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Maximum Degree Distance of Graphs with Exactly Two Cycles

by

Zhibin Du and Bo Zhou

Abstract

The degree distance of a connected graph G with vertex set $V(G)$ is defined as $D'(G) = \sum_{u\in V(G)} d_G(u)D_G(u)$, where $d_G(u)$ is the degree of vertex u and $D_G(u)$ is the sum of distances between u and all vertices of G. We determine the maximum degree distances in the class of connected graphs with exactly two vertex–disjoint cycles and in the class of connected graphs with exactly two cycles of a common vertex, respectively, and then the maximum degree distance in the class of connected graphs with exactly two cycles. The extremal graphs are characterized.

Key Words: Degree, distance, Wiener index, cycle, graph.
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1 Introduction

The topological indices are numbers associated with chemical structures via their hydrogen–depleted graphs. The topological indices especially those based on graph distance are widely used in modeling of structure–property relationships [2, 11].

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, let $d_G(u, v)$ be the distance between the vertices u and v in G, let $D_G(u)$ be the sum of distances between u and all vertices of G, i.e., $D_G(u) = \sum_{v\in V(G)} d_G(u, v)$. For $u \in V(G)$, let $d_G(u)$ be the degree of u in G. The *degree distance* of G is defined as [5, 6]

$$D'(G) = \sum_{u\in V(G)} d_G(u)D_G(u).$$

Besides as a topological index itself, the degree distance is also the non-trivial part of the molecular topological index (MTI) introduced by Schultz [10] for characterization of alkanes [9, 6, 8]. Some properties for the degree distance may be found, e.g., in [6, 8, 15] in the text of MTI.

The Wiener index of G is defined as [3, 4] $W(G) = \frac{1}{2} \sum_{u \in V(G)} D_G(u)$. Gutman [6] showed that if T is a tree with n vertices, then $D'(T) = 4W(T) - n(n-1)$. Thus, the study of the degree distance for trees is equivalent to the study of the Wiener index, which was summarized in [3].

A connected graph with n vertices is said to be unicyclic for $n \geq 3$ if it possesses n edges and bicyclic for $n \geq 4$ if it possesses $n+1$ edges. I. Tomescu [13] showed that the star is the unique graph with the minimum degree distance in the class of connected graphs with n vertices. A. I. Tomescu [12] characterized the unicyclic and bicyclic graphs with the minimum degree distances. I. Tomescu [14] gave properties of the graphs with the minimum degree distance in the class of connected graphs with n vertices and $m \geq n-1$ edges, which were determined recently by Bucicovschi and Cioabă [1]. Hou and Chang [7] characterized the unicyclic graph(s) with the maximum degree distance.

Let $\mathcal{B}_1(n)$ for $n \geq 6$ be the class of connected graphs on n vertices with exactly two vertex–disjoint cycles. Let $\mathcal{B}_2(n)$ for $n \geq 5$ be the class of connected graphs on n vertices with exactly two cycles of a common vertex. Obviously, the graphs in $\mathcal{B}_1(n)$ or $\mathcal{B}_2(n)$ are bicyclic graphs, and $\mathcal{B}_2(5)$ contains only the graph consisting of two triangles of a common vertex. In this paper, we determine the maximum degree distances in $\mathcal{B}_1(n)$, $\mathcal{B}_2(n)$ and $\mathcal{B}_1(n) \cup \mathcal{B}_2(n)$, respectively, for $n \geq 6$. We also characterize the extremal graphs.

2 Preliminaries

For edge subset E_1 of the graph G (the complement of G, respectively), $G - E_1$ ($G + E_1$, respectively) denotes the graph resulting from G by deleting (adding, respectively) the edges in E_1. A pendant vertex is a vertex of degree one. Let P_n be the path on n vertices.

For vertex–disjoint connected graphs Q_1 and Q_2 with $|V(Q_1)|, |V(Q_2)| \geq 2$, $x \in V(Q_1)$, $y \in V(Q_2)$ and integer $r \geq 1$, let H be the graph obtained from Q_1 and Q_2 by identifying x and y, and attaching a path P_r (at an end vertex) to this common vertex, and H_1 the graph obtained from Q_1 and Q_2 by joining x and y by a path of length r. Gutman [6] proved that $D'(H_1) > D'(H)$.

Lemma 1. *Let G and G^* be the graphs in Fig. 1, where M and N are vertex–disjoint connected graphs, T is a tree with $k \geq 3$ vertices, $V(M) \cap V(T) = \{u\}$, $V(N) \cap V(T) = \{v\}$, G^* is formed from G by setting the tree T to be P_k with end vertices u and v. Suppose that $G \neq G^*$.*

(i) If $V(N) = \{v\}$, then $D'(G) < D'(G^*)$.

(ii) If $|V(M)|,\ |V(N)| \geq 2$, then $D'(G) < D'(G^*)$.

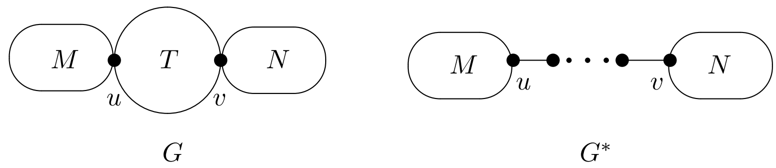

Fig. 1. The graphs G and G^*.

Proof: Note that T can not be a path from u to v. By proper choosing of Q_1 and Q_2, and applying the transformation from H to H_1 repeatedly, the results (i) and (ii) follow from Gutman's result mentioned above. □

Let G and H be connected graphs. Let $V_1(G) = \{x \in V(G) : d_G(x) \neq 2\}$. Then

$$\begin{aligned} D'(H) - D'(G) &= 4[W(H) - W(G)] + \sum_{x \in V_1(H)} (d_H(x) - 2) D_H(x) \\ &\quad - \sum_{x \in V_1(G)} (d_G(x) - 2) D_G(x), \end{aligned}$$

which will be used frequently to compare the degree distances of two related graphs.

Let C_n be the cycle on $n \geq 3$ vertices. Let s and t be integers with $s, t \geq 3$. Let $a_1, a_2, \ldots, a_{s-1}$ be nonnegative integers. Let U_1 be the unicyclic graph with cycle $C_s = u_0 u_1 \ldots u_{s-1} u_0$ such that $U_1 - E(C_s)$ consists of vertex–disjoint paths $P_1 \,(= u_0)$, P_{a_1+1}, P_{a_2+1}, $\ldots, P_{a_{s-1}+1}$ with u_i being an end vertex of the path P_{a_i+1} for $i = 1, 2, \ldots, s-1$, and U_2 a unicyclic graph with a vertex v_0 on its cycle. Let $G(a_1, \ldots, a_{s-1}; U_2)$ be the bicyclic graph obtained by joining u_0 of U_1 and v_0 of U_2 by a path of length at least one, and $H(a_1, \ldots, a_{s-1}; U_2)$ the bicyclic graph obtained by identifying u_0 of U_1 and v_0 of U_2.

Lemma 2. *For fixed i and j with $1 \leq i < j \leq s-1$ and fixed a_k for $k \neq i, j$, let G_{a_i, a_j} be the graph $G(a_1, \ldots, a_{s-1}; U_2)$. If $a_i, a_j \geq 1$, then*

$$D'\left(G_{a_i,a_j}\right) < \max\left\{D'\left(G_{a_i+a_j,0}\right), D'\left(G_{0,a_i+a_j}\right)\right\}.$$

Proof: Let $G = G_{a_i,a_j}$, $G_1 = G_{a_i+a_j,0}$. Then $G - E(C_s)$ consists of a unicyclic graph Q and $s-1$ paths. Let u_k^* be the pendant vertex of G in the path attached to u_k if $a_k \geq 1$, where $k = 1, 2, \ldots, s-1$. Denote by u the neighbor of u_j outside C_s. Obviously, $G_1 = G - \{uu_j\} + \{uu_i^*\}$. Note that

$$V_1(G_1) = (V_1(G_1) \cap V(Q)) \cup \left(\cup_{\substack{1 \leq k \leq s-1 \\ a_k \geq 1, k \neq i,j}} \{u_k, u_k^*\} \right) \cup \{u_i, u_j^*\}$$

and $V_1(G) = (V_1(G) \cap V(Q)) \cup \left(\cup_{\substack{1 \le k \le s-1 \\ a_k \ge 1}} \{u_k, u_k^*\} \right)$. It is easily seen that $D_{G_1}(x) - D_G(x) = D_{G_1}(u_0) - D_G(u_0)$ for $x \in V(Q)$, and thus

$$\begin{aligned}
& \sum_{x \in V_1(G_1) \cap V(Q)} (d_{G_1}(x) - 2) D_{G_1}(x) - \sum_{x \in V_1(G) \cap V(Q)} (d_G(x) - 2) D_G(x) \\
= & \sum_{x \in V(Q)} (d_G(x) - 2) \left[D_{G_1}(x) - D_G(x) \right] \\
= & \left[D_{G_1}(u_0) - D_G(u_0) \right] \cdot \left[\sum_{x \in V(Q)} (d_Q(x) - 2) + 2 \right] \\
= & 2 [D_{G_1}(u_0) - D_G(u_0)].
\end{aligned}$$

For $k \ne i, j$ with $a_k \ge 1$, $D_{G_1}(u_k) - D_G(u_k) = D_{G_1}(u_k^*) - D_G(u_k^*)$, and then

$$\begin{aligned}
& \sum_{x \in \{u_k, u_k^*\}} (d_{G_1}(x) - 2) D_{G_1}(x) - \sum_{x \in \{u_k, u_k^*\}} (d_G(x) - 2) D_G(x) \\
= & (3-2)[D_{G_1}(u_k) - D_G(u_k)] + (1-2)[D_{G_1}(u_k^*) - D_G(u_k^*)] = 0.
\end{aligned}$$

Also we have

$$\begin{aligned}
& \sum_{x \in \{u_i, u_j^*\}} (d_{G_1}(x) - 2) D_{G_1}(x) - \sum_{x \in \{u_i, u_j, u_i^*, u_j^*\}} (d_G(x) - 2) D_G(x) \\
= & (3-2) \left[D_{G_1}(u_i) - D_G(u_i) \right] + (1-2) [D_{G_1}(u_j^*) - D_G(u_j^*)] \\
& -(1-2) D_G(u_i^*) - (3-2) D_G(u_j) \\
= & \left[D_{G_1}(u_i) - D_{G_1}(u_j^*) \right] + \left[D_G(u_i^*) - D_G(u_i) \right] + \left[D_G(u_j^*) - D_G(u_j) \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
& D' \left(G_{a_i + a_j, 0} \right) - D' \left(G_{a_i, a_j} \right) \\
= & 4 \left[W(G_1) - W(G) \right] + 2 \left[D_{G_1}(u_0) - D_G(u_0) \right] \\
& + \left[D_{G_1}(u_i) - D_{G_1}(u_j^*) \right] + \left[D_G(u_i^*) - D_G(u_i) \right] + \left[D_G(u_j^*) - D_G(u_j) \right].
\end{aligned}$$

Let $G_2 = G - \{uu_j\} + \{uu_i\}$, $a_0 = |V(Q)| - 1$, $n = |V(G)|$, and $d(x, y) = d_G(x, y)$ for $x, y \in V(G)$.

Let Z be the set of vertices in the path from u to u_j^* in G, and W the set of vertices in the path from u_i to u_i^* in G. We have

$$\begin{aligned}
& W(G_1) - W(G_2) \\
= & \sum_{\substack{x\in Z\\ y\in W}} [d_{G_1}(x,y) - d_{G_2}(x,y)] + \sum_{\substack{x\in Z\\ y\in V(G)\setminus(Z\cup W)}} [d_{G_1}(x,y) - d_{G_2}(x,y)] \\
= & \ 0 + \sum_{\substack{x\in Z\\ y\in V(G)\setminus(Z\cup W)}} [d_{G_1}(x,y) - d_{G_2}(x,y)] \\
= & \sum_{\substack{x\in Z\\ y\in V(G)\setminus(Z\cup W)}} a_i = a_i a_j (n - a_i - a_j - 1),
\end{aligned}$$

$$\begin{aligned}
& W(G_2) - W(G) \\
= & \sum_{\substack{x\in Z\\ y\in V(C_s)}} [d_{G_2}(x,y) - d(x,y)] + \sum_{\substack{x\in Z\\ y\in V(G)\setminus(Z\cup V(C_s))}} [d_{G_2}(x,y) - d(x,y)] \\
= & \ 0 + \sum_{\substack{x\in Z\\ y\in V(G)\setminus(Z\cup V(C_s))}} [d_{G_2}(x,y) - d(x,y)] \\
= & \sum_{x\in Z} \sum_{\substack{0\le k\le s-1\\ k\neq j}} a_k\left[d(u_k,u_i) - d(u_k,u_j)\right] \\
= & \ a_j \sum_{\substack{0\le k\le s-1\\ k\neq j}} a_k\left[d(u_k,u_i) - d(u_k,u_j)\right],
\end{aligned}$$

and then

$$\begin{aligned}
& W(G_1) - W(G) \\
= & \ [W(G_1) - W(G_2)] + [W(G_2) - W(G)] \\
= & \ a_i a_j (n - a_i - a_j - 1) + a_j \sum_{\substack{0\le k\le s-1\\ k\neq j}} a_k\left[d(u_k,u_i) - d(u_k,u_j)\right].
\end{aligned}$$

Note also that

$$\begin{aligned}
D_{G_1}(u_0) - D_G(u_0) &= [D_{G_1}(u_0) - D_{G_2}(u_0)] + [D_{G_2}(u_0) - D_G(u_0)] \\
&= a_i a_j + a_j[d(u_0,u_i) - d(u_0,u_j)], \\
D_{G_1}(u_i) - D_{G_1}(u_j^*) &= -(a_i + a_j)(n - a_i - a_j - 1), \\
D_G(u_i^*) - D_G(u_i) &= a_i(n - a_i - 1), \\
D_G(u_j^*) - D_G(u_j) &= a_j(n - a_j - 1).
\end{aligned}$$

Then

$$\begin{aligned}
& D'\left(G_{a_i+a_j,0}\right) - D'\left(G_{a_i,a_j}\right) \\
= & \ 4a_i a_j (n - a_i - a_j) + 2a_j[d(u_0,u_i) - d(u_0,u_j)] \\
& + 4a_j \sum_{\substack{0\le k\le s-1\\ k\neq j}} a_k\left[d(u_k,u_i) - d(u_k,u_j)\right].
\end{aligned}$$

If $D'(G_{a_i+a_j,0}) \le D'(G_{a_i,a_j})$, then

$$2[d(u_0,u_j)-d(u_0,u_i)]+4\sum_{\substack{0\le k\le s-1\\ k\ne j}} a_k\,[d(u_k,u_j)-d(u_k,u_i)] \ge 4a_i(n-a_i-a_j),$$

and thus

$$\begin{aligned}
& D'(G_{0,a_i+a_j})-D'(G_{a_i,a_j})\\
= \;& 4a_ia_j(n-a_i-a_j)+2a_i[d(u_0,u_j)-d(u_0,u_i)]\\
& +4a_i\sum_{\substack{0\le k\le s-1\\ k\ne i}} a_k\,[d(u_k,u_j)-d(u_k,u_i)]\\
= \;& 4a_ia_j(n-a_i-a_j)-4a_i(a_i+a_j)d(u_i,u_j)\\
& +2a_i[d(u_0,u_j)-d(u_0,u_i)]+4a_i\sum_{\substack{0\le k\le s-1\\ k\ne j}} a_k\,[d(u_k,u_j)-d(u_k,u_i)]\\
\ge \;& 4a_ia_j(n-a_i-a_j)-4a_i(a_i+a_j)d(u_i,u_j)\\
& +a_i\cdot 4a_i(n-a_i-a_j)\\
= \;& 4a_i(a_i+a_j)[(n-a_i-a_j)-d(u_i,u_j)]\\
> \;& 4a_i(a_i+a_j)\left(s-\frac{s}{2}\right)=2a_i(a_i+a_j)s>0.
\end{aligned}$$

Now the result follows. □

Similar to Lemma 2, we have

Lemma 3. *For fixed i and j with $1 \le i < j \le s-1$ and fixed a_k with $k \ne i, j$, let H_{a_i,a_j} be the graph $H(a_1,\ldots,a_{s-1};U_2)$. If $a_i, a_j \ge 1$, then*

$$D'(H_{a_i,a_j}) < \max\left\{D'\left(H_{a_i+a_j,0}\right), D'\left(H_{0,a_i+a_j}\right)\right\}.$$

3 Degree distances of graphs in $\mathcal{B}_1(n)$

In this section, we determine the maximum degree distance in the class of connected graphs with exactly two vertex–disjoint cycles.

Let $U_{n,m}$ be the unicyclic graph obtained by attaching a path P_{n-m} to a vertex u of the cycle C_m, where $3 \le m \le n$. In particular, $U_{n,n} = C_n$. Denote by v the pendant vertex in $U_{n,m}$ if $3 \le m \le n-1$, and w a vertex on C_m with $d_{U_{n,m}}(u,w) = \lfloor \frac{m}{2} \rfloor$. Recall that $W(P_s) = \frac{s^3-s}{6}$ and $W(C_s) = \frac{s}{2}\lfloor \frac{s^2}{4} \rfloor$. By direct calculation, we have

$$\begin{aligned} W(U_{n,m}) &= \frac{n^3}{6} + \left(\left\lfloor \frac{m^2}{4} \right\rfloor - \frac{m^2}{2} + \frac{m}{2} - \frac{1}{6}\right) n \\ &\quad - \frac{m}{2}\left\lfloor \frac{m^2}{4} \right\rfloor + \frac{m^3}{3} - \frac{m^2}{2} + \frac{m}{6}, \quad (1) \\ D_{U_{n,m}}(u) &= \left\lfloor \frac{m^2}{4} \right\rfloor + \frac{1}{2}(n-m)(n-m+1), \quad (2) \\ D_{U_{n,m}}(v) &= \left\lfloor \frac{m^2}{4} \right\rfloor + \frac{1}{2}(n-m)(n+m-1), \quad (3) \\ D_{U_{n,m}}(w) &= \left\lfloor \frac{m^2}{4} \right\rfloor + \frac{1}{2}(n-m)\left(n-m+1+2\left\lfloor \frac{m}{2} \right\rfloor\right). \quad (4) \end{aligned}$$

Let $C_{s_1} = u_0 u_1 \ldots u_{s_1-1} u_0$ and $C_{s_2} = v_0 v_1 \ldots v_{s_2-1} v_0$ be two vertex–disjoint cycles. Let $G_n(s_1, s_2)$ be the bicyclic graph obtained by joining u_0 and v_0 by a path of length $n - s_1 - s_2 + 1$, where $s_1 + s_2 \le n$.

Lemma 4. *For integers n, m_1 and m_2 with $n \ge 6$, $m_1, m_2 \ge 3$ and $m_1+m_2 \le n$, we have $D'(G_n(m_1, m_2)) \le D'(G_n(3,3))$ with equality if and only if $(m_1, m_2) = (3,3)$.*

Proof: Suppose that $m_1 \ge 5$. Let $G = G_n(m_1, m_2)$ and $G_1 = G_n(m_1 - 2, m_2)$. Note that $V_1(G) = V_1(G_1) = \{u_0, v_0\}$. Then

$$\begin{aligned} & D'(G_n(m_1-2, m_2)) - D'(G_n(m_1, m_2)) \\ = \; & 4[W(G_1) - W(G)] + (3-2)[D_{G_1}(u_0) - D_G(u_0)] \\ & + (3-2)[D_{G_1}(v_0) - D_G(v_0)] \\ = \; & 4[W(G_1) - W(G)] + [D_{G_1}(u_0) - D_G(u_0)] + [D_{G_1}(v_0) - D_G(v_0)]. \end{aligned}$$

Denote by u_0^* the vertex outside C_{m_1-2} in G_1 with $d_{G_1}(u_0, u_0^*) = 2$. Let Q be the graph obtained from G by deleting the vertices of C_{m_1}. Then $|V(Q)| = n - m_1$.

Note that for $v \in V(Q)$, $d_{G_1}(u_0^*, v) = d_G(u_0, v)$, and then

$$\begin{aligned}
& \sum_{u\in V(U_{m_1,m_1-2})}\sum_{v\in V(Q)} d_{G_1}(u,v) - \sum_{u\in V(C_{m_1})}\sum_{v\in V(Q)} d_G(u,v) \\
= & \sum_{v\in V(Q)}\left[\sum_{u\in V(U_{m_1,m_1-2})}(d_{G_1}(u,u_0^*)+d_{G_1}(u_0^*,v))\right. \\
& \left. - \sum_{u\in V(C_{m_1})}(d_G(u,u_0)+d_G(u_0,v))\right] \\
= & \sum_{v\in V(Q)}\left[\sum_{u\in V(U_{m_1,m_1-2})} d_{G_1}(u,u_0^*) - \sum_{u\in V(C_{m_1})} d_G(u,u_0)\right] \\
& + \sum_{v\in V(Q)} m_1\cdot[d_{G_1}(u_0^*,v)-d_G(u_0,v)] \\
= & \sum_{v\in V(Q)}\left[\sum_{u\in V(U_{m_1,m_1-2})} d_{G_1}(u,u_0^*) - \sum_{u\in V(C_{m_1})} d_G(u,u_0)\right]+0 \\
= & (n-m_1)\left[D_{U_{m_1,m_1-2}}(u_0^*) - D_{C_{m_1}}(u_0)\right],
\end{aligned}$$

from which and Eqs. (1) and (3), we have

$$\begin{aligned}
& W(G_1)-W(G) \\
= & \left[W(U_{m_1,m_1-2})+W(Q)+\sum_{u\in V(U_{m_1,m_1-2})}\sum_{v\in V(Q)} d_{G_1}(u,v)\right] \\
& - \left[W(C_{m_1})+W(Q)+\sum_{u\in V(C_{m_1})}\sum_{v\in V(Q)} d_G(u,v)\right] \\
= & W(U_{m_1,m_1-2})-W(C_{m_1})+(n-m_1)\left[D_{U_{m_1,m_1-2}}(u_0^*)-D_{C_{m_1}}(u_0)\right] \\
= & \left\lfloor\frac{m_1^2}{4}\right\rfloor - \frac{3}{2}m_1^2 + \left(n+\frac{9}{2}\right)m_1 - 2n - 4.
\end{aligned}$$

For $v \in V(Q)$, $d_{G_1}(u_0, v) - d_G(u_0, v) = 2 + d_{G_1}(u_0^*, v) - d_G(u_0, v) = 2$. Using Eq.

(2),

$$\begin{aligned}
& D_{G_1}(u_0) - D_G(u_0) \\
= & \left[\sum_{v \in V(U_{m_1, m_1-2})} d_{G_1}(u_0, v) + \sum_{v \in V(Q)} d_{G_1}(u_0, v) \right] \\
& - \left[\sum_{v \in V(C_{m_1})} d_G(u_0, v) + \sum_{v \in V(Q)} d_G(u_0, v) \right] \\
= & \sum_{v \in V(U_{m_1, m_1-2})} d_{G_1}(u_0, v) - \sum_{v \in V(C_{m_1})} d_G(u_0, v) + \sum_{v \in V(Q)} 2 \\
= & D_{U_{m_1, m_1-2}}(u_0) - D_{C_{m_1}}(u_0) + 2(n - m_1) \\
= & 2n - 3m_1 + 4.
\end{aligned}$$

Using Eq. (3),

$$\begin{aligned}
& D_{G_1}(v_0) - D_G(v_0) \\
= & \sum_{v \in V(U_{m_1, m_1-2})} d_{G_1}(v_0, v) - \sum_{v \in V(C_{m_1})} d_G(v_0, v) \\
= & \sum_{v \in V(U_{m_1, m_1-2})} \left[d_{G_1}(v_0, u_0^*) + d_{U_{m_1, m_1-2}}(u_0^*, v) \right] \\
& - \sum_{v \in V(C_{m_1})} \left[d_G(v_0, u_0) + d_{C_{m_1}}(u_0, v) \right] \\
= & m_1 \cdot \left[d_{G_1}(v_0, u_0^*) - d_G(v_0, u_0) \right] \\
& + \left[\sum_{v \in V(U_{m_1, m_1-2})} d_{U_{m_1, m_1-2}}(u_0^*, v) - \sum_{v \in V(C_{m_1})} d_{C_{m_1}}(u_0, v) \right] \\
= & 0 + \left[D_{U_{m_1, m_1-2}}(u_0^*) - D_{C_{m_1}}(u_0) \right] = m_1 - 2.
\end{aligned}$$

Hence

$$\begin{aligned}
& D'\left(G_n(m_1 - 2, m_2)\right) - D'\left(G_n(m_1, m_2)\right) \\
= & 4 \left[\left\lfloor \frac{m_1^2}{4} \right\rfloor - \frac{3}{2} m_1^2 + \left(n + \frac{9}{2} \right) m_1 - 2n - 4 \right] \\
& + (2n - 3m_1 + 4) + (m_1 - 2) \\
= & \begin{cases} -5m_1^2 + 4(n+4)m_1 - 6n - 14 & \text{if } m_1 \text{ is even,} \\ -5m_1^2 + 4(n+4)m_1 - 6n - 15 & \text{if } m_1 \text{ is odd.} \end{cases}
\end{aligned}$$

Suppose that m_1 is even. Let $f(m_1) = -5m_1^2 + 4(n+4)m_1 - 6n - 14$ for $m_1 \geq 6$. Let r_1 and r_2 be the two roots of $f(m_1) = 0$ with $r_1 \leq r_2$. Since

$f(6) = 18n - 98 > 0$, we have $r_1 < 6 < r_2$. Thus, $f(m_1) \geq 0$ if $6 \leq m_1 \leq r_2$, and $f(m_1) < 0$ if $m_1 > r_2$. If $n - m_2$ is even, then $m_1 \leq n - m_2$, $D'(G_n(m_1, m_2))$ is maximum for fixed m_2 only if $m_1 = 4$ for $r_2 \geq n - m_2$ or $m_1 = 4, n - m_2$ for $r_2 < n - m_2$. Similarly, if $n - m_2$ is odd, then $m_1 \leq n - m_2 - 1$, $D'(G_n(m_1, m_2))$ is maximum for fixed m_2 only if $m_1 = 4, n - m_2 - 1$. Hence $D'(G_n(m_1, m_2))$ is maximum implies that $G_n(m_1, m_2) = G_n(4, m_2), G_n(n - m_2 - i, m_2)$, where $i = 0$ if $n - m_2$ is even and $i = 1$ if $n - m_2$ is odd. For fixed m_2, let $s = n - m_2 - i \geq 6$, $G_2 = G_n(4, m_2)$ and $G_3 = G_n(n - m_2 - i, m_2)$, by similar technique as in calculation of $D'(G_n(m_1 - 2, m_2)) - D'(G_n(m_1, m_2))$, we have

$$\begin{aligned}
& D'(G_n(4, m_2)) - D'(G_n(n - m_2 - i, m_2)) \\
= \;& 4[W(G_2) - W(G_3)] + [D_{G_2}(u_0) - D_{G_3}(u_0)] + [D_{G_2}(v_0) - D_{G_3}(v_0)] \\
= \;& 4\left[-\frac{5}{24}s^3 + \left(\frac{n}{4} + \frac{1}{2}\right)s^2 - \left(\frac{n}{2} + \frac{1}{6}\right)s - 2n + 6\right] \\
& + \left[-\frac{3}{4}s^2 + \left(n + \frac{1}{2}\right)s - 4n + 10\right] + \left(\frac{s^2}{4} - \frac{s}{2} - 2\right) \\
= \;& n(s^2 - s - 12) - \frac{5}{6}s^3 + \frac{3}{2}s^2 - \frac{2}{3}s + 32 \\
\geq \;& (s + 3)(s^2 - s - 12) - \frac{5}{6}s^3 + \frac{3}{2}s^2 - \frac{2}{3}s + 32 \\
= \;& \frac{s^3}{6} + \frac{7}{2}s^2 - \frac{47}{3}s - 4 > 0,
\end{aligned}$$

and thus, if $m_1 > 4$ is even, then $D'(G_n(m_1, m_2)) < D'(G_n(4, m_2))$. If $m_1 > 3$ is odd, then by similar arguments as above, we have $D'(G_n(m_1, m_2)) < D'(G_n(3, m_2))$. Since $D'(G_n(3, m_2)) - D'(G_n(4, m_2)) = 5(n - 4) > 0$, we have $D'(G_n(m_1, m_2)) < D'(G_n(3, m_2))$ if $m_1 > 3$, and if $m_2 > 3$, then

$$D'(G_n(3, m_2)) = D'(G_n(m_2, 3)) < D'(G_n(3, 3)).$$

The result follows. □

Theorem 1. *Let $G \in \mathcal{B}_1(n)$, where $n \geq 6$. Then*

$$D'(G) \leq \frac{2}{3}n^3 + n^2 - \frac{41}{3}n + 24$$

with equality if and only if $G = G_n(3, 3)$.

Proof: Suppose that G is a graph in $\mathcal{B}_1(n)$ with the maximum degree distance. Let $C_{m_1} = u_0u_1 \ldots u_{m_1-1}u_0$ and $C_{m_2} = v_0v_1 \ldots v_{m_2-1}v_0$ be the two vertex–disjoint cycles of G. Let $d_G(u_0, v_0) = \min\{d_G(u, v) : u \in V(C_{m_1}), v \in V(C_{m_2})\}$. By Lemma 1 (i) and (ii), $G - E(C_{m_1}) \cup E(C_{m_2})$ consists of vertex–disjoint paths, for which, by Lemma 2, except the path containing u_0 and v_0, there is at most one path containing a vertex in C_{m_1} and at most one path containing a vertex in C_{m_2} with length at least one.

Suppose that there is a path of length $a \geq 1$ containing u_s, $s \neq 0$. Denote by u_s^* the pendant vertex of G in this path, and u the neighbor of u_0 outside C_{m_1}. Let $G_1 = G - \{u_0u\} + \{uu_s^*\} \in \mathcal{B}_1(n)$.

Let Q be the component of $G-\{u_0\}$ containing u. Then Q is a unicyclic graph. Note that $V_1(G) \cap V(Q) = V_1(G_1) \cap V(Q) = \{v_0\}$ if all vertices on C_{m_2} except v_0 are of degree two, or $V_1(G) \cap V(Q) = V_1(G_1) \cap V(Q) = \{v_0, v_t, v_t^*\}$ if there is another vertex v_t on C_{m_2} with degree three, where v_t^* is the pendant vertex of the path attached to v_t. In either case, we have $D_{G_1}(x) - D_G(x) = D_{G_1}(u) - D_G(u)$ for $x \in V(Q)$, and thus

$$\begin{aligned}
& \sum_{x \in V_1(G_1) \cap V(Q)} (d_{G_1}(x) - 2) D_{G_1}(x) - \sum_{x \in V_1(G) \cap V(Q)} (d_G(x) - 2) D_G(x) \\
= & \sum_{x \in V(Q)} (d_G(x) - 2) \left[D_{G_1}(x) - D_G(x) \right] \\
= & \left[D_{G_1}(u) - D_G(u) \right] \cdot \left[\sum_{x \in V(Q)} (d_Q(x) - 2) + 1 \right] \\
= & D_{G_1}(u) - D_G(u).
\end{aligned}$$

It is easily seen that $V_1(G) = \{u_s, u_s^*, u_0\} \cup (V_1(G) \cap V(Q))$ and $V_1(G_1) = \{u_s\} \cup (V_1(G_1) \cap V(Q))$, and thus

$$\begin{aligned}
& D'(G_1) - D'(G) \\
= & 4[W(G_1) - W(G)] + (3-2)[D_{G_1}(u_s) - D_G(u_s)] \\
& -(1-2) D_G(u_s^*) - (3-2) D_G(u_0) + [D_{G_1}(u) - D_G(u)] \\
= & 4[W(G_1) - W(G)] + [D_{G_1}(u_s) - D_G(u_s)] \\
& +[D_G(u_s^*) - D_G(u_0)] + [D_{G_1}(u) - D_G(u)].
\end{aligned}$$

Let $G_2 = G - \{u_0u\} + \{uu_s\}$, $d_G(u_0, u_s) = c$ and $n_0 = |V(Q)| = n - m_1 - a$. Note that

$$\begin{aligned}
W(G_1) - W(G) &= [W(G_1) - W(G_2)] + [W(G_2) - W(G)] \\
&= an_0(m_1 - 1) - an_0c = an_0(m_1 - 1 - c), \\
D_{G_1}(u_s) - D_G(u_s) &= [D_{G_1}(u_s) - D_{G_2}(u_s)] + [D_{G_2}(u_s) - D_G(u_s)] \\
&= an_0 - cn_0 = (a - c)n_0, \\
D_G(u_s^*) - D_G(u_0) &= [D_G(u_s^*) - D_G(u_s)] + [D_G(u_s) - D_G(u_0)] \\
&= a(m_1 - 1 + n_0) + c(n_0 - a) \\
&= a(m_1 - 1 - c) + n_0(a + c), \\
D_{G_1}(u) - D_G(u) &= [D_{G_1}(u) - D_{G_2}(u)] + [D_{G_2}(u) - D_G(u)] \\
&= a(m_1 - 1) - ac = a(m_1 - 1 - c).
\end{aligned}$$

Then

$$\begin{aligned} D'(G_1) - D'(G) &= 2a[(2n_0+1)(m_1-1-c)+n_0] \\ &\geq 2a\left[(2n_0+1)\left(m_1-1-\frac{m_1}{2}\right)+n_0\right] \\ &= 2a\left[(2n_0+1)\left(\frac{m_1}{2}-1\right)+n_0\right] > 0, \end{aligned}$$

and thus $D'(G_1) > D'(G)$, a contradiction. Thus, there is no such path containing a vertex on the cycle C_{m_1}. Similarly, there is no such path containing a vertex on the cycle C_{m_2}. Then $G = G_n(m_1, m_2)$. By Lemma 4, we have $G = G_n(3,3)$. □

4 Degree distances of graphs in $\mathcal{B}_2(n)$

In this section, we determine the maximum degree distance in the class of connected graphs with exactly two cycles of a common vertex.

Let $C_{s_1} = u_0u_1 \ldots u_{s_1-1}u_0$ and $C_{s_2} = v_0v_1 \ldots v_{s_2-1}v_0$ be two vertex–disjoint cycles. Let $H_n(a, s_1, s_2)$ be the bicyclic graph obtained by identifying u_0 of C_{s_1} and v_0 of C_{s_2}, which is denoted by u_0, and attaching a path P_a and a path P_b to $u_{\lfloor s_1/2 \rfloor}$ and $v_{\lfloor s_2/2 \rfloor}$, respectively, where $a, b \geq 0$ and $a + b = n + 1 - s_1 - s_2$, and if $a = 0$ or $b = 0$, then no path is attached to $u_{\lfloor s_1/2 \rfloor}$ or $v_{\lfloor s_2/2 \rfloor}$. If $a \geq 1$, then let u be the pendant vertex of the path attached to $u_{\lfloor s_1/2 \rfloor}$. If $s_1 = s_2$, then $a \geq b$ is required.

Lemma 5. *For integers n, m_1 and m_2 with $n \geq 7$, $m_1, m_2 \geq 3$ and $m_1 + m_2 \leq n+1$, we have $D'(H_n(a, m_1, m_2)) \leq D'(H_n(n-5, 3, 3))$ with equality if and only if $(a, m_1, m_2) = (n-5, 3, 3)$.*

Proof: Suppose that $m_1 \geq 5$. Let $k = a + m_1$. For $G_1 = H_n(a, m_1, m_2)$ and $G_2 = H_n(a+2, m_1-2, m_2)$, using Eqs. (1) and (4), and by similar arguments as in the proof of Lemma 4, we have

$$\begin{aligned}
& D'\left(H_n(a+2,m_1-2,m_2)\right) - D'\left(H_n(a,m_1,m_2)\right) \\
= \; & 4[W(G_2)-W(G_1)] + (4-2)[D_{G_2}(u_0) - D_{G_1}(u_0)] \\
& +(3-2)\left[D_{G_2}\left(u_{\lfloor \frac{m_1}{2} \rfloor -1}\right) - D_{G_1}\left(u_{\lfloor \frac{m_1}{2} \rfloor}\right)\right] + (1-2)[D_{G_2}(u) - D_{G_1}(u)] \\
= \; & 4[W(G_2)-W(G_1)] + 2[D_{G_2}(u_0) - D_{G_1}(u_0)] \\
& + \left[D_{G_2}\left(u_{\lfloor \frac{m_1}{2} \rfloor -1}\right) - D_{G_2}(u)\right] + \left[D_{G_1}(u) - D_{G_1}\left(u_{\lfloor \frac{m_1}{2} \rfloor}\right)\right] \\
= \; & 4\left[-\frac{3m_1^2}{2} - \left(2n-3k-\frac{9}{2}\right)m_1 + \left\lfloor \frac{m_1^2}{4} \right\rfloor\right. \\
& \left. + 2\left\lfloor \frac{m_1}{2} \right\rfloor (n-k) + (k+2)(n-k-2)\right] + 2\left(2\left\lfloor \frac{m_1}{2} \right\rfloor - 2m_1 + k + 2\right) \\
& -(k-m_1+2)(n-k+m_1-3) + (k-m_1)(n-k+m_1-1) \\
= \; & \begin{cases} -5m_1^2 - (4n-8k-12)m_1 - 4k^2 + (4n-10)k + 6n - 6 \\ \qquad \text{if } m_1 \text{ is even,} \\ -5m_1^2 - (4n-8k-12)m_1 - 4k^2 + (4n-6)k + 2n - 9 \\ \qquad \text{if } m_1 \text{ is odd.} \end{cases}
\end{aligned}$$

Suppose that m_1 is even. Let $g(m_1) = -5m_1^2 - (4n-8k-12)m_1 - 4k^2 + (4n-10)k + 6n - 6$. Note that $k \geq 6$. It is easily seen that

$$\begin{aligned}
g(6) &= n(4k-18) - 4k^2 + 38k - 114 \\
&\geq (k+2)(4k-18) - 4k^2 + 38k - 114 = 28k - 150 > 0.
\end{aligned}$$

Let r_1 and r_2 be the two roots of $g(m_1) = 0$, where $r_1 \leq r_2$. It is easily seen that $r_1 < 6 < r_2$. Thus, $g(m_1) \geq 0$ if $6 \leq m_1 \leq r_2$, and $g(m_1) < 0$ if $m_1 > r_2$. If k is even, then $m_1 \leq k$, $D'\left(H_n(a,m_1,m_2)\right)$ is maximum for fixed m_2 only if $m_1 = 4$ for $r_2 \geq k$ or $m_1 = 4, k$ for $r_2 < k$, and by setting $G_3 = H_n(0,k,m_2)$, $G_4 = H_n(k-4,4,m_2)$, and using similar arguments as above, we have

$$\begin{aligned}
& D'\left(H_n(k-4,4,m_2)\right) - D'\left(H_n(0,k,m_2)\right) \\
= \; & 4[W(G_4)-W(G_3)] + 2[D_{G_4}(u_0) - D_{G_3}(u_0)] + [D_{G_4}(u_2) - D_{G_4}(u)] \\
= \; & 4\left[-\frac{5}{24}k^3 + \frac{1}{4}(n+6)k^2 - \frac{1}{6}(9n+25)k + 2n + 6\right] + 2\left(\frac{k^2}{4} - \frac{3}{2}k + 2\right) \\
& -(k-4)(n-k+3) \\
= \; & n(k^2-7k+12) - \frac{5}{6}k^3 + \frac{15}{2}k^2 - \frac{80}{3}k + 40 \\
\geq \; & (k+2)(k^2-7k+12) - \frac{5}{6}k^3 + \frac{15}{2}k^2 - \frac{80}{3}k + 40 \\
= \; & \frac{k^3}{6} + \frac{5}{2}k^2 - \frac{86}{3}k + 64 > 0,
\end{aligned}$$

implying that $D'\left(H_n(k-4,4,m_2)\right) > D'\left(H_n(0,k,m_2)\right)$. Similarly, if k is odd, then $m_1 \leq k-1$, $D'\left(H_n(a,m_1,m_2)\right)$ is maximum for fixed m_2 only if $m_1 = 4$

or $m_1 = k-1$, and thus $D'(H_n(k-4,4,m_2)) > D'(H_n(1,k-1,m_2))$. Let $b = n+1-m_1-m_2-a$. We conclude that, if $m_1 > 4$ is even, then

$$D'\left(H_n(a,m_1,m_2)\right) < D'\left(H_n(a+m_1-4,4,m_2)\right) = D'\left(H_n(b,m_2,4)\right).$$

If $m_1 > 3$ is odd, then by similar arguments as above, we have

$$D'\left(H_n(a,m_1,m_2)\right) < D'\left(H_n(a+m_1-3,3,m_2)\right) = D'\left(H_n(b,m_2,3)\right).$$

Thus, $D'(H_n(a,m_1,m_2))$ is maximum for fixed n only if $m_1, m_2 = 3,4$. For $0 \le s \le n-7$, let $t = n-7-s$. We may check that

$$\begin{aligned}
D'(H_n(n-5,3,3)) - D'(H_n(s,4,4)) &= 12st + 16s + 16t + 2 > 0,\\
D'(H_n(n-5,3,3)) - D'(H_n(0,3,4)) &= 5n - 34 > 0,\\
D'(H_n(n-5,3,3)) - D'(H_n(s+1,3,4)) &= 12st + 11s + 17t + 7 > 0,\\
D'(H_n(n-5,3,3)) - D'(H_n(s+1,3,3)) &= 12st + 12s + 12t + 12 > 0.
\end{aligned}$$

The result follows easily. □

Theorem 2. *Let $G \in \mathcal{B}_2(n)$, where $n \ge 6$. Then for $n = 6$, $D'(G) \le 112$ with equality if and only if $G = H_6(0,3,4)$, and for $n \ge 7$,*

$$D'(G) \le \frac{2}{3}n^3 + n^2 - \frac{83}{3}n + 94$$

with equality if and only if $G = H_n(n-5,3,3)$.

Proof: Obviously, there are only three graphs in $\mathcal{B}_2(6)$. By direct calculation, the result follows easily for $n = 6$. In the following, suppose that $n \ge 7$.

Suppose that G is a graph in $\mathcal{B}_2(n)$ with the maximum degree distance. Let $C_{m_1} = u_0u_1 \dots u_{m_1-1}u_0$ and $C_{m_2} = v_0v_1 \dots v_{m_2-1}v_0$ be the two cycles of G with $u_0 = v_0$. By Lemma 1 (i), we have $G - E(C_{m_1}) \cup E(C_{m_2})$ consists of vertex–disjoint paths, for which, by Lemma 3, at most three paths have length at least one: a path Q with u_0 as an end vertex, a path containing some vertex u_s in $C_{m_1}$with $s \ne 0$, and a path containing some vertex v_t in C_{m_2} with $t \ne 0$.

If there is such a path containing u_s with length $a \ge 1$, then denote by u_s^* the pendant vertex of G in this path. Otherwise, all vertices on C_{m_1} except u_0 have degree two and we set $a = 0$, $u_s = u_s^* = u_{\lfloor \frac{m_1}{2} \rfloor}$. If there is such a path containing v_t with length at least one, then denote by b the length of this path. Otherwise, all vertices on C_{m_2} except v_0 have degree two and we set $b = 0$.

Suppose that there is such a path Q containing u_0 with length at least one in G. Let u be the neighbor of u_0 outside the two cycles, and u^* the pendant vertex of G in Q. Suppose without loss of generality that $a \le b$. Let $G_1 = G - \{u_0u\} + \{u_s^*u\} \in \mathcal{B}_2(n)$. Let $c = d_G(u_0, u_s)$ and $k = d_G(u^*, u_0)$.

Let $G^* = G - \{u_0 u\} + \{u_s u\}$. Suppose first that $a \geq 1$. We have

$$\begin{aligned} W(G_1) - W(G) &= [W(G_1) - W(G^*)] + [W(G^*) - W(G)] \\ &= ka(n-a-k-1) + kc(b+m_2-1-a). \end{aligned}$$

Note also that

$$\begin{aligned} D_{G_1}(u_s) - D_{G_1}(u^*) &= -(k+a)(n-k-a-1), \\ D_G(u^*) - D_G(u_0) &= k(n-k-1), \\ D_G(u_s^*) - D_G(u_s) &= a(n-a-1), \\ D_{G_1}(u_0) - D_G(u_0) &= [D_{G_1}(u_0) - D_{G^*}(u_0)] + [D_{G^*}(u_0) - D_G(u_0)] \\ &= ka + kc = k(a+c). \end{aligned}$$

Then

$$\begin{aligned} & D'(G_1) - D'(G) \\ = \ & 4[W(G_1) - W(G)] + (3-2)[D_{G_1}(u_s) - D_G(u_s)] \\ & +(1-2)[D_{G_1}(u^*) - D_G(u^*)] + (4-2)D_{G_1}(u_0) \\ & -(5-2)D_G(u_0) - (1-2)D_G(u_s^*) \\ = \ & 4[W(G_1) - W(G)] + [D_{G_1}(u_s) - D_{G_1}(u^*)] + [D_G(u^*) - D_G(u_0)] \\ & +[D_G(u_s^*) - D_G(u_s)] + 2[D_{G_1}(u_0) - D_G(u_0)] \\ = \ & 4kc\left(b + m_2 - a - \frac{1}{2}\right) + 4ka(b + m_1 + m_2 - 1) > 0, \end{aligned}$$

and thus, $D'(G_1) > D'(G)$. If $a = 0$, then

$$\begin{aligned} & D'(G_1) - D'(G) \\ = \ & 4[W(G_1) - W(G)] + (1-2)[D_{G_1}(u^*) - D_G(u^*)] + (4-2)D_{G_1}(u_0) \\ & -(5-2)D_G(u_0) + (3-2)D_{G_1}\left(u_{\lfloor \frac{m_1}{2} \rfloor}\right) \\ = \ & 4[W(G_1) - W(G)] + \left[D_{G_1}\left(u_{\lfloor \frac{m_1}{2} \rfloor}\right) - D_{G_1}(u^*)\right] + [D_G(u^*) - D_G(u_0)] \\ & +2[D_{G_1}(u_0) - D_G(u_0)] \\ = \ & 4k\left\lfloor \frac{m_1}{2} \right\rfloor \left(b + m_2 - \frac{1}{2}\right) > 0, \end{aligned}$$

and thus, $D'(G_1) > D'(G)$. In either case, $D'(G_1) > D'(G)$, a contradiction. Thus, there is no such path Q containing u_0 in G, i.e., $d_G(u_0) = 4$.

Suppose that there is such a path containing u_s with length $a \geq 1$ and $d_G(u_0, u_s) = c < \lfloor \frac{m_1}{2} \rfloor$. Let u_{s_1} be the neighbor of u_s outside C_{m_1} in G. For

$G_2 = G - \{u_s u_{s_1}\} + \left\{u_{\lfloor \frac{m_1}{2} \rfloor} u_{s_1}\right\} \in \mathcal{B}_2(n)$,

$$\begin{aligned}
& D'(G_2) - D'(G) \\
= \; & 4\,[W(G_2) - W(G)] + (4-2)[D_{G_2}(u_0) - D_G(u_0)] \\
& +(1-2)\,[D_{G_2}(u_s^*) - D_G(u_s^*)] \\
& +(3-2)D_{G_2}\left(u_{\lfloor \frac{m_1}{2} \rfloor}\right) - (3-2)D_G(u_s) \\
= \; & 4\,[W(G_2) - W(G)] + 2[D_{G_2}(u_0) - D_G(u_0)] \\
& + \left[D_{G_2}\left(u_{\lfloor \frac{m_1}{2} \rfloor}\right) - D_{G_2}(u_s^*)\right] + [D_G(u_s^*) - D_G(u_s)] \\
= \; & 4 \cdot a \left(\left\lfloor \frac{m_1}{2} \right\rfloor - c\right)(n - a - m_1) + 2 \cdot a \left(\left\lfloor \frac{m_1}{2} \right\rfloor - c\right) \\
& -a(n-a-1) + a(n-a-1) \\
= \; & 2a \left(\left\lfloor \frac{m_1}{2} \right\rfloor - c\right)(2n - 2a - 2m_1 + 1) > 0,
\end{aligned}$$

and then $D'(G_2) > D'(G)$, a contradiction. Thus, if there is such a path containing u_s with length at least one, then $d_G(u_0, u_s) = \left\lfloor \frac{m_1}{2} \right\rfloor$. Similarly, if there is such a path containing v_t with length at least one, then $d_G(u_0, v_t) = \left\lfloor \frac{m_2}{2} \right\rfloor$. It follows that $G = H_n(a, m_1, m_2)$, and by Lemma 5, we have $G = H_n(n-5, 3, 3)$. □

5 Conclusions

It is easily checked that $D'(G_6(3,3)) - D'(H_6(0,3,4)) = 10 > 0$, and for $n \geq 7$, $D'(G_n(3,3)) - D'(H_n(n-5,3,3)) = 14n - 70 > 0$. By Theorems 1 and 2, we have the following conclusions:
(i) for $n \geq 6$, $G_n(3,3)$ is the unique graph in $\mathcal{B}_1(n)$ with the maximum degree distance $\frac{2}{3}n^3 + n^2 - \frac{41}{3}n + 24$;
(ii) for $n = 6$, $H_6(0,3,4)$ is the unique graph in $\mathcal{B}_2(6)$ with the maximum degree distance 112, and for $n \geq 7$, $H_n(n-5,3,3)$ is the unique graph in $\mathcal{B}_2(n)$ with the maximum degree distance $\frac{2}{3}n^3 + n^2 - \frac{83}{3}n + 94$;
(iii) for $n \geq 6$, $G_n(3,3)$ is the unique graph in $\mathcal{B}_1(n) \cup \mathcal{B}_2(n)$ with the maximum degree distance $\frac{2}{3}n^3 + n^2 - \frac{41}{3}n + 24$.

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Department of Mathematics,
South China Normal University,
Guangzhou 510631, P. R. China
E-mail: zhoubo@scnu.edu.cn