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The Beast and triangular Zeckendorf representations

by
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Abstract

The Zeckendorf representation of the number of the beast 666 is (almost) $F_{15} + F_{10} + F_1$, and the indices are all triangular numbers. Here, we show that the beast is the largest base 10 repdigit which is a sum of distinct Fibonacci numbers with triangular indices.

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1 Introduction

Let $\{F_n\}_{n\geq 1}$ be the Fibonacci sequence given by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. The Zeckendorf representation of the positive integer N is its representation of the form

$$N = F_{m_1} + F_{m_2} + \cdots + F_{m_k},$$

where $m_1, \ldots, m_k$ are positive integers with $m_i - m_{i+1} \geq 2$ for $i = 1, \ldots, k$. When $i = k$, the above inequality means that $m_k \geq 2$. We make a little amendment to this representation and replace $m_k = 2$ (whenever this occurs) by $m_k = 1$. The observation that started this note is that with this convention, the beast satisfies

$$666 = F_{15} + F_{10} + F_1,$$

and all indices in the above Zeckendorf representation of 666 are triangular numbers. Recall that a triangular number is a number of the form $m(m+1)/2$ for some positive integer m. We denote this number by T_m. With this notation, $T_5 = 15$, $T_4 = 10$ and $T_1 = 1$. Douglas Iannucci asked if there are any larger examples of repdigits in base 10; i.e., numbers whose base 10 representation is a string $\underbrace{dd\cdots d}_{m \text{ times}}$ consisting of the same base 10 digit d repeating m times, and whose Zeckendorf representation has only triangular indices. The answer is no.

Theorem 1. *The only solutions of the equation*

$$N = d\left(\frac{10^m - 1}{9}\right) = F_{T_{m_1}} + \cdots + F_{T_{m_k}}, \tag{1}$$

in positive integers $m,\ m_1 > \cdots > m_k,\ d \in \{1, \ldots, 9\}$ *are*

$$N \in \{1, 2, 3, 8, 9, 11, 55, 66, 666\}.$$

Problems of a similar flavor were studied in [2], [3], [5] and [6].

Note that technically, a representation of N of the form (1) may not be exactly the Zeckendorf representation of N, as in the examples $3 = F_3 + F_1$ and $11 = F_6 + F_3 + F_1$, which are not Zeckendorf representations. However, it is easy to see that if the representation (1) is not a Zeckendorf representation, then $k \geq 2$, its last two terms are $F_3 + F_1 = F_4$, and then $N = F_{T_{m_1}} + \cdots + F_{T_{m_{k-2}}} + F_4$ is the Zeckendorf representation of N. Hence, the Zeckendorf representation of a number N arising from (1) has at least $k - 1$ terms, and the first $k - 2$ of them are triangular.

2 The Proof

We started by performing a search in the range $1 \leq m \leq 500$ which turned up only the solution shown in the statement of Theorem 1. The way we searched was the following. Let N be a base 10 repdigit in this range whose Zeckendorf representation has at least four terms. We then checked whether the first two leading ones have triangular indices. This gave no solution with $m \in [6, 500]$. If on the other hand N has at most three terms in its Zeckendorf representation, then $m \leq 5$ also by the main result from [5]. Hence, if $m \leq 500$, then $m \leq 5$, and now the list of examples can be computed by hand.

Assume now that $m \geq 501$. Put $n := T_{m_1}$. Then

$$10^{501} \leq F_n + F_{n-1} + \cdots + F_1 = F_{n+2} - 1,$$

so

$$n \geq 2393. \tag{2}$$

Since $n = T_{m_1}$ is triangular, we get that $m_1 \geq 69$ and further

$$n - T_{m_2} \geq m_1(m_1+1)/2 - m_1(m_1-1)/2 = m_1 \geq \sqrt{n} \qquad (\geq 49). \tag{3}$$

Using the Binet formula

$$F_u = \frac{\alpha^u - \beta^u}{\alpha - \beta} \quad \text{for} \quad u = 1, 2, \ldots, \quad \text{with} \quad (\alpha, \beta) := \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right),$$

the equation (1) can be rewritten as

$$\frac{d10^m}{9} - \frac{\alpha^n}{\sqrt{5}} = -\frac{\beta^n}{\sqrt{5}} + \frac{d}{9} + \sum_{i=2}^{k} F_{T_{m_i}}. \tag{4}$$

Taking absolute values, we get

$$\left|\frac{d10^m}{9} - \frac{\alpha^n}{\sqrt{5}}\right| < 2 + \sum_{i \le T_{m_2}} F_i. \tag{5}$$

Using the fact that $F_k \le \alpha^{k-1}$ holds for all positive integers k as well as inequality (3), we get that the right hand side in (5) is bounded as

$$2 + \sum_{i \le T_{m_2}} F_i < 2\left(1 + \sum_{k=1}^{T_{m_2}} \alpha^{k-1}\right) = 2\left(1 + \frac{\alpha^{T_{m_2}} - 1}{\alpha - 1}\right) < \frac{2\alpha^{n-\sqrt{n}}}{\alpha - 1}. \tag{6}$$

Inserting the upper bound (6) into (5) and dividing both sides of the resulting inequality by $\alpha^n/\sqrt{5}$, we get

$$\left|\alpha^{-n}10^m(d\sqrt{5}/9) - 1\right| < \frac{2\sqrt{5}}{(\alpha-1)\alpha^{\sqrt{n}}} < \frac{1}{\alpha^{\sqrt{n}-5}}. \tag{7}$$

We now use a result of Matveev (see [7] or Theorem 9.4 in [1]), which asserts that if $\alpha_1,\ \alpha_2,\ \alpha_3$ are positive real algebraic numbers in an algebraic number field of degree D and $b_1,\ b_2,\ b_3$ are integers, then

$$|\alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1| > \exp\left(-1.4 \times 30^6 \times 3^{4.5} D^2(1+\log D)(1+\log B)A_1A_2A_3\right) \tag{8}$$

assuming that the left–hand side is nonzero, with $B := \max\{|b_1|, |b_2|, |b_3|\}$, and

$$A_i \ge \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\}, \qquad i = 1, 2, 3,$$

where $h(\gamma)$ is the logarithmic height of the algebraic number γ whose formula is

$$h(\gamma) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\gamma^{(i)}|, 1\}\right)\right),$$

with d being the degree of γ over $\mathbb{Q}$ and

$$f(X) := a_0 \prod_{i=1}^{d}(X - \gamma^{(i)}) \in \mathbb{Z}[X]$$

being the minimal polynomial over the integers having γ as a root. We shall apply this to the left–hand side of (7). Note first that this is not zero. Indeed, if the left hand side of (7) is zero, we then get $\alpha^n = 10^m d\sqrt{5}/9$, so $\alpha^{2n} \in \mathbb{Q}$, which is false. In (7), we take $\alpha_1 := \alpha,\ \alpha_2 := 10,\ \alpha_3 := (d\sqrt{5})/9,\ b_1 := -n,\ b_2 := m,\ b_3 := 1$. Since

$$10^{m-1} \le 10^{m-1} + \cdots + 1 \le d\left(\frac{10^m - 1}{9}\right) < F_{n+2} < \alpha^{n+1},$$

we get

$$n+1 > \left(\frac{\log 10}{\log \alpha}\right) m > 4.78m. \tag{9}$$

So, since $m \geq 101$, we definitely have $B = n$. Clearly $D = 2$. We can choose $A_1 := 0.5 > 2h(\alpha_1)$, $A_2 := 4.7 > 2\log \alpha_2$ and $A_3 := 6.1 > (\log 81 + 2\log \sqrt{5}) \geq 2h(\alpha_3)$. We thus get that

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2)(1+\log n) 0.5 \times 4.7 \times 6.1\right) < \frac{1}{\alpha^{\sqrt{n}-5}}$$

giving

$$\begin{aligned} \sqrt{n} &< 5 + \left(\frac{1}{\log \alpha}\right) \times 1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2) \times 0.5 \times 4.7 \times 6.1(1+\log n) \\ &< 2.9 \times 10^{13} \times (1+\log n), \end{aligned}$$

which implies that $n < 5 \times 10^{30}$. To lower the bound, put

$$\Lambda := m \log 10 - n \log \alpha + \log(d\sqrt{5}/9).$$

The right–hand side of (4) is obviously positive since n is large, so that $\Lambda > 0$. Using (7), we have $\Lambda < e^{\Lambda} - 1 < \alpha^{-\sqrt{n}+5}$. So, we get that

$$0 < \Lambda = m \log 10 - n \log \alpha + \log(d\sqrt{5}/9) < \frac{1}{\alpha^{\sqrt{n}-5}},$$

giving

$$0 < m\left(\frac{\log 10}{\log \alpha}\right) - n + \frac{\log(d\sqrt{5}/9}{\log \alpha} < \frac{1}{(\log \alpha)\alpha^{\sqrt{n}-5}} < \frac{1}{\alpha^{\sqrt{n}-7}}.$$

We put $\gamma := (\log 10)/(\log \alpha)$, $\mu := (\log(d\sqrt{5}/9))/(\log \alpha)$. We also put $M := 5 \times 10^{30}$. By the standard Baker-Davenport reduction lemma (see Lemma 5 in [4]), it follows that

$$\sqrt{n} \leq 7 + \frac{\log(q/\varepsilon)}{\log \alpha},$$

where $q > 2 \times 10^{31} > 6M$ is the denominator of a convergent to γ and $\varepsilon := \|\mu q\| - M\|\gamma q\| > 0$. We took $q := q_{68}$ to be the denominator of the sixty–eighth convergent to γ, where the continued fraction of γ is

$$[a_0, a_1, a_2, a_3, a_4, \ldots] = [4, 1, 3, 1, 1, \ldots],$$

whose convergents are $p_0/q_0 = [a_0]$, $p_1/q_1 = [a_0, a_1]$, ... Then $q > 8 \times 10^{33} > 6M$, and $\varepsilon > 0.01$ for all choices $d \in \{1, \ldots, 9\}$, giving

$$\sqrt{n} \leq 7 + \frac{\log(100 q_{68})}{\log \alpha} < 179,$$

so that $n \le 32000$.

We now repeat the process with $M := 32000$. We take $q := q_{13} = 3321060$ to be the denominator of the thirteenth convergent to γ. We compute again ε and we find that it exceeds 0.006 in all cases. Thus,

$$\sqrt{n} < 7 + \frac{\log(1000q_{13}/6)}{\log \alpha} < 49,$$

so $n \le 2385$, which contradicts (2).

3 Comments

The method of the proof is based on the fact that there are only finitely many positive integers which are *dominant* in two multiplicatively independent bases which are both algebraic integers. Namely, given nonzero algebraic numbers A, B, C, D, with B and D multiplicatively independent, there exists a constant $\kappa := \kappa(A, B, C, D)$ such that the equation

$$AB^n \left(1 + O\left(\frac{1}{n^K}\right)\right) = CD^m \left(1 + O\left(\frac{1}{m^L}\right)\right) \tag{10}$$

has only finitely many positive solutions (m, n) which are all effectively computable provided that $\min\{K, L\} > \kappa$. The upper bound on $\max\{m, n\}$ depends, of course, on A, B, C, D, and on the constants implied by the above O-symbols. The rep–units whose Zeckendorf representation has only triangular indices treated in this paper have the above property with $A := d/9$, where $d \in \{1, 2, \ldots, 9\}$, and $(B, C, D) := (10, 1/\sqrt{5}, (1 + \sqrt{5})/2)$, but the arguments can be applied to treat other problems of a similar flavor.

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