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On the critical group of matrices

by
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Abstract

Given a graph $\mathcal{G}$ with a distinguished vertex s, the critical group of $(\mathcal{G}, s)$ is the cokernel of their reduced Laplacian matrix $L(\mathcal{G}, s)$. In this article we generalize the concept of the critical group to the cokernel of any matrix with entries in a commutative ring with identity. In this article we find diagonal matrices that are equivalent to some matrices that generalize the reduced Laplacian matrix of the path, the cycle, and the complete graph over an arbitrary commutative ring with identity. We are mainly interested in those cases when the base ring is the ring of integers and some subrings of matrices. Using these equivalent diagonal matrices we calculate the critical group of the m-cones of the l-duplications of the path, the cycle, and the complete graph. Also, as byproduct, we calculate the critical group of another matrices, as the m-cones of the l-duplication of the bipartite complete graph with m vertices in each partition and the bipartite complete graph with $2m$ vertices minus a matching.

Key Words: Critical group, matrices, cartesian product, complete graph, path, cycle.
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1 Introduction

Let $\mathcal{G} = (V, E)$ be a finite connected graph without loops, but with multiple edges allowed. The *adjacency matrix* of $\mathcal{G}$, denoted by $A(\mathcal{G})$, is given by $A(\mathcal{G})_{u,v} = m_{u,v}$, where $m_{u,v}$ is the number of the edges between u and v in $\mathcal{G}$. The *Laplacian matrix* of $\mathcal{G}$ is the matrix $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ where

$$D(\mathcal{G})_{u,v} = \begin{cases} d_{\mathcal{G}}(u) & \text{if } u = v, \\ 0 & \text{otherwise,} \end{cases}$$

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and $d_{\mathcal{G}}(u)$ is the degree of the vertex u in $\mathcal{G}$. If s is a vertex of $\mathcal{G}$, the *reduced Laplacian matrix* of $(\mathcal{G}, s)$, denoted by $L(\mathcal{G}, s)$, is the matrix obtained from $L(\mathcal{G})$ by removing the row and column s. The *critical group* of $\mathcal{G}$, denoted by $K(\mathcal{G})$, is the cokernel of $L(\mathcal{G}, s)$. That is,

$$K(\mathcal{G}) = \mathbb{Z}^{\widetilde{V}}/\operatorname{Im} L(\mathcal{G}, s)^t,$$

where $\widetilde{V} = V \setminus s$.

The critical group is an abelian group that is isomorphic to the sandpile group introduced by Dhar in [10], which generalizes the case of a grid from [2]. The critical group has been studied by several authors, see for instance [1, 3, 4, 5, 6, 7, 12, 13, 14, 16, 17, 18, 19, 20].

The concept of critical group can be generalized easily to an arbitrary commutative ring with identity $\mathbb{A}$. More precisely, if $M \in M_{m\times n}(\mathbb{A})$, then the *critical module* of M, denoted by $K(M)$, is defined as:

$$K(M) := \mathbb{A}^n/M^t\mathbb{A}^m.$$

Given $H < GL_n(\mathbb{A})$ and $H' < GL_m(\mathbb{A})$, we say that $M, N \in M_{m\times n}(\mathbb{A})$ are (H, H')-*equivalent*, denoted by $N \sim_{(H,H')} M$, if there exist $P \in H$ and $Q \in H'$ such that $N = PMQ$. When $H = SL_n(\mathbb{A})$ and $H' = SL_m(\mathbb{A})$, then we simply say that M and N are *unitary equivalent* and will be denoted by $N \sim_u M$. Also, if H and H' are the subgroups generated by the elementary matrices, we simply say that M and N are *elementary equivalent* and will be denoted by $N \sim_e M$. Finally, if $H = GL_n(\mathbb{A})$ and $H = GL_n(\mathbb{A})$, then we simply say that M and N are equivalent and will be denoted by $N \sim_{\mathbb{A}} M$ or $M \sim N$ if the ring $\mathbb{A}$ is clear from the context.

Usually the critical group of a graph can be described in terms of a diagonal matrix called the Smith Normal Form of $L(\mathcal{G})$. Moreover, is not difficult to see that if M and N are equivalent, then

$$K(M) = \mathbb{A}^n/M^t\mathbb{A}^m \cong \mathbb{A}^n/N^t\mathbb{A}^m = K(N).$$

When the base ring $\mathbb{A}$ is a Principal Ideal Domain (PID), another description of the critical group of a matrix M is given by

$$K(M) = \bigoplus_{i=1}^{|V|} \mathbb{A}_{\Delta_i(M)/\Delta_{i-1}(M)},$$

where $\Delta_i(M)$ is the greatest common divisor of all the $i \times i$ minors of M.

This article is divided in two sections: In the first section, we find diagonal matrices over an arbitrary commutative ring with identity that are equivalent to some matrices that generalize the Laplacian matrices of the path, the cycle, and the complete graph. In the second section we apply the results obtained in the first section in the case when the base ring is the ring of integers and some

subrings of matrices. In particular we are able to calculate the critical group of the m-cones of the l-duplications of the path, the cycle, the complete graph, the bipartite complete graph with m vertices in each partition, the bipartite complete graph with $2m$ vertices minus a matching, etc.

In the following, every multigraph will be connected and will have a distinguished vertex $s_G \in V(G)$, called the *sink* of G. Sometimes we will simply write s instead of s_G. In [11] it can be seen any unexplained term of graph theory.

2 The critical module of matrices

In this section we will find diagonal matrices that are equivalent to some matrices that generalize the Laplacian matrices of the path, the cycle, and the complete graphs. After that, we will apply these results to calculate the critical group for several families of graphs.

For all $n \geq 2$ and $a, b \in \mathbb{A}$, let $K_n(a,b) = (a+b)I_n + bA(\mathcal{K}_n)$, $T_n(a,b) = aI_n + bA(\mathcal{P}_n)$, $C_n(a,b) = aI_n + bA(\mathcal{C}_n)$, and

$$P_n(a,b) = \begin{pmatrix} a+b & b & 0 & \dots & 0 \\ b & a & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a & b \\ 0 & \cdots & 0 & b & a+b \end{pmatrix}.$$

where $I_n \in M_{n\times n}(\mathbb{A})$ is the identity matrix on order n.

Since the critical module of a matrix is invariant under equivalency classes, then in order to determine the critical module of a matrix, it is enough to find an equivalent diagonal matrix.

Theorem 2.1. *Let $a, b \in \mathbb{A}$ such that the equation $ax + by = 1$ has solution in $\mathbb{A}$ and $f_n(x,y)$ are polynomials in $\mathbb{A}[x,y]$ that satisfy the recurrence relation*

$$f_n(x,y) = xf_{n-1}(x,y) - y^2 f_{n-2}(x,y)$$

with initial values $f_{-1}(x,y) = 0$ and $f_0(x,y) = 1$. Then

(i) *$T_n(a,b) \sim_u \operatorname{diag}(1,\dots,1,f_n(a,b))$ for all $n \geq 2$,*

(ii) *$P_n(a,b) \sim_u \operatorname{diag}(1,\dots,1,(a+2b)f_{n-1}(a,b))$ for all $n \geq 2$,*

(iii) *$K_n(a,b) \sim_u \operatorname{diag}(1,a,\dots,a,a(a+nb))$ for all $n \geq 2$, and*

(iv) *$C_n(a,b) \sim_u I_{n-2} \oplus C$ for all $n \geq 4$, where*

$$C = \begin{cases} f_q(a,b)\begin{pmatrix} a & 2b \\ 2b & a \end{pmatrix} & \text{if } n-2 = 2q, \\ \big(f_{q+1}(a,b) - bf_q(a,b)\big)\begin{pmatrix} 1 & 0 \\ 0 & a+2b \end{pmatrix} & \text{if } n-2 = 2q+1. \end{cases}$$

Proof. (i) For all $l \geq 2$, $n \geq 2$, and $1 \leq k \leq l-1$, let

$$Z_{k,l}(a,b) = \begin{pmatrix} f_k & bf_{k-1} & \mathbf{0}_{l-2} \\ b & & \\ \mathbf{0} & & T_{l-1}(a,b) \end{pmatrix} \in M_l(\mathbb{A}), \text{ and } Z_{n,1}(a,b) = (f_n),$$

where $f_k := f_k(a,b)$ for all $k \geq -1$ and $\mathbf{0}$ is the matrix with all the entries equal to 0. Note that $Z_{1,n}(a,b) = T_n(a,b)$

Now we will prove the following statement:

Claim 2.2. *For all $l \geq 2$ and $1 \leq k \leq l-1$*

$$Z_{k,l}(a,b) \sim_u I_1 \oplus Z_{k+1,l-1}(a,b).$$

Proof. Let $x_1, y_1 \in \mathbb{A}$ be a solution of the equation $ax + by = 1$. Moreover, for all $k \geq 1$, let

$$x_k = x_1^k \text{ and } y_k = \sum_{i=1}^{k} \binom{k}{i} a^i b^{k-1-i} x_1^i y_1^{k-i} - x_1^k (f_k - a^k)/b,$$

that is, x_k and y_k are a solution of the equation $f_k x_k + b y_k = 1$.

Since

$$\begin{pmatrix} x_k & y_k & \mathbf{0} \\ -b & f_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n-k-1} \end{pmatrix} Z_{k,l}(a,b) = \begin{pmatrix} 1 & * & * & * \\ 0 & af_k - b^2 f_{k-1} & bf_k & \mathbf{0} \\ 0 & b & & \\ \mathbf{0} & \mathbf{0} & & T_{n-k-1}(a,b) \end{pmatrix}$$

and $\det \begin{pmatrix} x_k & y_k \\ -b & f_k \end{pmatrix} = f_k x_k + b y_k = 1$, then $Z_{k,l}(a,b) \sim_u I_1 \oplus Z_{k+1,l-1}(a,b)$. $\square$

Applying claim 2.2, we get that $T_n(a,b) = Z_{1,n}(a,b) \sim_u I_{n-1} \oplus Z_{n,1}(a,b)$.

(ii) For all $n \geq 0$, $m \geq 1$, let

$$Z_{k,l}(a,b) = \begin{pmatrix} f_k + bf_{k-1} & b(f_{k-1} + bf_{k-2}) & 0 & \cdots & 0 \\ b & a & b & \cdots & 0 \\ 0 & b & & & 0 \\ 0 & 0 & & T_{l-3}(a,b) & \mathbf{0} \\ \vdots & \vdots & & & b \\ 0 & 0 & \mathbf{0} & b & a+b \end{pmatrix} \in M_l(\mathbb{A}).$$

Also, let $x'_k = x_k$ and $y'_k = y_k - f_{k-1}x_k$, that is, x_k and y_k are a solution of the equation $(f_k + bf_{k-1})x'_k + by'_k = 1$. Since

$$\begin{pmatrix} x'_k & y'_k & \mathbf{0} \\ -b & f_k + bf_{k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_m \end{pmatrix} Z_{k,l}(a,b) = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & f_{k+1} + bf_k & b(f_k + bf_{k-1}) & \mathbf{0} & 0 \\ 0 & b & & & 0 \\ 0 & 0 & & T_{l-4}(a,b) & \mathbf{0} \\ \vdots & \vdots & & & \\ 0 & 0 & \mathbf{0} & b & a+b \end{pmatrix}$$

for all $l \geq 2$ and $\det \begin{pmatrix} x'_k & y'_k \\ -b & f_k + bf_{k-1} \end{pmatrix} = (f_k + bf_{k-1})x_k + by_k = 1$, then $Z_{k,l}(a,b) \sim_u I_1 \oplus Z_{k+1,l-1}(a,b)$ for all $l \geq 2$. Thus

$$P_n(a,b) = Z_{1,n}(a,b) \sim_u I_n \oplus \begin{pmatrix} f_{n-1} + bf_{n-2} & b(f_{n-2} + bf_{n-3}) \\ b & a+b \end{pmatrix}.$$

Therefore, $P_{n+2}(a,b) \sim_u \operatorname{diag}(1,\ldots,1,(a+2b)f_{n-1}(a,b))$ because

$$\begin{pmatrix} x'_{n-1} & y'_{n-1} \\ -b & f_{n-1} + bf_{n-2} \end{pmatrix} \begin{pmatrix} f_{n-1} + bf_{n-2} & b(f_{n-2} + bf_{n-3}) \\ b & a+b \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & (a+2b)f_{n-1} \end{pmatrix} \sim_e \begin{pmatrix} 1 & 0 \\ 0 & (a+2b)f_{n-1}(a,b) \end{pmatrix}.$$

(*iii*) We begin proving the following statement:

Claim 2.3. *If $n \geq 2$, then $K_n(a,b) \sim_e \begin{pmatrix} a & b \\ na & -a \end{pmatrix} \oplus aI_{n-2}$.*

Proof. It turns out because

$$\begin{aligned} P_n K_n(a,b) Q_n &= \begin{pmatrix} b & b & \cdots & b & b & a+b \\ a & a & a & \cdots & a & -(n-1)a \\ 0 & a & -a & 0 & \cdots & 0 \\ 0 & 0 & a & -a & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a & -a \end{pmatrix} Q_n \\ &= \begin{pmatrix} a & b \\ -na & a \end{pmatrix} \oplus aI_{n-2}, \end{aligned}$$

where

$$P_n = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & -(n-1) \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$$

and

$$Q_n = \begin{pmatrix} -n+1 & 1 & -1 & -2 & \cdots & -n+2 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 1 & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 1 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

are elementary matrices. □

On the other hand, $K_n(a,b) \sim_u \operatorname{diag}(1,a,\dots,a,a(a+nb))$ because

$$\begin{pmatrix} a & b \\ -na & a \end{pmatrix}\begin{pmatrix} x_1 & -b \\ y_1 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & a(a+nb) \end{pmatrix} \sim_e \begin{pmatrix} 1 & 0 \\ 0 & a(a+nb) \end{pmatrix}$$

and $\det \begin{pmatrix} x_1 & -b \\ y_1 & a \end{pmatrix} = ax_1 + by_1 = 1$.

(*iv*) For all $l \geq 4$ and $k \geq 1$, let

$$Z_{k,l}(a,b) := \begin{pmatrix} f_k & bf_{k-1} & \mathbf{0}_{l-4} & b^2f_{k-2} & bf_{k-1} \\ b & & & & \mathbf{0} \\ & & T_{l-2}(a,b) & & \\ \mathbf{0} & & & & b \\ bf_{k-1} & b^2f_{k-2} & \mathbf{0}_{l-4} & bf_{k-1} & f_k \end{pmatrix} \in M_l(\mathbb{A})$$

and

$$Z'_{k,l}(a,b) := \begin{pmatrix} f_{k+1} & bf_k & \mathbf{0}_{l-4} & -b^3f_{k-2} & -b^2f_{k-1} \\ b & & & & \mathbf{0} \\ & & T_{l-2}(a,b) & & \\ \mathbf{0} & & & & b \\ -f_k & -bf_{k-1} & \mathbf{0}_{l-4} & bf_{k-1} & f_k \end{pmatrix} \in M_l(\mathbb{A}).$$

Also, let

$$P_{k,l}(a,b) := \begin{pmatrix} x_k & y_k & \mathbf{0} \\ -b & f_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \\ 0 & -f_{k-1} & I_{l-2} \end{pmatrix}$$

and

$$Q_{k,l}(a,b) := I_1 \oplus \begin{pmatrix} I_{l-3} & bf_{k-1} & 0 \\ & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & f_k & -b \\ \mathbf{0} & y_k & x_k \end{pmatrix}.$$

Since

$$\begin{aligned} P_{k,l}(a,b)\cdot Z_{k,l}(a,b) &= \begin{pmatrix} 1 & * & * & & * & * \\ 0 & f_{k+1} & bf_k & \mathbf{0}_{l-5} & -b^3f_{k-2} & -b^2f_{k-1} \\ 0 & b & & & & \mathbf{0} \\ \mathbf{0} & & & T_{l-3}(a,b) & & \\ \mathbf{0} & \mathbf{0} & & & & b \\ 0 & -f_k & -bf_{k-1} & \mathbf{0}_{l-5} & bf_{k-1} & f_k \end{pmatrix} \\ &\sim_u I_1 \oplus Z'_{k,l-1}(a,b) \end{aligned}$$

for all $l \geq 5$,

$$Q_{k,l}(a,b)\cdot(I_1 \oplus Z'_{k,l-1}(a,b)) = I_1 \oplus \begin{pmatrix} f_{k+1} & bf_k & \mathbf{0}_{l-6} & b^2f_{k-1} & bf_k & 0 \\ b & & & & \mathbf{0} & \\ & & T_{l-4}(a,b) & & & \\ \mathbf{0} & & & & b & \\ bf_k & b^2f_{k-1} & \mathbf{0}_{l-6} & bf_k & f_{k+1} & 0 \\ * & * & & * & * & 1 \end{pmatrix}$$

for all $l-1 \geq 5$, and $\det(P_{k,l}(a,b)) = \det(Q_{k,l}(a,b)) = 1$, then

$$Z_{k,l}(a,b) \sim_u I_1 \oplus Z_{k+1,l-2}(a,b) \oplus I_1 \text{ for all } l \geq 6.$$

Moreover, since $Z_{1,n}(a,b) = C_n(a,b)$, then

$$C_n(a,b) \sim_u \begin{cases} Z_{q,4}(a,b) & \text{if } n-2 = 2q, \\ Z'_{q,4}(a,b) & \text{if } n-2 = 2q+1. \end{cases}$$

Finally, using similar reductions we get that

$$\begin{aligned} Z_{q,4}(a,b) &\sim_u \begin{pmatrix} f_q & bf_{q-1} & b^2f_{q-2} & bf_{q-1} \\ b & a & b & 0 \\ 0 & b & a & b \\ 0 & -f_q & 0 & f_q \end{pmatrix} \\ &\sim_u \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f_{q+1} & bf_q - b^3f_{q-2} & -b^2f_{q-1} \\ 0 & b & a & b \\ 0 & -f_q & 0 & f_q \end{pmatrix} \\ &\sim_u \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & af_q & 2bf_q & 0 \\ 0 & b & a & b \\ 0 & -f_q & 0 & f_q \end{pmatrix} \sim_u \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & af_q & 2bf_q & 0 \\ 0 & 2bf_q & af_q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\sim_u I_2 \oplus f_q \begin{pmatrix} a & 2b \\ 2b & a \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} Z'_{q,4}(a,b) &\sim_u \begin{pmatrix} f_{q+1} & bf_q + b^2f_{q-1} & abf_{q-1} - b^3f_{q-2} & 0 \\ b & a & b & 0 \\ 0 & b & a & b \\ -f_q & -bf_{q-1} & bf_{q-1} & f_q \end{pmatrix} \\ &\sim_u \begin{pmatrix} f_{q+1} & bf_q + b^2f_{q-1} & bf_q & 0 \\ b & a & b & 0 \\ bf_q & bf_q + b^2f_{q-1} & f_{q+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\sim_u \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & af_{q+1} - b^2f_q - b^3f_{q-1} & bf_{q+1} - b^2f_q & 0 \\ 0 & bf_q + b^2f_{q-1} - af_q & f_{q+1} - bf_q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\sim_u I_2 \oplus \left(f_{q+1} - bf_q\right) \begin{pmatrix} a+b & b \\ -1 & 1 \end{pmatrix} \sim_u I_2 \oplus \left(f_{q+1} - bf_q\right) \begin{pmatrix} 1 & 0 \\ 0 & a+2b \end{pmatrix}. \end{aligned}$$

□

Remark 2.4. *In [8, pag. 44], a simpler proof of theorem 2.1 (iii) when $\mathbb{A}$ is a principal ideal domain can be found .*

The next Lemma give us some useful properties of the polynomials $f_n(x,y)$.

Lemma 2.5. *If $n \geq 1$, then*

(i) $f_n(x,y) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} x^{n-2i} y^{2i}$,

(ii) $f_n(x+y,-1) - f_n(y,-1) = x \sum_{i=0}^{n-1} f_i(x+y,-1) f_{n-i}(y,-1)$,

(iii) $f_n(x,y) = f_k(x,y) f_{n-k}(x,y) - y^2 f_{k-1}(x,y) f_{n-k-1}(x,y)$,

(iv) $x^k f_n(x,y) = \sum_{i=0}^{k} \binom{k}{i} x^{2i} f_{n+k-2i}(x,y)$.

Proof. It follows using induction on n. □

Remark 2.6. *Note that, $f_n(x,0) = x^n$.*

If $\mathbb{A}$ is a principal ideal domain and $a, b \in \mathbb{A}$, then the equation $ax + by = 1$ has a solution if and only if $\gcd(a,b) = 1$.

Corollary 2.7. *Let $\mathbb{A}$ be a principal ideal domain, $a, b \in \mathbb{A}$ with $r = \gcd(a,b)$, $a = ra'$, and $b = rb'$. Then*

(i) $T_n(a,b) \sim_u \operatorname{diag}(r, \ldots, r, r f_n(a',b'))$ *for all $n \geq 2$,*

(ii) $P_n(a,b) \sim_u \operatorname{diag}(r, \ldots, r, (a+2b) f_{n-1}(a',b'))$ *for all $n \geq 2$,*

(iii) $K_n(a,b) \sim_u \operatorname{diag}(r, a, \ldots, a, a'(a+nb))$ *for all $n \geq 2$, and*

(iv) $C_n(a,b) \sim_u rI_{n-2} \oplus C$ *for all $n \geq 4$, where*

$$C = \begin{cases} f_q(a',b') \begin{pmatrix} a & 2b \\ 2b & a \end{pmatrix} & \text{if } n-2 = 2q, \\ \big(f_{q+1}(a',b') - b' f_q(a',b')\big) \begin{pmatrix} r & 0 \\ 0 & a+2b \end{pmatrix} & \text{if } n-2 = 2q+1. \end{cases}$$

Proof. Let $X_n(a,b)$ be either one of the matrices $T_n(a,b)$, $P_n(a,b)$, $K_n(a,b)$, or $C_n(a,b)$, then $X_n(a,b) = rX_n(a',b')$. On the other hand, since $r = \gcd(a,b)$ if and only if $1 = \gcd(a',b')$ if and only if the equation $a'x + b'y = 1$ has solution in $\mathbb{A}$. Then, we get the result applying theorem 2.1 to $X_n(a',b')$. □

3 Some applications

In this section we will apply the equivalences of the matrices $K_n(a,b)$, $C_n(a,b)$, $P_n(a,b)$ and $T_n(a,b)$ obtained in the previous section to the cases when a and b are in the ring of integers and in the subring $K_n(\mathbb{A})$ of matrices of the form $K_n(a,b)$ where a,b are in a commutative ring with identity $\mathbb{A}$.

At this point we need to introduce some definitions. Given a simple graph $\mathcal{G}$ and a natural number $l \geq 1$, the l-duplication of $\mathcal{G}$, denoted by $\mathcal{G}(l)$, is the multigraph obtained from $\mathcal{G}$ when we replace every edge of $\mathcal{G}$ by l parallel edges. Note that $L(\mathcal{G}(l)) = lL(\mathcal{G})$ for any graph $\mathcal{G}$.



Figure 1: The simple cycle C_4 and its 2-duplication.

Given a graph $\mathcal{G}$ and a natural number k, the k-cone of $\mathcal{G}$, denoted by $c_k(\mathcal{G})$, is the multigraph obtained from $\mathcal{G}$ by adding a new vertex s and adding k parallel edges between s and all the vertices of $\mathcal{G}$. Note that $L(c_k(\mathcal{G}), s) = L(\mathcal{G}) + kI_{|V(\mathcal{G})|}$ for any graph $\mathcal{G}$.



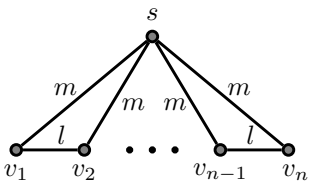
Figure 2: The cycle C_4 and its 1-cone.

The most direct application of the results obtained in the first section is when the base ring $\mathbb{A}$ is the ring of integers and the matrices are the reduced Laplacian matrix of the n-cone of the thick path, the thick cycle and the thick complete graph.

Corollary 3.1. *For all $m \geq 0$, $l \geq 1$, and $n \geq 2$, let $c_m(\mathcal{P}_n(l))$ be the m-cone of the thick path with all the edges with multiplicities equal to l. Then*

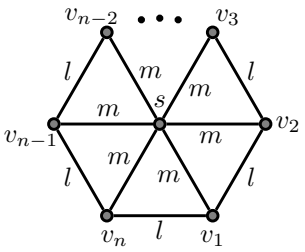
$$K(c_m(\mathcal{P}_n(l))) = \mathbb{Z}_r^{n-1} \oplus \mathbb{Z}_{mf_{n-1}(m+2l,-l)/r^{n-1}},$$

where $r = \gcd(l,m)$.



Proof. Since the reduced Laplacian matrix of $c_m(\mathcal{P}_n(l))$, $L(c_m(\mathcal{P}_n(l)), s)$, is equal to $P_n(m+2l, -l)$ and $\gcd(m+2l, -l) = \gcd(l, m) = r$, then by corollary 2.7 (ii) we get the result. □

Corollary 3.2. *For all $m \geq 0$, $l \geq 1$, and $n \geq 4$, let $c_m(\mathcal{C}_n(l))$ be the m-cone of*



the thick cycle where all the edges has multiplicity equal to l. Then

$$K(c_m(\mathcal{C}_n(l))) = \begin{cases} \mathbb{Z}_r^{n-2} \oplus \mathbb{Z}_{rs_q} \oplus \mathbb{Z}_{ms_q} & \text{if } n-2 = 2q+1, \\ \mathbb{Z}_r^{n-2} \oplus \mathbb{Z}_{rt_q} \oplus \mathbb{Z}_{m(m+4l)t_q/r} & \text{if } n-2 = 2q \text{ and } m/r \text{ is odd}, \\ \mathbb{Z}_r^{n-2} \oplus \mathbb{Z}_{2rt_q} \oplus \mathbb{Z}_{m(m+4l)t_q/2r} & \text{if } n-2 = 2q \text{ and } m/r \text{ is even}, \end{cases}$$

where $r = \gcd(l, m)$, $s_q = (f_{q+1}(m+2l, -l) + l f_q(m+2l, -l))/r^{q+1}$, and $t_q = f_q(m+2l, -l)/r^q$.

Proof. Since $L(c_m(\mathcal{C}_n(l)), s)$ is equal to $C_n(m+2l, -l)$ and $\gcd(m+2l, -l) = \gcd(l, m) = r$, then by corollary 2.7 (iv), $C_n(m+2l, -l) \sim_u rI_{n-2} \oplus C$, where

$$C = \begin{cases} f_q(m+2l, -l)/r^q \begin{pmatrix} m+2l & -2l \\ -2l & m+2l \end{pmatrix} & \text{if } n-2 = 2q, \\ \big(f_{q+1}(m+2l, -l) + l f_q(m+2l, -l)\big)/r^{q+1} \begin{pmatrix} r & 0 \\ 0 & m \end{pmatrix} & \text{if } n-2 = 2q+1. \end{cases}$$

Finally,

$$\begin{pmatrix} m+2l & -2l \\ -2l & m+2l \end{pmatrix} \sim_u \begin{cases} \begin{pmatrix} r & 0 \\ 0 & (m^2+4ml)/r \end{pmatrix} & \text{if } m/r \text{ is odd}, \\ \begin{pmatrix} 2r & 0 \\ 0 & (m^2+4ml)/2r \end{pmatrix} & \text{if } m/r \text{ is even}. \end{cases}$$

□

Corollary 3.3. *For all $m \geq 0$, $l \geq 1$, and $n \geq 4$, let $c_m(\mathcal{K}_n(l))$ be the m-cone of the thick complete graph where all the edges have multiplicity equal to l. Then*

$$K(c_m(\mathcal{K}_n(l))) = \mathbb{Z}_r \oplus \mathbb{Z}_{m+nl}^{n-2} \oplus \mathbb{Z}_{m(m+nl)/r},$$

where $r = \gcd(l, m)$.

Proof. Since the reduced Laplacian matrix, $L(c_m(\mathcal{K}_n(l)), s)$, is equal to $K_n(m + nl, -l)$ and $\gcd(m + nl, -l) = \gcd(l, m) = r$, then by corollary 2.7 (*iii*) we get the result. □

3.1 The subring $K_n(\mathbb{A})$

In this part we will turn our attention to the case when $\mathbb{A}$ is the subring of matrices given by

$$K_n(\mathbb{A}) = \{K_n(a, b) \,|\, a, b \in \mathbb{A}\} \subset M_n(\mathbb{A}).$$

At first, we will prove that $K_n(\mathbb{A})$ is a subalgebra of $M_n(\mathbb{A})$.

Lemma 3.4. *If $a, b, c, d, \alpha \in \mathbb{A}$, then*

(i) $\alpha \cdot K_n(a, b) = K_n(\alpha \cdot a, \alpha \cdot b)$,

(ii) $K_n(a, b) + K_n(c, d) = K_n(a + c, b + d)$,

(iii) $K_n(a, b) \cdot K_n(c, d) = K_n(ac, ad + bc + nbd)$,

(iv) $K_n(a, b)^m = K_n(a^m, p_m(a, b))$,

where the polynomials $p_{m,n}(x, y) \in \mathbb{A}[x, y]$ satisfy the recurrence relation

$$p_m^n(x, y) = (x + ny)p_{m-1}^n(x, y) + yx^{m-1}$$

with initial value $p_0^n(x, y) = 0$.

Proof. The parts (*i*) and (*ii*) are straightforward. (*iii*) Since $A(\mathcal{K}_n)^2 = (n-1)I_n + (n-2)A(\mathcal{K}_n)$, then $K_n(a, b) \cdot K_n(c, d) = ((a+b)I_n + bA(\mathcal{K}_n)) \cdot ((c+d)I_n + dA(\mathcal{K}_n)) = (ac + ad + bc + nbd)I_n + (ad + bc + nbd)A(\mathcal{K}_n) = K_n(ac, ad + bc + nbd)$.

(*iv*) We will use induction on m. The result is clear for $m = 1$ because $p_1^n(a, b) = (a + nb)p_0^n(a, b) + ba^0 = b$. Assume that the result is true for all the natural numbers less or equal to $m - 1$. Thus

$$\begin{aligned} K_n(a, b)^m &= K_n(a, b)^{m-1} \cdot K_n(a, b) = K_n(a^{m-1}, p_{m-1}^n(a, b)) \cdot K_n(a, b) \\ &= K_n(a^m, (a + nb)p_{m-1}^n(a, b) + ba^{m-1}) = K_n(a^m, p_m^n(a, b)). \end{aligned}$$

□

Remark 3.5. *Note that $K_n(\mathbb{A})$ is a commutative ring with identity because $K_n(1,0) = I_n$ for all $n \in \mathbb{N}$. On the other hand, since $K_n(0,1)K_n(-n,1) = 0 = K_n(0,0)$, then $K_n(\mathbb{A})$ is not a principal ideal domain because it has zero divisors.*

Remark 3.6. *Using induction on n is not difficult to see that*

$$p_m^n(x,y) = \sum_{i=1}^{m} n^{i-1}\binom{m}{i}x^{m-i}y^i.$$

Also, since $p_m^n(n,-1) = (n-n)p_{m-1}^n(n,-1) - (n)^{m-1} = -n^{m-1}$, then

$$K_n(n,-1)^m = n^{m-1}K_n(n,-1) \text{ for all } m \in \mathbb{N}.$$

Now, we will apply theorem 2.1 (*iii*) when the base ring is the subring of matrices with entries in $K_n(\mathbb{A})$ to obtain a theorem that will be a powerful tool to calculate the critical group of several graphs.

Given $A = (a_{i,j}), B = (b_{i,j}) \in M_n(\mathbb{A})$ and $m \geq 2$, let

$$\Phi_m(A,B) = \begin{pmatrix} K_m(a_{1,1},b_{1,1}) & \cdots & K_m(a_{1,n},b_{1,n}) \\ \vdots & \ddots & \vdots \\ K_m(a_{n,1},b_{n,1}) & \cdots & K_m(a_{n,n},b_{n,n}) \end{pmatrix} \in M_n(K_m(\mathbb{A})) \subseteq M_{nm}(\mathbb{A}).$$

As the next theorem will show, if a matrix has the block structure of $\Phi_m(A,B)$, then we can get a simpler equivalent matrix.

Theorem 3.7. *Let $A = (a_{i,j}), B = (b_{i,j}) \in M_n(\mathbb{A})$ and $n \geq 2$, then*

$$\Phi_m(A,B) \sim_e \left[\bigoplus_{i=1}^{m-2} A\right] \oplus \begin{pmatrix} A & B \\ 0 & A+mB \end{pmatrix}$$

Proof. Let P_n and Q_n be as in claim 2.3, then $I_m \otimes P_n, I_m \otimes Q_n \in M_{nm}(\mathbb{A})$ are elementary matrices and

$$\begin{aligned} (I_m \otimes P_n)\cdot\Phi_m(A,B)\cdot(I_m \otimes Q_n) &= \begin{pmatrix} D_m(a_{1,1},b_{1,1}) & \cdots & D_m(a_{1,n},b_{1,n}) \\ \vdots & \ddots & \vdots \\ D_m(a_{n,1},b_{n,1}) & \cdots & D_m(a_{n,n},b_{n,n}) \end{pmatrix} \\ &\sim_e \left[\bigoplus_{i=1}^{m-2} A\right] \oplus \begin{pmatrix} A & B \\ 0 & A+mB \end{pmatrix}, \end{aligned}$$

where $D_m(a_{i,j},b_{i,j}) = \begin{pmatrix} a_{i,j} & b_{i,j} \\ -ma_{i,j} & a_{i,j} \end{pmatrix} \oplus a_{i,j}I_{m-2}$. □

In the last part of this paper, we will use theorem 3.7 in order to find the critical group of some graphs whose Laplacian matrix is given by $\Phi_m(A, B)$ for some $A, B \in M_n(\mathbb{A})$. The simplest case is when $n = 2$. If $\Phi_m(A, B)$ is the Laplacian matrix of a graph, then the vertex set of the graph can be partitioned in two sets and the incidence structure between these sets is given by a matrix in $K_m(\mathbb{Z})$. In this sense, we will define the following families of graphs:

Let $U = \{u_1, u_2, \ldots, u_m\}$, $V = \{v_1, v_2, \ldots, v_m\}$, and $\mathcal{X}_{y,z}$ be the graph with $U \cup V$ as vertex set and edge set equal to $E_y \cup E_z \cup E_{y,z}$, where

$$E_y = \begin{cases} \emptyset & \text{if } y = m, \\ u_i u_j \text{ for all } i \neq j \in \{1, 2, \cdots, m\} & \text{if } y = M, \end{cases}$$

similarly for E_z, and

$$E_{y,z} = \begin{cases} u_i u'_j \text{ for all } i, j \in \{1, 2, \cdots, m\} & \text{if } \mathcal{X} = \mathcal{K}, \\ u_i u'_j \text{ for all } i \neq j \in \{1, 2, \cdots, m\} & \text{if } \mathcal{X} = \mathcal{L}, \\ u_i u'_i \text{ for all } i \in \{1, 2, \cdots, m\} & \text{if } \mathcal{X} = \mathcal{M}. \end{cases}$$

Note that $\mathcal{K}_{m,m}$ is the bipartite complete graph with m vertices in each partition and $\mathcal{L}_{m,m}$ is the bipartite complete graph with $2m$ vertices minus a matching. The Laplacian matrix of all these graphs can be represented by $\Phi_m(A, B)$ for some two by two matrices A and B.

For instance, the graph $\mathcal{M}_{M,m}$ is illustrated in figure 3.

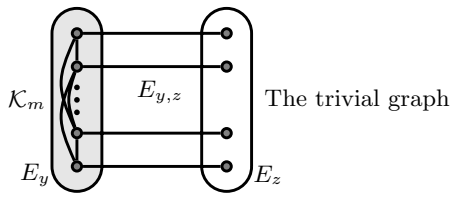


Figure 3: The graph $\mathcal{M}_{M,m}$.

Before using theorem 3.7 in order to calculate the critical group of the n-cones of graphs $\mathcal{X}_{y,z}$, we will introduce a theorem that gives an equivalent matrix of $\mathcal{X}_{y,z}$ when the base ring is a general commutative ring with identity.

Corollary 3.8. *Let $m \geq 2$, $a, b \in \mathbb{A}$ such that the equation $ax + by = 1$ has a solution in $\mathbb{A}$, and $X_{y,y}(a, b) = aI_{2m} + bA(\mathcal{X}_{y,y})$. Then*

(i) $K_{m,m}(a, b) \sim_u I_2 \oplus aI_{2(m-2)} \oplus a \begin{pmatrix} a & mb \\ mb & a \end{pmatrix}$,

(ii) $L_{m,m}(a, b) \sim_u I_m \oplus (a^2 - b^2)I_{m-2} \oplus \begin{pmatrix} a^2 & (m-2)ab \\ (m-2)ab & a^2 - (m-1)b^2 \end{pmatrix}$,

(iii) $L_{M,M}(a,b) \sim_u I_{m-1} \oplus a(a-2b)I_{m-2} \oplus \begin{pmatrix} a(a-2b) & ab & 0 \\ 0 & a+2(m-1)b & 0 \\ 0 & (m-1)b & a \end{pmatrix}$,

(iv) $M_{M,M}(a,b) \sim_u I_{m+1} \oplus a(a-2b)I_{m-2} \oplus \begin{pmatrix} a(a-2b) & -b^2(2a+(m-2)b) \\ 0 & (a+(m-1)b)^2-b^2 \end{pmatrix}$.

Proof. (i) Since $K_{m,m}(a,b) = \Phi_m(A,B)$ for $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$, then by theorem 3.7

$$K_{m,m}(a,b) \sim_e \left[\bigoplus_{i=1}^{m-2} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right] \oplus \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & 0 & a & mb \\ 0 & 0 & mb & a \end{pmatrix}.$$

Moreover, since

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & 0 & a & mb \\ 0 & 0 & mb & a \end{pmatrix} \begin{pmatrix} x & 0 & 0 & -b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y & 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ mby & 0 & a & mab \\ ay & 0 & mb & a^2 \end{pmatrix}$$

$$\sim_e \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & mab \\ 0 & 0 & mb & a^2 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b & 0 \\ 0 & a & mab \\ 0 & mb & a^2 \end{pmatrix} \begin{pmatrix} x & -b & 0 \\ y & a & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ ay & a^2 & mab \\ mby & mab & a^2 \end{pmatrix}$$

$$\sim_e \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^2 & mab \\ 0 & mab & a^2 \end{pmatrix},$$

then

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & 0 & a & mb \\ 0 & 0 & mb & a \end{pmatrix} \sim_u I_2 \oplus a \begin{pmatrix} a & mb \\ mb & a \end{pmatrix}.$$

Hence

$$K_{m,m}(a,b) \sim_u I_2 \oplus aI_{2(m-2)} \oplus a \begin{pmatrix} a & mb \\ mb & a \end{pmatrix}.$$

(*ii*) Since $L_{m,m}(a,b) = \Phi_m(A,B)$ for $A = \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$, then by theorem 3.7

$$L_{m,m}(a,b) \sim_e \left[\bigoplus_{i=1}^{m-2} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}\right] \oplus \begin{pmatrix} a & -b & 0 & b \\ -b & a & b & 0 \\ 0 & 0 & a & (m-1)b \\ 0 & 0 & (m-1)b & a \end{pmatrix}.$$

Moreover, since

$$\begin{pmatrix} a & -b \\ -b & a \end{pmatrix}\begin{pmatrix} x & b \\ -y & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -by-ay & a^2-b^2 \end{pmatrix} \sim_e \begin{pmatrix} 1 & 0 \\ 0 & a^2-b^2 \end{pmatrix},$$

$$\begin{pmatrix} a & -b & 0 & b \\ -b & a & b & 0 \\ 0 & 0 & a & (m-1)b \\ 0 & 0 & (m-1)b & a \end{pmatrix} \sim_e \begin{pmatrix} a & 0 & 0 & b \\ -b & a & b & 0 \\ 0 & (m-1)b & a & (m-1)b \\ 0 & a & (m-1)b & a \end{pmatrix},$$

$$\begin{pmatrix} a & 0 & 0 & b \\ -b & a & b & 0 \\ 0 & (m-1)b & a & (m-1)b \\ 0 & a & (m-1)b & a \end{pmatrix}\begin{pmatrix} x & 0 & 0 & -b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y & 0 & 0 & a \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ * & a & b & b^2 \\ * & (m-1)b & a & (m-1)ab \\ * & a & (m-1)b & a^2 \end{pmatrix} \sim_e I_1 \oplus \begin{pmatrix} a & b & 0 \\ (m-1)b & a & (m-2)ab \\ a & (m-1)b & a^2-(m-1)b^2 \end{pmatrix},$$

and

$$\begin{pmatrix} a & b & 0 \\ (m-1)b & a & (m-2)ab \\ a & (m-1)b & a^2-(m-1)b^2 \end{pmatrix}\begin{pmatrix} x & -b & 0 \\ y & a & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & a^2 & (m-2)ab \\ * & (m-2)ab & a^2-(m-1)b^2 \end{pmatrix} \sim_e I_1 \oplus \begin{pmatrix} a^2 & (m-2)ab \\ (m-2)ab & a^2-(m-1)b^2 \end{pmatrix},$$

then

$$L_{m,m}(a,b) \sim_u I_m \oplus (a^2-b^2)I_{m-2} \oplus \begin{pmatrix} a^2 & (m-2)ab \\ (m-2)ab & a^2-(m-1)b^2 \end{pmatrix}.$$

(*iii*) Since $L_{M,M}(a,b) = \Phi_m(A,B)$ for $A = \begin{pmatrix} a-b & -b \\ -b & a-b \end{pmatrix}$ and $B = \begin{pmatrix} b & b \\ b & b \end{pmatrix}$, then by theorem 3.7

$$L_{M,M}(a,b) \sim_e \left[\bigoplus_{i=1}^{m-2} \begin{pmatrix} a-b & -b \\ -b & a-b \end{pmatrix}\right] \oplus \begin{pmatrix} a-b & -b & b & b \\ -b & a-b & b & b \\ 0 & 0 & a+(m-1)b & (m-1)b \\ 0 & 0 & (m-1)b & a+(m-1)b \end{pmatrix}.$$

Moreover, since

$$\begin{pmatrix} x & -(x+y) \\ b & a-b \end{pmatrix}\begin{pmatrix} a-b & -b \\ -b & a-b \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & a(a-2b) \end{pmatrix} \sim_e \begin{pmatrix} 1 & 0 \\ 0 & a(a-2b) \end{pmatrix},$$

$$\begin{pmatrix} x & -(x+y) & 0 \\ b & a-b & 0 \\ 0 & 0 & I_2 \end{pmatrix}\begin{pmatrix} a-b & -b & b & b \\ -b & a-b & b & b \\ 0 & 0 & a+(m-1)b & (m-1)b \\ 0 & 0 & (m-1)b & a+(m-1)b \end{pmatrix} =$$

$$\begin{pmatrix} 1 & * & * & * \\ 0 & a(a-2b) & ab & ab \\ 0 & 0 & a+(m-1)b & (m-1)b \\ 0 & 0 & (m-1)b & a+(m-1)b \end{pmatrix},$$

and

$$\begin{pmatrix} a(a-2b) & ab & ab \\ 0 & a+(m-1)b & (m-1)b \\ 0 & (m-1)b & a+(m-1)b \end{pmatrix} \sim_e \begin{pmatrix} a(a-2b) & ab & 0 \\ 0 & a+2(m-1)b & 0 \\ 0 & (m-1)b & a \end{pmatrix},$$

then the result is followed.

(*iv*) Since $M_{M,M}(a,b) = \Phi_m(A,B)$ for $A = \begin{pmatrix} a-b & b \\ b & a-b \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$, then by theorem 3.7

$$M_{M,M}(a,b) \sim_e \left[\bigoplus_{i=1}^{m-2}\begin{pmatrix} a-b & b \\ b & a-b \end{pmatrix}\right] \oplus \begin{pmatrix} a-b & b & b & 0 \\ b & a-b & 0 & b \\ 0 & 0 & a+(m-1)b & b \\ 0 & 0 & b & a+(m-1)b \end{pmatrix}.$$

Moreover, since

$$\begin{pmatrix} x & x+y & 0 \\ -b & a-b & 0 \\ 0 & 0 & I_2 \end{pmatrix}\begin{pmatrix} a-b & b & b & 0 \\ b & a-b & 0 & b \\ 0 & 0 & a+(m-1)b & b \\ 0 & 0 & b & a+(m-1)b \end{pmatrix} =$$

$$\begin{pmatrix} 1 & * & * & * \\ 0 & a(a-2b) & -b^2 & (a-b)b \\ 0 & 0 & a+(m-1)b & b \\ 0 & 0 & b & a+(m-1)b \end{pmatrix}$$

and

$$\begin{pmatrix} a(a-2b) & -b^2 & (a-b)b \\ 0 & a+(m-1)b & b \\ 0 & b & a+(m-1)b \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & a+(m-1)b & y-(m-1)x \\ 0 & -b & x \end{pmatrix} =$$

$$\begin{pmatrix} a(a-2b) & -b^2(2a+(m-2)b) & * \\ 0 & (a+(m-1)b)^2-b^2 & * \\ 0 & 0 & 1 \end{pmatrix},$$

then the result is followed. □

Remark 3.9. *Note that $\mathcal{K}_{M,M}$ is the complete graph with $2m$ vertices and $\mathcal{M}_{m,m}$ is the disjoint union of m copies of $\mathcal{K}_2$.*

When a graph $\mathcal{G}$ is not regular, then there is not a straightforward way to define their matrix $G(a,b)$. Thus, we will define $K_{m,M}(a,b)$ as $a(I_m \oplus 2I_m) + bA(\mathcal{K}_{m,M})$, $L_{m,M}(a,b)$ as $a(I_m \oplus 2I_m) + bA(\mathcal{L}_{m,M})$, and $M_{m,M}(a,b)$ as $a(I_m \oplus (m+1)I_m) + bA(\mathcal{M}_{m,M})$. Now, we have the following equivalent matrices of $K_{m,M}(a,b)$, $L_{m,M}(a,b)$, and $M_{m,M}(a,b)$.

Corollary 3.10. *Let $m \geq 2$, $a, b \in \mathbb{A}$ such that the equation $ax + by = 1$ has a solution in $\mathbb{A}$, then*

(i) $K_{m,M}(a,b) \sim_u I_2 \oplus aI_{m-2} \oplus 2aI_{m-2} \oplus a \begin{pmatrix} 2a & -(a-mb)b \\ 2m & 2a \end{pmatrix}$,

(ii) $M_{m,M}(a,b) \sim_u I_m \oplus uI_{m-2} \oplus \begin{pmatrix} u & b \\ 0 & (a+b)(b-(m+1)a) \end{pmatrix}$,

(iii) $L_{m,M}(a,b) \sim_u I_{m-1} \oplus (2a^2-b^2)I_{m-2} \oplus \begin{pmatrix} 2a^2-b^2 & 0 & ab+mb^2 \\ 0 & a & (m-1)b \\ 0 & (m-1)b & 2a+mb \end{pmatrix}$,

where $u = (m+1)a^2 - b^2$.

Proof. (i) Since $K_{m,M}(a,b) = \Phi_m(A,B)$ for $A = \begin{pmatrix} a & 0 \\ 0 & 2a \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b \\ b & b \end{pmatrix}$, then by theorem 3.7

$$K_{m,M}(a,b) \sim_e \left[\bigoplus^{m-2} \begin{pmatrix} a & 0 \\ 0 & 2a \end{pmatrix}\right] \oplus \begin{pmatrix} a & 0 & 0 & b \\ 0 & 2a & b & b \\ 0 & 0 & a & mb \\ 0 & 0 & mb & 2a+mb \end{pmatrix}.$$

Moreover,

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & 2a & b & b \\ 0 & 0 & a & mb \\ 0 & 0 & mb & 2a+mb \end{pmatrix} \begin{pmatrix} x & 0 & 0 & -b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y & 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 2a & b & ab \\ * & 0 & a & mab \\ * & 0 & mb & 2a^2+mab \end{pmatrix}$$

$$\sim_e \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 2a & ab \\ 0 & a & 0 & mab \\ 0 & 0 & -2ma & 2a^2 \end{pmatrix}$$

and

$$\begin{pmatrix} y & x & 0 \\ -a & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b & 2a & ab \\ a & 0 & mab \\ 0 & -2ma & 2a^2 \end{pmatrix} = \begin{pmatrix} 1 & * & * \\ 0 & -2a^2 & mab^2 - a^2b \\ 0 & -2ma & 2a^2 \end{pmatrix}$$

$$\sim_e I_1 \oplus \begin{pmatrix} 2a^2 & ab(mb-a) \\ 2ma & 2a^2 \end{pmatrix}.$$

(*ii*) Since $M_{m,M}(a,b) = \Phi_m(A,B)$ for $A = \begin{pmatrix} a & b \\ b & (m+1)a \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, then by theorem 3.7

$$M_{m,M}(a,b) \sim \left[\bigoplus^{m-2} \begin{pmatrix} a & b \\ b & (m+1)a \end{pmatrix}\right] \oplus \begin{pmatrix} a & b & 0 & 0 \\ b & (m+1)a & 0 & b \\ 0 & 0 & a & b \\ 0 & 0 & b & (m+1)a+mb \end{pmatrix}$$

Moreover,

$$\begin{pmatrix} a & b \\ b & (m+1)a \end{pmatrix}\begin{pmatrix} x & -b \\ y & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & (m+1)a^2-b^2 \end{pmatrix} \sim_e \begin{pmatrix} 1 & 0 \\ 0 & (m+1)a^2-b^2 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b & 0 & 0 \\ b & (m+1)a & 0 & b \\ 0 & 0 & a & b \\ 0 & 0 & b & (m+1)a+mb \end{pmatrix} \sim_u \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & u & 0 & b \\ * & 0 & a & b \\ * & 0 & b & (m+1)a+mb \end{pmatrix}$$

$$\sim_e I_1 \oplus \begin{pmatrix} b & (m+1)a+mb & 0 \\ a & b & 0 \\ 0 & b & u \end{pmatrix}$$

$$\sim_u I_2 \oplus \begin{pmatrix} (a+b)(b-(m+1)a) & 0 \\ b & u \end{pmatrix},$$

where $u = (m+1)a^2 - b^2$.

(*iii*) Since $L_{m,M}(a,b) = \Phi_m(A,B)$ for $A = \begin{pmatrix} a & -b \\ -b & 2a \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b \\ b & b \end{pmatrix}$, then by theorem 3.7

$$L_{m,M}(a,b) \sim \left[\bigoplus^{m-2} \begin{pmatrix} a & -b \\ -b & 2a \end{pmatrix}\right] \oplus \begin{pmatrix} a & -b & 0 & b \\ -b & 2a & b & b \\ 0 & 0 & a & (m-1)b \\ 0 & 0 & (m-1)b & 2a+mb \end{pmatrix}.$$

Moreover,

$$\begin{pmatrix} x & -y \\ b & a \end{pmatrix}\begin{pmatrix} a & -b \\ -b & 2a \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 2a^2-b^2 \end{pmatrix} \sim_e \begin{pmatrix} 1 & 0 \\ 0 & 2a^2-b^2 \end{pmatrix},$$

$$\begin{pmatrix} a & -b & 0 & b \\ -b & 2a & b & b \\ 0 & 0 & a & (m-1)b \\ 0 & 0 & (m-1)b & 2a+mb \end{pmatrix} \sim_u \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2a^2-b^2 & ab & ab+b^2 \\ 0 & 0 & a & (m-1)b \\ 0 & 0 & (m-1)b & 2a+mb \end{pmatrix},$$

and

$$\begin{pmatrix} 2a^2-b^2 & ab & ab+b^2 \\ 0 & a & (m-1)b \\ 0 & (m-1)b & 2a+mb \end{pmatrix} \sim_e \begin{pmatrix} 2a^2-b^2 & 0 & ab+mb^2 \\ 0 & a & (m-1)b \\ 0 & (m-1)b & 2a+mb \end{pmatrix}.$$

□

The next corollaries calculate the critical group of the n-cone of the graphs $\mathcal{X}_{y,z}$.

Corollary 3.11. *Let $m \geq 2$, $l \geq 1$, and $n \geq 0$, then*

$$K(c_n(\mathcal{K}_{m,m}(l))) = \mathbb{Z}_r^2 \oplus \mathbb{Z}_{n+ml}^{2(m-2)} \oplus \mathbb{Z}_{(n+ml)s/r} \oplus \mathbb{Z}_{n(n+ml)(n+2ml)/rs},$$

where $r = \gcd(l,n)$ and $s = \gcd(ml,n)$.

Proof. Since $L(c_n(\mathcal{K}_{m,m}(l)),s) = K_{m,m}(n+ml,-l)$ and $K_{m,m}(n+ml,-l) = rK_{m,m}((n+ml)/r,-l/r)$ with $\gcd((n+ml)/r,-l/r) = 1$, then applying corollary 3.10 (i) to $K_{m,m}((n+ml)/r,-l/r)$ we get that

$$K_{m,m}(n+ml,-l) \sim_u rI_2 \oplus (n+ml)I_{2(m-2)} \oplus ((n+ml)/r) \begin{pmatrix} n+ml & -ml \\ -ml & n+ml \end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix} n+ml & -ml \\ -ml & n+ml \end{pmatrix} \sim_u \begin{pmatrix} s & 0 \\ 0 & n(n+2ml)/s \end{pmatrix},$$

where $s = \gcd(ml,n)$ and we get the result. □

Remark 3.12. *Note that $\mathcal{K}_{m,m}$ is the complete bipartite graph with m vertices in each partition. Lorenzini in [15] calculated that*

$$K(\mathcal{K}_{m,m}) = \mathbb{Z}_m^{2(m-2)} \oplus \mathbb{Z}_{m^2},$$

which agrees with the corollary 3.11 for $l = 1$ and $n = 0$. Also note that $K(c_n(\mathcal{K}_{m,m}(l)))$ has $2m-2$ invariant factors different to 1.

Corollary 3.13. *Let $m \geq 3$, $l \geq 1$, and $n \geq 0$, then*

$$K(c_n(\mathcal{L}_{m,m}(l))) = \mathbb{Z}_r^m \oplus \mathbb{Z}_{(s^2-l^2)/r}^{m-2} \oplus \mathbb{Z}_{rt} \oplus \mathbb{Z}_{u/r^3t},$$

where $r = \gcd(l,n)$, $s = n+(m-1)l$, $t = \gcd(m-1,n)/\gcd(l,m-1,n)$, and $u = s^2(n^2+2n(m-1)l+(m-2)l^2)$.

Proof. In a similar way that in corollary 3.11, $L(c_n(\mathcal{L}_{m,m}(l)),s) = L_{m,m}(n+(m-1)l,-l)$ and applying corollary 3.10 (ii) to $L_{m,m}(a/r,b/r)$ with $a = n+(m-1)l$ and $b = -l$

$$L_{m,m}(n+(m-1)l,-l) \sim_u rI_m \oplus (s^2-l^2)/rI_{m-2} \oplus \begin{pmatrix} a^2/r & (m-2)ab/r \\ (m-2)ab/r & v/r \end{pmatrix},$$

where $s = n+(m-1)l$ and $v = a^2-(m-1)b^2$.

On the other hand, it is not difficult to see that

$$\begin{pmatrix} a^2/r & (m-2)ab/r \\ (m-2)ab/r & (a^2-(m-1)b^2)/r \end{pmatrix} \sim_u \begin{pmatrix} rt & 0 \\ 0 & u/r^3t \end{pmatrix},$$

where $t = \gcd(m-1,n)/\gcd(l,m-1,n)$ and $u = s^2(n^2+2n(m-1)l+(m-2)l^2)$. □

Corollary 3.14. *Let $m \geq 2$, $l \geq 1$, and $n \geq 0$, then*

$$K(c_n(\mathcal{L}_{M,M}(l))) = \mathbb{Z}_r^{m-1} \oplus \mathbb{Z}_{st/r}^{m-2} \oplus \mathbb{Z}_u \oplus \mathbb{Z}_{sv/u} \oplus \mathbb{Z}_{nst/rv},$$

where $r = \gcd(l,n)$, $s = n+2(m-1)l$, $t = n+2ml$, $u = \gcd(n,(m-1)l)$, and $v = \gcd(n,2(m-1)l^2/r)$.

Proof. Since $L(c_n(\mathcal{L}_{M,M}(l)),s) = L_{M,M}(n+2(m-1)l,-l)$ and $r = \gcd(n+2(m-1)l,-l)$, then applying corollary 3.10 (iii) to $L_{M,M}(a/r,b/r)$ with $a = n+2(m-1)l$ and $b = -l$ we get that

$$L_{M,M}(n+2(m-1)l,-l) \sim_u rI_{m-1} \oplus st/rI_{m-2} \oplus \begin{pmatrix} st/r & -sl/r & 0 \\ 0 & n & 0 \\ 0 & -(m-1)l & s \end{pmatrix},$$

where $r = \gcd(l,n)$, $s = n+2(m-1)l$, and $t = n+2ml$.

On the other hand, it is not difficult to see that

$$\begin{pmatrix} st/r & -sl/r & 0 \\ 0 & n & 0 \\ 0 & -(m-1)l & s \end{pmatrix} \sim_u u \oplus sv/u \oplus nst/rv,$$

where $u = \gcd(n,(m-1)l)$ and $v = \gcd(n,2(m-1)l^2/r)$. □

Corollary 3.15. *Let $m \geq 2$, $l \geq 1$, and $n \geq 0$, then*

$$K(c_n(\mathcal{M}_{M,M}(l))) = \mathbb{Z}_r^{m+1} \oplus \mathbb{Z}_{w/r}^{m-2} \oplus \mathbb{Z}_u \oplus \mathbb{Z}_{n(n+2l)w/ur^2},$$

where $r = \gcd(l,n)$, $u = \gcd(n(n+2l),lv(n+t))/r$, $v = \gcd(m,l/r)$, and $w = (n+ml)(n+(m+2)l)$.

Proof. Since $L(c_n(\mathcal{M}_{M,M}(l)), s) = M_{M,M}(n+ml, -l)$ and $r = \gcd(n+ml, -l)$, then applying corollary 3.10 (iii) to $M_{M,M}(a/r, b/r)$ with $a = n+ml$ and $b = -l$ we get that

$$M_{M,M}(n+ml,-l) \sim_u rI_{m+1} \oplus st/rI_{m-2} \oplus \begin{pmatrix} st/r & -l^2(n+t)/r^2 \\ 0 & n(n+2l)/r, \end{pmatrix}$$

where $r = \gcd(l, n)$, $s = n + ml$, and $t = n + (m+2)l$.

On the other hand, it is not difficult to see that

$$\begin{pmatrix} st/r & -l^2(n+t)/r^2 \\ 0 & n(n+2l)/r \end{pmatrix} \sim_u u \oplus stn(n+2l)/ur^2,$$

where $u = \gcd(n(n+2l), lv(n+t))/r$ and $v = \gcd(m, l/r)$. □

Remark 3.16. *Note that $\mathcal{M}_{M,M}(l)$ is the cartesian product of $\mathcal{K}_2(l)$ and $\mathcal{K}_n(l)$. A deep analysis of the cartesian product of matrices can be found in [9].*

Corollary 3.17. *Let $m \geq 2$, $l \geq 1$, and $n \geq 0$, then*

$$K(c_n(\mathcal{K}_{m,M}(l))) = \mathbb{Z}_{ml+n}^{m-2} \oplus \mathbb{Z}_{2ml+n}^{m-2} \oplus \mathbb{Z}_r^2 \oplus \mathbb{Z}_{s(n+2ml)/r^2} \oplus \mathbb{Z}_{n(n+ml)(n+2ml)/s},$$

where $r = \gcd(l, n)$ and $s = \gcd(n^2, mlr)$.

Proof. Since $L(c_n(\mathcal{K}_{m,M}(l))) = \Phi_m(A, B)$ for $A = \begin{pmatrix} ml+n & 0 \\ 0 & 2ml+n \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -l \\ -l & -l \end{pmatrix}$, then the result is followed by theorem 3.10 (iv) because

$$\begin{pmatrix} ml+n & 0 & 0 & -l \\ 0 & 2ml+n & -l & -l \\ 0 & 0 & ml+n & -ml \\ 0 & 0 & -ml & ml+n \end{pmatrix} \sim_u rI_2 \oplus (n+2ml)s/r^2 \oplus u/s,$$

where $r = \gcd(l, n)$, $s = \gcd(n^2, mlr)$ and $u = n(n+ml)(n+2ml)$. □

Remark 3.18. *Note that $\mathcal{K}_{m,M}$ is the graph $\mathcal{K}_{2m} \setminus \mathcal{K}_m$. In general the expression for $K(c_n(\mathcal{K}_{m,M}(l)))$ given in corollary 3.17 does not give us the invariant factors of $K(c_n(\mathcal{K}_{m,M}(l)))$. Also note that $K(c_n(\mathcal{K}_{m,M}(l)))$ has $2m-2$ invariant factors different to 1.*

Corollary 3.19. *Let $m \geq 2$, $l \geq 1$, and $n \geq 0$, then*

$$K(c_n(\mathcal{M}_{m,M}(l))) = \mathbb{Z}_r^{m+1} \oplus \mathbb{Z}_{s/r}^{m-2} \oplus \mathbb{Z}_{n(n+2l)s/r^3},$$

where $r = \gcd(l, n)$ and $s = n^2 + ml^2 + nl(m+2)$.

Proof. Since $L(c_n(\mathcal{M}_{m,M}(l))) = \Phi_m(A,B)$ for $A = \begin{pmatrix} l+n & -l \\ -l & (m+1)l+n \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & -l \end{pmatrix}$, then the result is followed by theorem 3.10 (v) because

$$\begin{pmatrix} l+n & -l \\ -l & (m+1)l+n \end{pmatrix} \sim_u r \oplus s/r$$

and

$$\begin{pmatrix} l+n & -l & 0 & 0 \\ -l & (m+1)l+n & 0 & -l \\ 0 & 0 & l+n & -l \\ 0 & 0 & -l & l+n \end{pmatrix} \sim_u rI_3 \oplus n(n+2l)s/r^3,$$

where $r = \gcd(l,n)$ and $s = n^2 + ml^2 + nl(m+2)$. □

We conclude the article with the critical group of the graph $\mathcal{L}_{m,M}$.

Corollary 3.20. *Let $m \geq 2$, $l \geq 1$, and $n \geq 0$, then*

$$K(\mathcal{L}_{m,M}) = \mathbb{Z}_r^m \oplus \mathbb{Z}_{s/r}^{m-2} \oplus \mathbb{Z}_t \oplus \mathbb{Z}_{n(n+2(m-1)l)s/tr^2},$$

where $r = \gcd(l,n)$, $s = n^2 + (3m-2)nl + m(2m-3)l^2$, and $t = \gcd(n, l^3(m-1)(2m-3)/r^2)$.

Proof. Since $L(c_n(\mathcal{L}_{m,M}(l))) = \Phi_m(A,B)$ for $A = \begin{pmatrix} (m-1)l+n & l \\ l & (2m-1)l+n \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -l \\ -l & -l \end{pmatrix}$, then the result is followed by theorem 3.10 (vi) because

$$\begin{pmatrix} (m-1)l+n & l \\ l & (2m-1)l+n \end{pmatrix} \sim r \oplus s/r,$$

where $r = \gcd(l,n)$, $s = n^2 + (3m-2)nl + m(2m-3)l^2$, and

$$\begin{pmatrix} (m-1)l+n & l & 0 & -l \\ l & (2m-1)l+n & -l & -l \\ 0 & 0 & (m-1)l+n & -(m-1)l \\ 0 & 0 & -(m-1)l & (m-1)l+n \end{pmatrix} \sim_u rI_2 \oplus t \oplus u/tr^2,$$

where $t = \gcd(n, l^3(m-1)(2m-3)/r^2)$ and $u = n(n+2(m-1)l)s$. □

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