mathfrak{p}$, as desired. □

Lemma 2.6. *Suppose that every primal submodule of a non-zero R-module M is primary. Let $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of R. Then every $\mathfrak{p}$-primary submodule of M is contained in every $\mathfrak{q}$-primary submodule of M.*

Proof: Let N be any $\mathfrak{p}$-primary submodule of M and let L be any $\mathfrak{q}$-primary submodule of M. Suppose that $N \nsubseteq L$. Let $r \in Z(M/(N \cap L))$. Then there exists $m \in M \backslash (N \cap L)$ such that $rm \in N \cap L$. Either $m \notin N$ in which case $r \in \mathfrak{p}$ or $m \notin L$ in which case $r \in \mathfrak{q}$. In any case $r \in \mathfrak{q}$. Thus $Z(M/(N \cap L)) \subseteq \mathfrak{q}$. Now suppose that $s \in \mathfrak{q}$. Let $x \in N \backslash L$. There exists a positive integer k such that $s^k x \in N \cap L$. It follows that $s \in Z(M/(N \cap L))$. Thus $Z(M/(N \cap L)) = \mathfrak{q}$, so that $N \cap L$ is a primal submodule of M, and hence primary, say $\mathfrak{r}$-primary, by assumption. Then $\mathrm{Ass}_f(M/(N \cap L)) = \{\mathfrak{r}\}$. As in the proof of Lemma 2.5, we have $\mathfrak{q} = Z(M/(N \cap L)) = \bigcup_{\mathfrak{n} \in \mathrm{Ass}_f(M/(N \cap L))} \mathfrak{n} = \mathfrak{r}$, so $N \cap L$ is a $\mathfrak{q}$-primary submodule of M. Let $a \in \mathfrak{q}$, and let $y \in M \backslash N$. There exists a positive integer n such that $a^n y \in N \cap L \subseteq N$. Because N is $\mathfrak{p}$-primary, $a^n \in \mathfrak{p}$ and hence $a \in \mathfrak{p}$. This implies that $\mathfrak{q} \subseteq \mathfrak{p}$ and hence $\mathfrak{p} = \mathfrak{q}$, a contradiction. It follows that $N \subseteq L$, as desired. □

Theorem 2.7. *Let R be a Noetherian ring. Then the following statements are equivalent for a non-zero R-module M.*

(1) *Every primal submodule of M is a primary submodule of M.*

(2) *For all prime ideals $\mathfrak{p} \subset \mathfrak{q}$ of R, every $\mathfrak{p}$-primary submodule of M is contained in every $\mathfrak{q}$-primary submodule of M.*

(3) *For any submodules P and Q of M such that* $\mathrm{Ass}(M/P) = \{\mathfrak{p}\}$, $\mathrm{Ass}(M/Q) = \{\mathfrak{q}\}$, *and* $\mathfrak{p} \subset \mathfrak{q}$, *one has* $P \subset Q$.

Proof: (1) $\Longrightarrow$ (2): By Lemma 2.6.

(2) $\Longrightarrow$ (1): Let N be any primal submodule of M. Note that condition (2) passes from M to M/N so that we can suppose without loss of generality that $N = 0$. Every non-zero submodule of M contains a coirreducible submodule. Let $\{U_i \,|\, i \in I\}$ be a maximal independent collection of coirreducible submodules of M. Then $L = \bigoplus_{i \in I} U_i$ is an essential submodule of M. For each $i \in I$ let $\mathfrak{p}_i := \mathrm{Ass}(U_i)$ and let N_i be a submodule of M which is maximal in the collection of submodules H of M such that $\bigoplus_{j \neq i} U_j \subseteq H$ and $H \cap U_i = 0$. In this situation it is a standard fact that N_i is an irreducible submodule of M for all $i \in I$. By Lemma 2.5, N_i is a primary submodule of M for all $i \in I$. Moreover, for each $i \in I$, $U_i \subseteq \bigcap_{j \neq i} N_j$ and $L \cap \left(\bigcap_{i \in I} N_i\right) = 0$. Thus $0 = \bigcap_{i \in I} N_i$ is an irredundant intersection of primary submodules of M.

Let $\mathfrak{p} = Z(M)$. Note that $\mathfrak{p}_i = Z(M/N_i)$ for all $i \in I$, according to Lemma 2.5. By Lemma 2.4, $\mathfrak{p} = \bigcup_{i \in I} \mathfrak{p}_i$. But $\mathfrak{p}$ is finitely generated, so that $\mathfrak{p} = \bigcup_{j \in J} \mathfrak{p}_j$ for some finite subset J of I. It follows that $\mathfrak{p} = \mathfrak{p}_j$ for some j in J. Let $i \in I$. Then

$\mathfrak{p}_i \subseteq \mathfrak{p}_j$. If $\mathfrak{p}_i \neq \mathfrak{p}_j$ then $N_i \subseteq N_j$ by (2), which contradicts the irredundancy of $0 = \bigcap_{i\in I} N_i$. Thus $\mathfrak{p}_i = \mathfrak{p}_j$ for all $i \in I$. We have proved that $\mathfrak{p} = \mathrm{Ass}(U_i)$ for all $i \in I$. Since $L = \bigoplus_{i\in I} U_i$ is an essential submodule of M, it follows that $\mathrm{Ass}_f(M) = \mathrm{Ass}(M) = \mathrm{Ass}(\bigoplus_{i\in I} U_i) = \bigcup_{i\in I} \mathrm{Ass}(U_i) = \{\mathfrak{p}\}$. This shows precisely that 0 is a $\mathfrak{p}$-primary submodule of M, as desired.

(2) $\Longleftrightarrow$ (3): Since R is a Noetherian ring, one has $\mathrm{Ass}(V) = \mathrm{Ass}_f(V)$ for any R-module V, and so $N \leqslant M$ is a $\mathfrak{p}$-primary submodule of M if and only if $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$. □

As usual, for any non-empty subset X of a ring R the *annihilator* of X in R will be denoted by $\mathrm{Ann}_R(X)$, i.e., $\mathrm{Ann}_R(X) := \{\, r \in R \mid rx = 0 \text{ for all } x \in X \,\}$.

Lemma 2.8. *Let R be a Noetherian ring such that every primal ideal of R is a primary ideal of R. Then $R = \mathfrak{p} + \mathrm{Ann}_R(\mathfrak{p})$ for every non-maximal prime ideal $\mathfrak{p}$ of R. Moreover, in this case R is one-dimensional.*

Proof: Let $\mathfrak{p}$ be any non-maximal prime ideal of R, and suppose that $R \neq \mathfrak{p}+\mathrm{Ann}_R(\mathfrak{p})$. Let $\mathfrak{q}$ be any maximal ideal of R such that $\mathfrak{p}+\mathrm{Ann}_R(\mathfrak{p}) \subseteq \mathfrak{q}$. For each positive integer n, $\mathfrak{q}^n$ is a $\mathfrak{q}$-primary ideal of R. Because $\mathfrak{p} \subset \mathfrak{q}$, Theorem 2.7 gives that $\mathfrak{p} \subseteq \bigcap_{n=1}^{\infty} \mathfrak{q}^n$. By Krull's Intersection Theorem (see, for example, Kaplansky [8, Theorems 74 and 76]), there exists $q \in \mathfrak{q}$ such that $(1-q)\mathfrak{p} = 0$. But this implies that $1 - q \in \mathrm{Ann}_R(\mathfrak{p}) \subseteq \mathfrak{q}$, a contradiction. Thus $R = \mathfrak{p} + \mathrm{Ann}_R(\mathfrak{p})$, as required. In particular, there exists $p \in \mathfrak{p}$ such that $(1-p)\mathfrak{p} = 0$. If $\mathfrak{r}$ is a prime ideal of R such that $\mathfrak{r} \subseteq \mathfrak{p}$ then $(1-p)\mathfrak{p} \subseteq \mathfrak{r}$ so that $\mathfrak{p} = \mathfrak{r}$. It follows that R is one-dimensional. □

Theorem 2.9. *The following statements are equivalent for a Noetherian ring R.*

(1) *Every primal ideal of R is a primary ideal of R.*

(2) $R = \mathfrak{p} + \mathrm{Ann}_R(\mathfrak{p})$ *for every non-maximal prime ideal $\mathfrak{p}$ of R.*

(3) *R is a finite direct product of Artinian rings and one-dimensional domains.*

Proof: (1) $\Longrightarrow$ (2): By Lemma 2.8.

(2) $\Longrightarrow$ (3): It is easy to check that condition (2) goes over to every ring homomorphic image of R. Thus, without loss of generality, we can suppose that R is an indecomposable ring. If every prime ideal of R is maximal then it is well known that R is an Artinian ring. Suppose that R contains a non-maximal prime ideal $\mathfrak{p}$. Let $\mathfrak{n} = \mathfrak{p} \cap \mathrm{Ann}_R(\mathfrak{p})$. By (2), $\mathfrak{p}/\mathfrak{n}$ is a direct summand of $R/\mathfrak{n}$ and hence is generated by an idempotent $\overline{e}$. Because $\mathfrak{n}^2 = 0$, we can lift $\overline{e}$ to an idempotent of R, that is, $\overline{e} = e+\mathfrak{n}$ for some idempotent e in R. Because R is indecomposable, $e = 1$ or $e = 0$. Clearly $e \neq 1$. Thus $e = 0$ and hence $R = \mathrm{Ann}_R(\mathfrak{p})$, so that

$\mathfrak{p} = 0$. We have proved that R is a domain and R is one-dimensional by Lemma 2.8.

(3) $\Longrightarrow$ (2): Assume that R is isomorphic to a direct product $R_1 \times \cdots \times R_n$ of rings R_i, $1 \leqslant i \leqslant n$, for some positive integer n, such that R_i is Artinian or a one-dimensional domain for each $1 \leqslant i \leqslant n$. Without loss of generality we may consider that $R = R_1 \times \cdots \times R_n$. Let $\mathfrak{g}$ be a non-maximal prime ideal of R. Then

$$\mathfrak{g} = R_1 \times \cdots \times R_{i-1} \times 0 \times R_{i+2} \times \cdots \times R_n,$$

for some $1 \leqslant i \leqslant n$. It is easy to check that $R = \mathfrak{g} + \mathrm{Ann}_R(\mathfrak{g})$.

(2) $\Longrightarrow$ (1): Let $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of R, and let $\mathfrak{a}$ be any $\mathfrak{q}$-primary ideal of R. Note that $\mathfrak{a} \subseteq \mathfrak{q}$. By (2), there exists $p \in \mathfrak{p}$ such that $(1-p)\mathfrak{p} = 0$. Hence $(1-p)\mathfrak{p} \subseteq \mathfrak{a}$, so that $\mathfrak{p} \subseteq \mathfrak{a}$. It is now clear that every $\mathfrak{p}$-primary ideal of R is contained in every $\mathfrak{q}$-primary ideal of R. By Theorem 2.7, every primal ideal of R is primary, as required. □

Remark 2.10. Observe that if R is a Noetherian ring such that $\mathcal{P}(R) = \mathcal{Q}(R)$, i.e., every primal ideal of R is a primary ideal of R, then, this does not imply in general that $\mathcal{P}(M) = \mathcal{Q}(M)$ for every non-zero R-module M. To see this, take $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$. □

3 Irredundant intersections

In this section we examine irredundant decompositions of a submodule of a module over an arbitrary ring as an intersection of irreducible, completely irreducible, or primal submodules. Thus, we extend from ideals to modules some results of Fuchs, Heinzer, and Olberding [7], and similar to the characterization, due to Fort [4], of modules rich in coirreducibles via irredundant irreducible decompositions, we characterize modules rich in completely coirreducibles via irredundant completely irreducible decompositions.

Definition 3.1. *Let M_R be a module, and let $A \leqslant C$ be submodules of M. We say that C is a* relevant completely irreducible divisor*, abbreviated an* RCID*, of A if A has a decomposition as an intersection of completely irreducibles submodules of M in which C appears and is* relevant*, i.e., cannot be omitted.* □

Proposition 3.2. *Let $P \leqslant N$ be submodules of a module M such that N is a* CI *submodule of M. Then N is an* RCID *of P if and only if the submodule N/P of M/P is not essential in M/P.*

Proof: Apply Albu [1, Proposition 1.19] to the lattice $L = \mathcal{L}(M)$ of all submodules of M. □

If N is a completely irreducible submodule of M then $\mathrm{Soc}(M/N) = N^*/N$ is a simple essential submodule of M/N by Lemma 0.2. The submodule N^* is called the *cover of* N.

Corollary 3.3. *Let* $P \leqslant N$ *be submodules of a module* M *such that* N *is a* CI *submodule of* M*. Then,* N^*/P *is an essential submodule of* M/P*, where* N^* *is the cover of* N*.*

Proof: Apply Albu [1, Corollary 1.20] to the lattice $L = \mathcal{L}(M)$. □

Corollary 3.4. *Let* N *be a proper submodule of a module* M*. Then there exists an* RCID *of* N *if and only if* $\mathrm{Soc}(M/N) \neq 0$*.*

Proof: Apply Albu [1, Corollary 1.21] to the lattice $L = \mathcal{L}(M)$. □

As in Fort [4], a right module M_R is said to be *rich in coirreducibles*, abbreviated RC, if $M \neq 0$ and each of its non-zero submodules contains a coirreducible (or uniform) submodule. The next result characterizes the RC modules.

Theorem 3.5. (Fort [4, Théoréme 3]). *The following statements are equivalent for a non-zero module* M_R*.*

(1) M *is* RC.

(2) M *is an essential extension of a direct sum of coirreducible submodules of* M*.*

(3) *The injective hull* $E_R(M)$ *of* M *is an essential extension of a direct sum of indecomposable injective modules.*

(4) 0 *has an irredundant irreducible decomposition in any non-zero submodule of* M*.* □

It is natural to ask whether condition (4) in Theorem 3.5 can be replaced by the weaker one: *0 has an irredundant irreducible decomposition in M* (see also Section 4, Problem 1). We guess that the answer in *no*, but do not have any counterexample. Such a counterexample will be a module M_R that is not rich in coirreducibles such that 0 has an irredundant irreducible decomposition in M. According to Fort [4, Théorème 1, Proposition 5], the module M has to be a direct sum of a module without any coirreducibles with a module that is a maximal direct sum of coirreducibles.

If we replace "coirreducibles" by "completely coirreducibles" in the definition of a module rich in coirreducibles one obtains the concept of a module rich in completely coirreducibles. More precisely, a module M is said to be *rich in completely coirreducibles*, abbreviated RCC, if $M \neq 0$ and for any $0 \neq N \leqslant M$ there exists a completely coirreducible submodule C of M such that $C \leqslant N$.

Recall that in Albu and Smith [2] a module M is called *completely coirreducible* provided M is non-zero and the zero submodule of M is completely irreducible. The next result gives a characterization of RCC modules similar to that for RC modules in Theorem 3.5.

Theorem 3.6. *The following statements are equivalent for a non-zero module* M_R.

(1) M *is* RCC.

(2) *Every non-zero submodule of* M *contains a simple submodule.*

(3) *The socle* $\mathrm{Soc}(M)$ *of* M *is essential in* M.

(4) M *is an essential extension of a direct sum of completely coirreducible submodules of* M.

(5) M *is an essential extension of a direct sum of simple submodules of* M.

(6) *The injective hull* $E_R(M)$ *of* M *is an essential extension of a direct sum of injective hulls of simple* R*-modules.*

(7) *For every* $0 \neq N \leqslant M$ *there exists a nonempty set* I_N *such that* 0 *can written as an irredundant intersection*

$$0 = \bigcap_{i \in I_N} N_i$$

of CI *submodules* $N_i \, (i \in I_N)$ *of* N.

(8) *For every* $0 \neq N \leqslant M$ *there exists a nonempty set* J_N *such that* 0 *can written as an irredundant intersection*

$$0 = \bigcap_{i \in J_N} K_i$$

of maximal submodules $K_i \, (i \in J_N)$ *of* N, *in other words the Jacobson radical* $\mathrm{Rad}\,(N)$ *of* N *is zero and an irredundant intersection of maximal submodules of* N.

Proof: Apply Albu [1, Theorem 1.16] to the lattice $L = \mathcal{L}(M_R)$. □

From now on, R will be a *commutative ring* with a non-zero identity element, and M a unital R-module.

Proposition 3.7. *Let* M *be an* R*-module, let* P *be a proper submodule of* M, *and let* $\mathfrak{m} \in \mathrm{Max}(R)$. *Then* $\mathfrak{m} \in \mathrm{Ass}(M/P)$ *if and only if there exists an* RCID *of* P *that is* $\mathfrak{m}$*-primal.*

Proof: "$\Longrightarrow$": Suppose that $\mathfrak{m} \in \mathrm{Ass}(M/P)$. There exists $x \in M \setminus P$ such that $(P : x) = \mathfrak{m}$. Let $N \leqslant M$ be maximal with $P \leqslant N$ and $x \notin N$. Then it is easy to check that N is a CI submodule of M with cover $N^* = N + Rx$. We have $\mathfrak{m} = (P : x) \subseteq (N : x) \neq R$, so $\mathfrak{m} = (N : x)$ and $\mathfrak{m} \in \mathrm{Ass}(M/N)$. Then $\mathrm{Ass}(M/N) = \{\mathfrak{m}\}$ because the socle $N^*/N \simeq R/\mathfrak{m}$ of the completely coirreducible module M/N is essential in M/N. Thus, N is $\mathfrak{m}$-primal by Albu and Smith [2, Lemma 3.3]. Moreover, $P \leqslant N \cap (P + Rx) < P + Rx$ and $(P + Rx)/P \simeq R/(P : x) = R/\mathfrak{m}$, so that $(P + Rx)/P$ is simple. Thus $N \cap (P + Rx) = P$. By Proposition 3.2, N is an RCID of P.

"$\Longleftarrow$": Conversely, suppose that N is an RCID of P that is $\mathfrak{m}$-primal. Then N is a CI submodule of M and, by Proposition 3.2, $N \cap L = P$ for some submodule L of M properly containing P. It follows that $N < N + L$ and hence $N^* \leqslant N + L$. Thus $N^* = N + (N^* \cap L)$ so that $P < N^* \cap L$. Let $z \in (N^* \cap L) \setminus P$. Then $\mathfrak{m}z \subseteq N \cap L = P$ which implies that $\mathfrak{m} = (P : z) \in \mathrm{Ass}(M/P)$ as desired. □

As we know, for any R-module M one has $Z(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_f(M)} \mathfrak{p}$. The next result says that in some cases, the set $\mathrm{Ass}_f(M)$ can be replaced by its subset $\mathrm{Ass}(M)$.

Proposition 3.8. *Let N be a proper submodule of an R-module M. If N is an irredundant intersection $N = \bigcap_{i \in I} N_i$ of* CI *submodules N_i $(i \in I)$ of M, then*

$$Z(M/N) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M/N)} \mathfrak{p} = \bigcup_{i \in I} \mathrm{adj}\, N_i .$$

Proof: For each $i \in I$ let $\mathfrak{p}_i = Z(M/N_i) = \mathrm{adj}\, N_i$. Then $\mathfrak{p}_i \in \mathrm{Ass}(M/N)$ for each $i \in I$ by Proposition 3.7. Let $r \in Z(M/N)$. Then $rm \in N$ for some $m \in M \setminus N$. There exists $j \in I$ such that $m \notin N_j$ and hence $r \in \mathfrak{p}_j$. Thus

$$Z(M/N) \subseteq \bigcup_{i \in I} \mathfrak{p}_i \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M/N)} \mathfrak{p} \subseteq Z(M/N),$$

and the result follows. □

Corollary 3.9. *Let $N = \bigcap_{i \in I} N_i$ be an irredundant intersection of* CI *submodules of a module M. Then N is $\mathfrak{m}$-primal if and only if N_i is $\mathfrak{m}$-primal for all $i \in I$.*

Proof: Set $\mathfrak{m}_i := \mathrm{adj}\, N_i$ for each $i \in I$. By Proposition 3.8, we have $Z(M/N) = \bigcup_{i \in I} \mathfrak{m}_i$, so that if all the N_i are $\mathfrak{m}$-primal, then $\mathfrak{m}_i = \mathfrak{m}$ for all $i \in I$, and then $Z(M/N) = \mathfrak{m}$, i.e., N is $\mathfrak{m}$-primal.

Conversely, if N is $\mathfrak{m}$-primal, then again by Proposition 3.8, $\mathfrak{m} = \bigcup_{i \in I} \mathfrak{m}_i$, and then necessarily $\mathfrak{m}_i = \mathfrak{m}$ for all $i \in I$ because the $\mathfrak{m}_i$ are all maximal ideals of R, so all the N_i are $\mathfrak{m}$-primal. □

4 Seven open problems

In this section we present a list of seven open questions mainly related with the opposite inclusions in the tower of inclusions

$$\mathcal{I}^c(M) \subseteq \mathcal{I}(M) \subseteq \mathcal{P}(M)$$

of Albu and Smith [2, Lemma 1.3] associated with any R-module over a commutative ring R.

1. A classical result of Fort [4, Théoréme 3] (see also Theorem 3.5) states: *A module M over a not necessarily commutative ring R is rich in coirreducibles (RC) $\Longleftrightarrow$ 0 has an irredundant irreducible meet decomposition in any non-zero submodule of M.*

   It is natural to ask whether the right hand condition above can be replaced by the weaker one: 0 *has an irredundant irreducible meet decomposition in M.*

   We guess that the answer in *no*, but no counterexample is available so far.

2. *Characterize M_R such that $\mathcal{I}^c(M) = \mathcal{I}(M)$.* Note that

   $$M_R \text{ is semi-Artinian } \Longrightarrow \mathcal{I}^c(M) = \mathcal{I}(M),$$

   but not conversely.

3. *Characterize M_R such that $\mathcal{I}(M) \subseteq \mathcal{Q}(M)$.* Note that the inclusion holds for any *Noetherian* module.

4. *Characterize M_R such that $\mathcal{Q}(M) \subseteq \mathcal{I}(M)$.*

5. *Characterize M_R such that $\mathcal{Q}(M) = \mathcal{P}(M)$ for an* arbitrary *commutative ring R.*

6. *Characterize M_R such that $\mathcal{I}^c(M) = \mathcal{P}(M)$.*

7. We have seen that $\mathcal{I}^c(M) \subseteq \mathcal{Q}(M) \Longleftrightarrow M$ is a module *with primary decomposition. Characterize M_R such that $\mathcal{Q}(M) \subseteq \mathcal{I}^c(M)$.*

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