Slant submersions from almost Hermitian manifolds

by

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Abstract

As a generalization of almost Hermitian submersions and anti-invariant submersions, we introduce slant submersions from almost Hermitian manifolds onto Riemannian manifolds. We give examples, investigate the geometry of foliations which are arisen from the definition of a Riemannian submersion and check the harmonicity of such submersions. We also find necessary and sufficient conditions for a slant submersion to be totally geodesic. Moreover, we obtain a decomposition theorem for the total manifold of such submersions.

Key Words: Riemannian submersion, Hermitian manifold, almost Hermitian submersion, anti-invariant Riemannian submersion, slant submersion.

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1 Introduction

Let $\bar{M}$ be an almost Hermitian manifold with complex structure $J$ and $M$ a Riemannian manifold isometrically immersed in $\bar{M}$. We note that submanifolds of a Kähler manifold are determined by the behavior of the tangent bundle of the submanifold under the action of the complex structure of the ambient manifold. A submanifold $M$ is called holomorphic (complex) if $J(T_p M) \subset T_p M$, for every $p \in M$, where $T_p M$ denotes the tangent space to $M$ at the point $p$. $M$ is called totally real if $J(T_p M) \subset T_p M^\perp$ for every $p \in M$, where $T_p M^\perp$ denotes the normal space to $M$ at the point $p$. As a generalization of holomorphic and totally real submanifolds, slant submanifolds were introduced by Chen in [5]. We recall that the submanifold $M$ is called slant [5] if for any $p \in M$ and any $X \in T_p M$, the angle between $JX$ and $T_p M$ is a constant $\theta(X) \in [0, \frac{\pi}{2}]$, i.e. it does not depend on the choice of $p \in M$ and $X \in T_p M$. It follows that invariant and totally real immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively.
A slant immersion which is neither invariant nor totally real is called a proper slant immersion.

On the other hand, Riemannian submersions between Riemannian manifolds were studied by O’Neill [13] and Gray [10]. Later such submersions were considered between manifolds with differentiable structures. As an analogue of holomorphic submanifolds, Watson defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases [15]. We note that almost Hermitian submersions have been extended to the almost contact manifolds [6], locally conformal Kähler manifolds [12] and quaternion Kähler manifolds [11].

Let \( M \) be a complex \( m \)-dimensional almost Hermitian manifold with Hermitian metric \( g_M \) and almost complex structure \( J_M \) and \( N \) be a complex \( n \)-dimensional almost Hermitian manifold with Hermitian metric \( g_N \) and almost complex structure \( J_N \). A Riemannian submersion \( F : M \to N \) is called an almost Hermitian submersion if \( F \) is an almost complex map, i.e., \( F^* J_M = J_N F^* \). The main result of this notion is that the vertical and horizontal distributions are \( J_M \)-invariant. On the other hand, Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds have been studied by many authors under the assumption that the vertical spaces of such submersions are invariant with respect to the complex structure. For instance, Escobales [8] studied Riemannian submersions from complex projective space onto a Riemannian manifold under the assumption that the fibers are connected, complex, totally geodesic submanifolds. One can see that this assumption implies that the vertical distribution is invariant.

Recently, we introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds and investigated the geometry of such submersions [14]. We now recall the definition of anti-invariant Riemannian submersions. Let \( M \) be a complex \( m \)-dimensional almost Hermitian manifold with Hermitian metric \( g_M \) and almost complex structure \( J \) and \( N \) a Riemannian manifold with Riemannian metric \( g_N \). Suppose that there exists a Riemannian submersion \( F : M \to N \) such that \((\ker F^*)^\perp\) is anti-invariant with respect to \( J \), i.e., \( J(\ker F^*) \subseteq (\ker F^*)^\perp \). Then we say that \( F \) is an anti-invariant Riemannian submersion. In this paper, as a generalization of Hermitian submersions and anti-invariant submersions, we define and study slant submersions from almost Hermitian manifolds onto Riemannian manifolds.

The paper is organized as follows: In section 2, we present the basic information needed for this paper. In section 3, we give definition of slant Riemannian submersions, provide examples and give a sufficient condition for slant submersions to be harmonic. We also investigate the geometry of leaves of the dis-
tributions and obtain a decomposition theorem. Finally we give necessary and sufficient conditions for such submersions to be totally geodesic.

2 Preliminaries

In this section, we define almost Hermitian manifolds, recall the notion of Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions.

Let \((\bar{M}, g)\) be an almost Hermitian manifold. This means \[16\] that \(\bar{M}\) admits a tensor field \(J\) of type (1, 1) on \(\bar{M}\) such that, \(\forall X, Y \in \Gamma(T\bar{M})\), we have

\[J^2 = -I, \quad g(X, Y) = g(JX, JY). \tag{2.1}\]

An almost Hermitian manifold \(\bar{M}\) is called Kähler manifold if

\[(\bar{\nabla}_X J)Y = 0, \forall X, Y \in \Gamma(T\bar{M}), \tag{2.2}\]

where \(\bar{\nabla}\) is the Levi-Civita connection on \(\bar{M}\). Let \((M^m, g_M)\) and \((N^n, g_N)\) be Riemannian manifolds, where \(\dim(M) = m, \dim(N) = n\) and \(m > n\). A Riemannian submersion \(F : M \to N\) is a map from \(M\) onto \(N\) satisfying the following axioms:

(S1) \(F\) has maximal rank.

(S2) The differential \(F_*\) preserves the lengths of horizontal vectors.

For each \(q \in N\), \(F^{-1}(q)\) is an \((m - n)\) dimensional submanifold of \(M\). The submanifolds \(F^{-1}(q)\) are called fibers. A vector field on \(M\) is called horizontal if it is always tangent to fibers. A vector field on \(M\) is called horizontal if it is always orthogonal to fibers. A vector field \(X\) on \(M\) is called basic if \(X\) is horizontal and \(F-\) related to a vector field \(X_*\) on \(N\), i.e., \(F_*X_p = X_*F(p)\) for all \(p \in M\). Note that we denote the projection morphisms on the distributions \((kerF_*)\) and \((kerF_*^N)'\) by \(\mathcal{V}\) and \(\mathcal{H}\), respectively.

We recall the following lemma from O’Neill [13].

Lemma 2.1 Let \(F : M \to N\) be a Riemannian submersion between Riemannian manifolds and \(X, Y\) be basic vector fields of \(M\). Then

(a) \(g_M(X, Y) = g_N(X_*, Y_*) \circ F\),

(b) the horizontal part \([X, Y]^H\) of \([X, Y]\) is a basic vector field and corresponds to \([X_*, Y_*]\), i.e., \(F_*([X, Y]^H) = [X_*, Y_*]\).

(c) \([V, X]\) is vertical for any vector field \(V\) of \((kerF_*)\).

(d) \((\nabla^N_X Y)^H\) is the basic vector field corresponding to \(\nabla^N_X Y_*\).
The geometry of Riemannian submersions is characterized by O'Neill’s tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields $E, F$ on $M$ by

$$\mathcal{A}_E F = \mathcal{H} \nabla_{\mathcal{H}E} V F + \mathcal{V} \nabla_{\mathcal{H}E} \mathcal{H} F \quad (2.3)$$

$$\mathcal{T}_E F = \mathcal{H} \nabla_{\mathcal{V}E} V F + \mathcal{V} \nabla_{\mathcal{V}E} \mathcal{H} F, \quad (2.4)$$

where $\nabla$ is the Levi-Civita connection of $g_M$. It is easy to see that a Riemannian submersion $F : M \longrightarrow N$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM)$, $\mathcal{A}_E$ and $\mathcal{T}_E$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_V E$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_E = \mathcal{A}_H E$. We note that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T} U W = \mathcal{T} W U, \forall U, W \in \Gamma(ker F_*) \quad (2.5)$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2} \mathcal{V}[X, Y], \forall X, Y \in \Gamma((ker F_*)^\perp). \quad (2.6)$$

On the other hand, from (2.3) and (2.4) we have

$$\nabla_V W = \mathcal{T}_V W + \mathcal{\hat{V}}_V W \quad (2.7)$$

$$\nabla_V X = \mathcal{H} \nabla_V X + \mathcal{T}_V X \quad (2.8)$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V} \nabla_X V \quad (2.9)$$

$$\nabla_X Y = \mathcal{H} \nabla_X Y + \mathcal{A}_X Y \quad (2.10)$$

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $V, W \in \Gamma(ker F_*)$, where $\mathcal{\hat{V}}_V W = \mathcal{V} \nabla_V W$. If $X$ is basic, then $\mathcal{H} \nabla_V X = \mathcal{A}_X V$.

Finally, we recall the notion of harmonic maps between Riemannian manifolds. Let $(M, g_M)$ and $(N, g_N)$ be Riemannian manifolds and suppose that $\varphi : M \longrightarrow N$ is a smooth map between them. Then the differential $\varphi_*$ of $\varphi$ can be viewed a section of the bundle $\text{Hom}(TM, \varphi^{-1}TN) \longrightarrow M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibers $(\varphi^{-1}TN)_p = T_{\varphi(p)}N, p \in M$. $\text{Hom}(TM, \varphi^{-1}TN)$ has a connection $\nabla$ induced from the Levi-Civita connection $\nabla^M$ and the pullback connection $\nabla^\varphi$. Then the second fundamental form of $\varphi$ is given by

$$(\nabla \varphi_*)(X, Y) = \nabla^\varphi_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y) \quad (2.11)$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \longrightarrow (N, g_N)$ is said to be harmonic if $\text{trace}(\nabla \varphi_*) = 0$. On the other hand, the tension field of $\varphi$ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = \text{div} \varphi_* = \sum_{i=1}^m (\nabla \varphi_*)(e_i, e_i), \quad (2.12)$$

where $\{e_1, ..., e_m\}$ is the orthonormal frame on $M$. Then it follows that $\varphi$ is harmonic if and only if $\tau(\varphi) = 0$, for details, see [1].
3 Slant Submersions

In this section, we define slant submersions from an almost Hermitian manifold onto a Riemannian manifold by using the definition of a slant distribution given in [3]. We give examples and check the harmonicity of slant submersions. Then we investigate the geometry of leaves of distributions and obtain a decomposition theorem for the total manifold. We also obtain a necessary and sufficient condition for such submersions to be totally geodesic map.

We first recall the definition of the slant distribution. Given a submanifold $M$, isometrically immersed in an almost Hermitian manifold $(\bar{M}, \bar{g}, J)$, a differentiable distribution $D$ on $M$ is said to be a slant distribution if for any non-zero vector $X \in D_p$; $p \in M$, the angle between $JX$ and the vector space $D_p$ is constant, that is, it is independent of the choice of $p \in M$ and $X \in D_p$. This constant angle is called the slant angle of the slant distribution [3]. Inspiring from the above definition we present the following notion.

**Definition 3.1.** Let $F$ be a Riemannian submersion from an almost Hermitian manifold $(M_1, g_1, J_1)$ onto a Riemannian manifold $(M_2, g_2)$. If for any non-zero vector $X \in (\ker F^*_p)$; $p \in M_1$, the angle $\theta(X)$ between $JX$ and the space $(\ker F^*_p)$ is a constant, i.e. it is independent of the choice of the point $p \in M_1$ and choice of the tangent vector $X$ in $(\ker F^*_p)$, then we say that $F$ is a slant submersion. In this case, the angle $\theta$ is called the slant angle of the slant submersion.

It is known that the distribution $(\ker F^*_p)$ is integrable for a Riemannian submersion between Riemannian manifolds. In fact, its leaves are $F^{-1}(q)$, $q \in M_1$, i.e., fibers. Thus it follows from above definition that the fibers of a slant submersion are slant submanifolds of $M_1$, for slant submanifolds, see: [4].

We first give some examples of slant submersions.

**Example 1.** Every almost Hermitian submersion from an almost Hermitian manifold onto an almost Hermitian manifold is a slant submersion with $\theta = 0$.

**Example 2.** Every anti-invariant Riemannian submersion from an almost Hermitian manifold to a Riemannian manifold is a slant submersion with $\theta = \frac{\pi}{2}$.

A slant submersion is said to be proper if it is neither Hermitian nor anti-invariant Riemannian submersion.

**Example 3.** Consider the following Riemannian submersion given by

$$F : \ R^4 \rightarrow \ R^2 \quad (x_1, x_2, x_3, x_4) \mapsto \ (x_1 \sin \alpha - x_3 \cos \alpha, x_4).$$
Then for any $0 < \alpha < \frac{\pi}{2}$, $F$ is a slant submersion with slant angle $\alpha$.

**Example 4.** Consider the following Riemannian submersion given by

$$F : \mathbb{R}^4 \rightarrow \mathbb{R}^2 \quad (x_1, x_2, x_3, x_4) \rightarrow \left(\frac{x_1-x_4}{\sqrt{2}}, x_2\right).$$

Then $F$ is a slant submersion with slant angle $\theta = \frac{\pi}{4}$.

Let $F$ be a Riemannian submersion from an almost Hermitian manifold $(M_1, g_1, J_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then for $X \in \Gamma(kerF_*)$, we write

$$JX = \phi X + \omega X,$$

where $\phi X$ and $\omega X$ are vertical and horizontal parts of $JX$. Also for $Z \in \Gamma((kerF_*)^\perp)$, we have

$$JZ = BZ + CZ,$$

where $BZ$ and $CZ$ are vertical and horizontal components of $JZ$. Using (2.7), (2.8), (3.1) and (3.2) we obtain

$$(\nabla_X \omega)Y = CT_X Y - T_X \phi Y$$

$$(\nabla_X \phi)Y = BT_X Y - T_X \omega Y,$$

where $\nabla$ is the Levi-Civita connection on $M_1$ and

$$(\nabla_X \omega)Y = H \nabla_X \omega Y - \omega \nabla_X Y$$

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y$$

for $X, Y \in \Gamma(kerF_*)$. Let $F$ be a proper slant submersion from an almost Hermitian manifold $(M_1, g_1, J_1)$ onto a Riemannian manifold $(M_2, g_2)$, then we say that $\omega$ is parallel with respect to the Levi-Civita connection $\nabla$ on $(kerF_*)$ if its covariant derivative with respect to $\nabla$ vanishes, i.e., we have

$$(\nabla_X \omega)Y = \nabla_X \omega Y - \omega(\nabla_X Y) = 0$$

for $X, Y \in \Gamma(kerF_*)$.

The proof of the following result is exactly same with slant immersions (see [4] or [2] for Sasakian case), therefore we omit its proof.

**Theorem 3.1.** Let $F$ be a Riemannian submersion from an almost Hermitian manifold $(M_1, g_1, J)$ onto a Riemannian manifold $(M_2, g_2)$. Then $F$ is a proper slant submersion if and only if there exists a constant $\lambda \in [-1, 0]$ such that

$$\phi^2 X = \lambda X$$
for $X \in \Gamma(\ker F)$. If $F$ is a proper slant submersion, then $\lambda = -\cos^2 \theta$.

By using above theorem, it is easy to see the following lemma.

**Lemma 3.1.** Let $F$ be a proper slant submersion from an almost Hermitian manifold $(M_1, g_1, J_1)$ onto a Riemannian manifold $(M_2, g_2)$ with slant angle $\theta$. Then, for any $X, Y \in \Gamma(\ker F)$, we have

\[
g_1(\phi X, \phi Y) = \cos^2 \theta g_1(X, Y),
\]

(3.5)

\[
g_1(\omega X, \omega Y) = \sin^2 \theta g_1(X, Y).
\]

(3.6)

We now denote the orthogonal complementary distribution to $\omega(\ker F)$ in $(\ker F)^{\perp}$ by $\mu$. Then we have the following.

**Proposition 3.1.** Let $F$ be a proper slant submersion from an almost Hermitian manifold $(M_1, g_1, J_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then $\mu$ is invariant with respect to $J_1$.

**Proof:** For $V \in \Gamma(\mu)$, using (2.1), we have

\[
g_1(J_1 V, \omega X) = g_1(J_1 V, J_1 X) - g_1(J_1 V, \phi X) = -g_1(J_1 V, \phi X)
\]

for $X \in \Gamma(\ker F)$. Then, using Theorem 3.1, we get

\[
g_1(J_1 V, \omega X) = g_1(V, J_1 \phi X) = g_1(V, \phi^2 X) + g_1(V, \omega \phi X) = -\cos^2 \theta g_1(V, X) + g_1(V, \omega \phi X) = g_1(V, \omega \phi X) = 0
\]

due to $\mu$ is orthogonal to $\omega(\ker F)$. In a similar way, we have $g_1(J_1 V, Y) = -g_1(V, J_1 Y) = 0$ due to $J_1 Y \in \Gamma((\ker F) \oplus \omega(\ker F))$ for $V \in \Gamma(\mu)$ and $Y \in \Gamma(\ker F)$. Thus proof is complete. 

**Corollary 3.1.** Let $F$ be a proper slant submersion from an almost Hermitian manifold $(M_1^m, g_1, J_1)$ onto a Riemannian manifold $(M_2^n, g_2)$. Let

\[
\{e_1, \ldots, e_{m-n}\}
\]

be a local orthonormal basis of $(\ker F)$, then $\{\text{csc } \omega e_1, \ldots, \text{csc } \omega e_{m-n}\}$ is a local orthonormal basis of $\omega(\ker F)$. 

Proof: It will be enough to show that $g_1(\csc \theta \omega e_i, \csc \theta \omega e_j) = \delta_{ij}$, for any $i, j \in \{1, \ldots, \frac{m-n}{2}\}$. By using (3.6), we have

$$g_1(\csc \theta \omega e_i, \csc \theta \omega e_j) = \csc^2 \theta \sin^2 \theta g_1(e_i, e_j) = \delta_{ij},$$

which proves the assertion.

We note that above Proposition 3.1 tells that the distributions $\mu$ and $(\ker F^*) \oplus \omega(\ker F^*)$ are even dimensional. In fact it implies that the distribution $(\ker F^*)$ is even dimensional. More precisely, we have the following result whose proof is similar to the above corollary.

Lemma 3.2. Let $F$ be a proper slant submersion from an almost Hermitian manifold $(M^m_1, g_1, J_1)$ onto a Riemannian manifold $(M^m_2, g_2)$. If $e_1, \ldots, e_{\frac{m-n}{2}}$ are orthogonal unit vector fields in $(\ker F^*)$, then

$$\{e_1, \sec \theta \phi e_1, e_2, \sec \theta \phi e_2, \ldots, e_{\frac{m-n}{2}}, \sec \theta \phi e_{\frac{m-n}{2}}\}$$

is a local orthonormal basis of $(\ker F^*)$.

Let $F$ be a proper slant submersion from an almost Hermitian manifold $(M^m_1, J_1, g_1)$ onto a Riemannian manifold $(M^m_2, g_2)$. As in slant immersions, we call such an orthonormal frame

$$\{e_1, \sec \theta \phi e_1, e_2, \sec \theta \phi e_2, \ldots, e_n, \sec \theta \phi e_n, \csc \theta \omega e_1, \csc \theta \omega e_2, \ldots, \csc \theta \omega e_n\}$$

an adapted slant frame for slant submersions.

In the sequel, we show that the slant submersion puts some restrictions on the dimensions of the distributions and the base manifold.

Proposition 3.2. Let $F$ be a proper slant submersion from an almost Hermitian manifold $(M^m_1, g_1, J_1)$ onto a Riemannian manifold $(M^m_2, g_2)$. Then $\dim(\mu) = 2n - m$. If $\mu = \{0\}$, then $n = \frac{m}{2}$.

Proof: First note that $\dim(\ker F^*) = m - n$. Thus using Corollary 3.1, we have $\dim((\ker F^*) \oplus \omega(\ker F^*)) = 2(m - n)$. Since $M_1$ is $m-$ dimensional, we get $\dim(\mu) = 2n - m$. Second assertion is clear.

We now check the harmonicity of slant submersions. But we first give a preparatory lemma.
Lemma 3.3. Let $F$ be a proper slant submersion from a Kähler manifold onto a Riemannian manifold. If $\omega$ is parallel with respect to $\nabla$ on $(\ker F^*)$, then we have
\[ T_{\phi X} \phi X = -\cos^2 \theta T_X X \] (3.7)
for $X \in \Gamma(\ker F^*)$.

Proof: If $\omega$ is parallel, then from (3.3) we have $C T_X Y = T_X \phi Y$ for $X, Y \in \Gamma(\ker F^*)$. Interchanging the role of $X$ and $Y$, we get $C T_Y X = T_Y \phi X$. Thus we have
\[ C T_X Y - C T_Y X = T_X \phi Y - T_Y \phi X. \]
Using (2.5) we derive
\[ T_X \phi Y = T_Y \phi X. \] (3.8)
Then substituting $Y$ by $\phi X$ we get $T_X \phi^2 X = T_{\phi X} \phi X$. Finally using Theorem 3.1 we obtain (3.7).

We now give a sufficient condition for a proper slant submersion to be harmonic.

Theorem 3.2. Let $F$ be a proper slant submersion from a Kähler manifold onto a Riemannian manifold. If $\omega$ is parallel with respect to $\nabla$ on $(\ker F^*)$, then $F$ is a harmonic map.

Proof: Since
\[ (\nabla F_*)(Z_1, Z_2) = 0 \] (3.9)
for $Z_1, Z_2 \in \Gamma((\ker F^*)^\perp)$. A proper slant submersion $F$ is harmonic if and only if
\[ \sum_{i=1}^{m-n} (\nabla F_*)(\tilde{e}_i, \tilde{e}_i) = -\sum_{i=1}^{m-n} F_*(T e_i, T e_i) = 0, \]
where $\{\tilde{e}_i\}_{i=1}^{m-n}$ is an orthonormal basis of $(\ker F^*)$. Thus using Lemma 3.2, we can write
\[ \tau = -\sum_{i=1}^{m-n} F_*(T e_i, e_i + T_{\sec \theta \phi e_i, \sec \theta \phi e_i}). \]
Hence we have
\[ \tau = -\sum_{i=1}^{m-n} F_*(T e_i, e_i + \sec^2 \theta T_{\phi e_i, \phi e_i}). \]
Then using (3.7) we arrive at
\[ \tau = -\sum_{i=1}^{m-n} F_*(T e_i, e_i - T e_i, e_i) = 0 \]
which shows that $F$ is harmonic.
We now investigate the geometry of the leaves of the distributions \((\ker F^*)\) and \((\ker F^*)^\perp\).

**Theorem 3.3.** Let \(F\) be a proper slant submersion from a Kähler manifold \((M_1, g_1, J_1)\) onto a Riemannian manifold \((M_2, g_2)\). Then the distribution \((\ker F^*)\) defines a totally geodesic foliation on \(M_1\) if and only if

\[
g_1(\mathcal{H}_X \omega Y, Z) = g_1(\mathcal{H}_X \omega Y, CZ) + g_1(T_X \omega Y, BZ)
\]

for \(X, Y \in \Gamma(\ker F^*)\) and \(Z \in \Gamma((\ker F^*)^\perp)\).

**Proof:** For \(X, Y \in \Gamma(\ker F^*)\) and \(Z \in \Gamma((\ker F^*)^\perp)\), from (2.1) and (3.1) we have

\[
g_1(\nabla_X Y, Z) = g_1(\nabla_X \phi Y, JZ) + g_1(\nabla_X \omega Y, JZ).
\]

Using (2.1), (3.1) and (3.2) we get

\[
g_1(\nabla_X Y, Z) = -g_1(\nabla_X \phi^2 Y, Z) - g_1(\nabla_X \omega Y, Z) + g_1(\nabla_X \omega Y, BZ) + g_1(\nabla_X \omega Y, CZ).
\]

Then from (2.8) and Theorem 3.1 we obtain

\[
g_1(\nabla_X Y, Z) = \cos^2 \theta g_1(\nabla_X Y, Z) - g_1(\mathcal{H}_X \omega Y, Z) + g_1(O_X \omega Y, BZ) + g_1(\mathcal{H}_X \omega Y, CZ).
\]

Hence we have

\[
sin^2 \theta g_1(\nabla_X Y, Z) = -g_1(\mathcal{H}_X \omega Y, Z) + g_1(O_X \omega Y, BZ) + g_1(\mathcal{H}_X \omega Y, CZ)
\]

which proves assertion. \(\Box\)

In a similar way we have the following.

**Theorem 3.4.** Let \(F\) be a proper slant submersion from a Kähler manifold \((M_1, g_1, J_1)\) onto a Riemannian manifold \((M_2, g_2)\). Then the distribution \((\ker F^*)^\perp\) defines a totally geodesic foliation on \(M_1\) if and only if

\[
g_1(\mathcal{H}_Z, Z_2, \omega Y) = g_1(A_Z, BZ_2 + \mathcal{H}_Z, CZ_2, \omega Y)
\]

for \(X \in \Gamma(\ker F^*)\) and \(Z_1, Z_2 \in \Gamma((\ker F^*)^\perp)\).

From Theorem 3.3 and Theorem 3.4 we have the following result.
Corollary 3.1. Let $F$ be a proper slant submersion from a Kähler manifold $(M_1, g_1, J_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then $M_1$ is a locally product Riemannian manifold if and only if

\[
g_1(\mathcal{H}\nabla Z_1 Z_2, \omega \phi X) = g_1(A Z_1 B Z_2 + \mathcal{H}\nabla Z_1 C Z_2, \omega X)
\]

and

\[
g_1(\mathcal{H}\nabla X \omega \phi Y, Z_1) = g_1(\mathcal{H}\nabla X \omega Y, C Z_1) + g_1(T X \omega Y, B Z_1)
\]

for $X, Y \in \Gamma(\ker F^*)$ and $Z_1, Z_2 \in \Gamma((\ker F^*)^\perp)$.

Finally we give necessary and sufficient conditions for a proper slant submersion to be totally geodesic. Recall that a differentiable map $F$ between Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ is called a totally geodesic map if $(\nabla F^*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM_1)$.

Theorem 3.5. Let $F$ be a proper slant submersion from a Kähler manifold $(M_1, g_1, J_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then $F$ is totally geodesic if and only if

\[
g_1(T X \omega Y, B Z_1) + g_1(\mathcal{H}\nabla X \omega Y, C Z_1) = g_1(\mathcal{H}\nabla X \omega \phi Y, Z_1)
\]

and

\[
g_1(A Z_1 B Z_2 + \mathcal{H}\nabla Z_1 C Z_2, \omega X) = -g_1(\mathcal{H}\nabla Z_1 \omega \phi X, Z_2)
\]

for $Z_1, Z_2 \in \Gamma((\ker F^*)^\perp)$ and $X, Y \in \Gamma(\ker F^*)$.

Proof: First of all, since $F$ is a Riemannian submersion we have

\[
(\nabla F^*)(Z_1, Z_2) = 0
\]

for $Z_1, Z_2 \in \Gamma((\ker F^*)^\perp)$. Thus it is enough to show that $(\nabla F^*)(X, Y) = 0$ and $(\nabla F^*)(X, Z) = 0$ for $X, Y \in \Gamma(\ker F^*)$ and $Z \in \Gamma((\ker F^*)^\perp)$. For $X, Y \in \Gamma(\ker F^*)$ and $Z_1 \in \Gamma((\ker F^*)^\perp)$, since $F$ is a Riemannian submersion, from (2.1), (3.1) and (3.2) we have

\[
g_2((\nabla F^*)(X, Y), F Z) = g_1(\nabla X \phi Y, Z) - g_1(\nabla X \omega Y, J Z).
\]

Using again (3.1) and (3.2) we get

\[
g_2((\nabla F^*)(X, Y), F Z) = g_1(\nabla X \phi^2 Y, Z) + g_1(\nabla X \omega \phi Y, Z) - g_1(\nabla X \omega Y, B Z) - g_1(\nabla X \omega Y, C Z).
\]

Then Theorem 3.1, (2.7) and (2.8) imply that

\[
g_2((\nabla F^*)(X, Y), F Z) = -\cos^2 \theta g_1(\nabla X Y, Z) + g_1(\nabla X \omega \phi Y, Z) - g_1(T X \omega Y, B Z) - g_1(\mathcal{H}\nabla X \omega Y, C Z).
\]
Hence we obtain
\[ \sin^2 \theta g_2((\nabla F)(X,Y), F_* Z) = g_1(\nabla_X \phi Y, Z) - g_1(T_X Y, B Z) - g_1(\mathcal{H}\nabla_X Y, C Z). \] (3.10)

In a similar way, we get
\[ \sin^2 \theta g_2((\nabla F)(X, Z_1), F_* (Z_2)) = g_1(A Z_1 B Z_2 + \mathcal{H}\nabla Z_1 C Z_2, \omega X) + g_1(\mathcal{H}\nabla Z_1 \omega \phi X, Z_2). \] (3.11)

Then proof follows from (3.10) and (3.11).

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References


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