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Acute Triangulations of Pentagons

by
Liping Yuan

Abstract

An acute triangulation is a triangulation whose triangles have all their angles less than $\frac{\pi}{2}$. In this paper we prove that *i*) every planar pentagon can be triangulated into at most 54 acute triangles, and *ii*) every double pentagon can be triangulated into at most 76 acute triangles.

Key Words: Acute triangulation; pentagon; double pentagon; Helly's theorem.
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1 Introduction

A *triangulation* of a two-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments*, i.e., shortest paths between the corresponding vertices. We are interested only in *geodesic triangulations*, all the members of which are, by definition, geodesic triangles. An *acute* triangulation is a triangulation whose triangles have all their angles less than $\frac{\pi}{2}$. The number of triangles in a triangulation is called its *size*.

The interest for acute (or non-obtuse) triangulation first appeared in applied mathematics. In 1953 R. H. MacNeal [11] needed them when investigating the discretization of partial differential equations. Also, for problems in Numerical Analysis, very flat (and very sharp) angles are not desirable (see for example [3]). In 1960, Burago and Zalgaller [1] investigated in considerable depth acute triangulations of polygonal complexes, being led to them by the problem of their isometric embedding into $\mathbb{R}^3$. (However, their method could not give an estimate on the number of triangles used.) The question whether an obtuse triangle can be acutely triangulated was also independently asked and solved (see [4], [5], [6]). In 1980, Cassidy and Lord [2] considered acute triangulations of the square.

Recently, Maehara investigated acute triangulations of quadrilaterals [12] and other polygons [13], and a result of the latter was improved by Yuan [16], where it was proved that every n-gon admits an acute triangulation with size at most $24(106n - 216)$. This is the first concrete upper bound on the size of acute triangulations of n-gons depending on n. In 2010, Yuan [17] considered the acute triangulations of trapezoids.

On the other hand, compact convex surfaces have also been triangulated. Acute triangulations of all Platonic surfaces, which are surfaces of the five well-known Platonic solids, were investigated in [7], [9], and [10]. Furthermore, some other famous surfaces have also been acutely triangulated, such as flat Möbius strips [18] and flat tori [8].

In 2009, Saraf [15] gave a new proof for the existence of acute triangulations of general polyhedral surfaces, but there is still no estimate on the size of the existing acute triangulations. The following problem first raised in [7] is natural, and not easy.

Problem 1. Does there exist a number N such that every compact convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles?

As remarked in [10], Problem 1 can be transfered to other families of Alexandrov surfaces, with or without boundary.

In this paper we discuss the acute triangulations of pentagons and doubly covered pentagons. The doubly covered pentagon, or simply the *double pentagon*, is a (degenerate) convex polyhedral surface (homeomorphic to the 2-sphere) consisting of two isometric convex pentagonal (planar) sides glued along the boundaries in the obvious way (according to the isometry). Acute triangulations of double triangles [20] and double quadrilaterals [19] have already been investigated. In Section 2, we present some preparatory propositions. In Section 3, we prove that every pentagon can be triangulated into at most 54 acute triangles (the upper bound $24(106n-216)$ gives 7536 triangles in this case). In Section 4, we consider the acute triangulations of double pentagons. We prove that any double pentagon can be triangulated into at most 76 acute triangles. (Note that the glued a edges of the pentagons need not be edges of the triangulation.)

2 Preliminaries

A (simple) polygon Γ is a planar set homeomorphic to a compact disc, having as boundary bdΓ a finite union of line-segments. Each endpoint of such a line-segment is called a vertex of Γ. A vertex of Γ is called an *acute corner* if Γ has an acute angle at this vertex.

Let $\mathcal{T}$ be an acute triangulation of a polygon Γ. A vertex P of $\mathcal{T}$ is called

− a *corner vertex* if P is a vertex of Γ,

− a *side vertex* if P lies on bdΓ but is not a corner vertex,

− an *interior vertex* otherwise.

For any set $A \subset \mathbb{R}^d$, let int$A$ denote the interior of A and relint$A$ the relative interior of A. (Here the "relative interior" of a set A is defined as its interior within the affine hull of A.)

Several results obtained by Maehara [12] will be very useful.

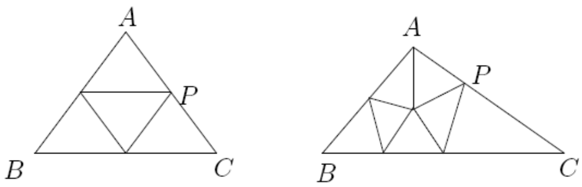

Figure 1: Illustrations of Proposition 2.1

Proposition 2.1. ([12]) *Let ABC be a triangle with acute angles at B and C, and let $P \in \mathrm{relint} AC$. If the angle at A is acute (resp. non-acute), then there is an acute triangulation $\mathcal{T}$ of ABC with size 4 (resp. 7) such that P is the only side vertex on AC. Further, there is (resp. are) exactly 1 (resp. 2) new vertex introduced on BC and exactly 1 new vertex introduced on AB.*

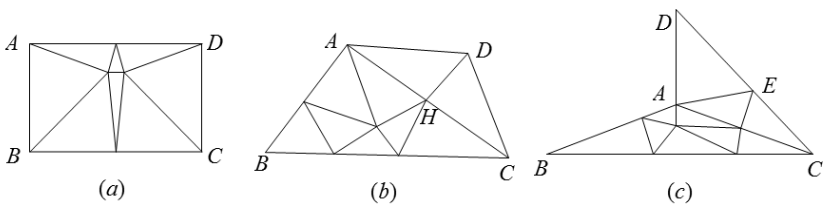

Figure 2: Illustrations of Proposition 2.2 and 2.3

Proposition 2.2. ([12]) *Let $ABCD$ be a convex quadrilateral. If $\angle B < \frac{\pi}{2}$ and $\angle D \geq \frac{\pi}{2}$, then there is an acute triangulation $\mathcal{T}$ of $ABCD$ of size at most 9 such that there is no side vertex in $CD \cup DA$. Further, if $\angle ACB$(resp. $\angle BAC$) $< \frac{\pi}{2}$, then there is exactly 1 new vertex introduced on AB (resp. BC); if the angle $\angle ACB$ (resp. $\angle BAC$) $\geq \frac{\pi}{2}$, then there are exactly 2 new vertex introduced on AB (resp. BC).*

Proposition 2.3. ([12]) *Every quadrilateral admits an acute triangulation of size at most 10, such that there are at most two new vertices introduced on each side.*

The following results will also be useful.

Proposition 2.4. *Let ABC be a triangle with $\angle B < \frac{\pi}{2}$ and let $M, N \in \mathrm{relint} AC$. Then ABC admits a non-obtuse triangulation of size at most 11, with M, N as the only side vertices on AC, so that the angles at all vertices different from M and N are acute.*

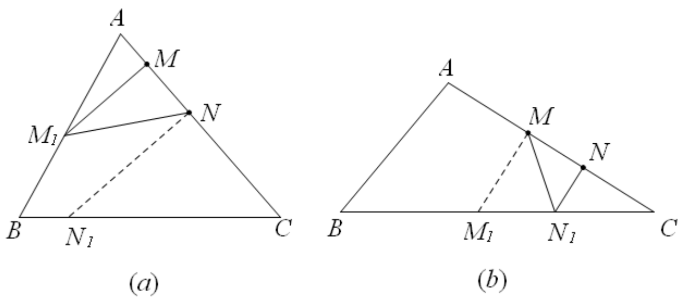

Figure 3: A non-obtuse triangulation of ABC with two new vertices on AC

Proof. Consider $M_1, N_1 \in AB \cup BC$ with $M_1M \perp AC$, $N_1N \perp AC$. We may assume without loss of generality that $\angle C < \frac{\pi}{2}$, $M \in \text{relint}AN$ and $N_1 \in \text{relint}BC$, as shown in Figure 3 (a) and (b). If $M_1 \in AB$ (resp. $M_1 \in BC$) then, by Proposition 2.2 the quadrilateral $BCNM_1$ (resp. ABN_1M) can be triangulated into at most 9 acute triangles with no new vertex introduced on $BM_1 \cup M_1N$ (resp. $AM \cup MN_1$). Hence ABC admits a non-obtuse triangulation of size at most 11, with M, N the only side vertices on AC. □

Let Γ be a convex polygon. A point $P \in \Gamma$ and an edge XY of Γ are said to be *facing* each other *in* Γ, if the points P, X, Y are the vertices of a non-degenerate triangle contained in Γ and $\angle PXY$, $\angle PYX$ are both less than or equal to $\frac{\pi}{2}$. A point $P \in \text{int}\Gamma$ is called a *pivot* of Γ if all edges of Γ are facing P in Γ.

Motivated by [13], we obtain the following refined result; the similar proof is omitted.

Proposition 2.5. *If a convex polygon Γ has a pivot $P \in \text{int}\Gamma$, then it admits an acute triangulation in which the vertices newly introduced on the edges facing P are the orthogonal projections of P. Furthermore, if Γ has n vertices, m non-obtuse angles and r edges, the orthogonal projection of P on each of which is a corner of Γ, then the number of triangles in this acute triangulation is at most $4n + 2m - r$.*

Now we give two examples to illustrate the acute triangulations described in Proposition 2.5.

In Figure 4 (a), P is a pivot of a right triangle ABC. By Proposition 2.5, ABC can be triangulated into $4 \times 3 + 2 \times 3 - 0 = 18$ acute triangles, where P_1, P_2, P_3 are orthogonal projections of P on AB, BC, CA respectively.

In Figure 4 (b), P is a pivot of a right trapezoid $ABCD$ with $PD \perp AD$. By Proposition 2.5, $ABCD$ can be triangulated into $4 \times 4 + 2 \times 3 - 1 = 21$ acute

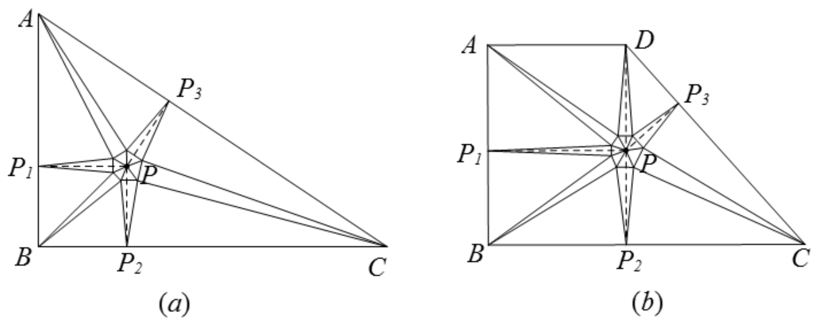

Figure 4: Acute triangulations by using pivots

triangles, where P_1, P_2, P_3 are orthogonal projections of P on AB, BC, CD respectively.

Remark. Let ABC be an acute triangle. If we replace a vertex, say A, by a vertex A' which is close to A enough, then clearly the triangle $A'BC$ is still acute. In other words, if we slightly slide A in any direction, then the triangle obtained is still acute.

3 Acute triangulations of pentagons

Let $\Gamma = ABCDE$ be a pentagon with an acute corner B. If Γ can be divided into a triangle ABC and a simple quadrilateral $ACDE$, then B is said to be a *good acute corner* of Γ. In order to prove Theorem 3.5, we first present some lemmas.

Lemma 3.1. *Every pentagon with at least one acute corner can be triangulated into at most* 32 *acute triangles.*

Proof. Let $\Gamma = ABCDE$ be a pentagon with at least one acute corner.

Case 1. Γ has a good acute corner.

Suppose that B is a good acute corner. By Proposition 2.3, $ACDE$ admits an acute triangulation $\mathcal{T}$ with $|\mathcal{T}| \leq 10$ such that there are at most 2 side vertices on AC.

Subcase 1.1. There is no side vertex on AC. Let ACM be the acute triangle in $\mathcal{T}$ which contains AC. Let H be the orthogonal projection of M on AC. By Proposition 2.1, ABC can be triangulated into at most 7 acute triangles with H as the only side vertex on AC. Then we can slightly slide H in direction $\overrightarrow{MH}$ such that both triangles MAH and MCH become acute, and obtain an acute triangulation of Γ whose size is at most 18.

Subcase 1.2. There is precisely one side vertex on AC. Then by the similar discussion in Subcase 1.1 we know that Γ can be triangulated into at most 17 acute triangles.

Subcase 1.3. There are exactly two side vertices, say M and N, on AC. Use Proposition 2.4 to triangulate Γ into at most 21 non-obtuse triangles. Finally we can slightly slide M, N away from ABC in direction perpendicular to AC such that all the triangles become acute.

Case 2. Γ has no good acute corner.

Let B be an acute corner of Γ. We suppose without loss of generality that $\angle BCA < \frac{\pi}{2}$. Since B is not good, D, E can not lie both outside the triangle ABC.

Subcase 2.1. D lies outside the triangle ABC.

Then $E \in \mathrm{int}ABC \cup \mathrm{relint}AC$ (here $\mathrm{int}ABC$ denotes the interior of the triangle ABC). Recalling that Γ has no good acute corner, we have $\angle BAE \geq \frac{\pi}{2}$, $\angle CDE \geq \frac{\pi}{2}$. In fact, if $\angle BAE < \frac{\pi}{2}$, then A is a good acute corner; if $\angle CDE < \frac{\pi}{2}$, then D is a good acute corner.

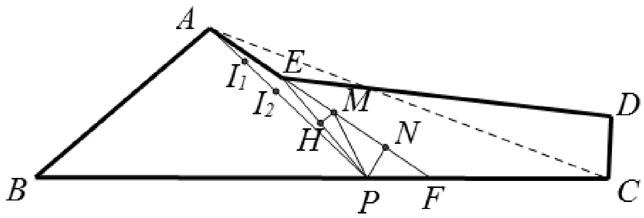

Figure 5: Two new vertices introduced on EF

If $E \in \mathrm{int}ABC$, then the supporting line of AE must intersect $\mathrm{relint}BC$ at a point F. Thus Γ can be divided into a triangle ABF and a simple quadrilateral $EFCD$, see Figure 5. By Proposition 2.3, $EFCD$ can be triangulated into at most 10 acute triangles such that there are at most 2 new vertices introduced on EF. If there is no (resp. precisely one) new vertex introduced on EF, then similarly to Subcase 1.2 (resp. 1.3), Γ admits an acute triangulation with size at most 17 (resp. 21). If there are precisely 2 new vertices, say M and N, introduced on AP, then let P be the point on BC such that $PN \perp AF$. Since $\angle BAF \geq \frac{\pi}{2}$, $P \in \mathrm{relint}BF$, see again Figure 5. Clearly, $\angle EMP > \frac{\pi}{2}$. Let H be the orthogonal projection of M on EP. Then $H \in \mathrm{relint}EP$. Furthermore, since $\angle EAP < \frac{\pi}{2}$ and $\angle AEP > \frac{\pi}{2}$, by Proposition 2.1, the triangle EAP can be triangulated into 7 acute triangles such that H is the only side vertex on EP, and there are 2 new vertices, say I_1 and I_2, introduced on AP. By Proposition 2.4, ABP admits a non-obtuse triangulation of size at most 11, such that I_1, I_2 are the only side vertices on AP. Now we slightly slide I_1, I_2 away the triangle ABP in the direction perpendicular to AP, and then slightly slide H in direction $\overrightarrow{MH}$ and N in direction $\overrightarrow{PN}$, thus obtaining an acute triangulation of Γ with size at most 32.

If $E \in \mathrm{relint}AC$, then we can triangulate triangle EDC into 7 acute triangles such that there are 2 new vertices introduced on EC. By the above discussion we know that Γ can be triangulated into 29 acute triangles.

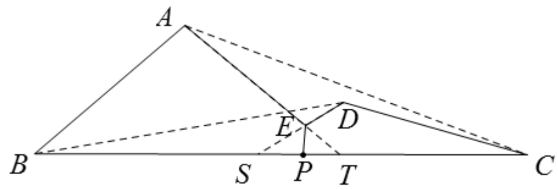

Figure 6: Illustration of Subcase 2.2

Subcase 2.2. $D \in \mathrm{int}ABC \cup \mathrm{relint}AC$.

Since $\angle BCD < \frac{\pi}{2}$ and Γ has no good acute corner, we have $E \in \mathrm{int}BDC \cup \mathrm{relint}BD$, $\angle BAE \geq \frac{\pi}{2}$, $\angle CDE \geq \frac{\pi}{2}$. Clearly, E is a concave corner of Γ. Thus there is a point $P \in \mathrm{relint}ST$ such that $\angle AEP > \frac{\pi}{2}$, $\angle DEP > \frac{\pi}{2}$, as shown in Figure 6. By Proposition 2.2, both quadrilaterals $ABPE$ and $EPCD$ can be triangulated into 9 acute triangulations such that there is no new vertex introduced on PE, which implies that Γ admits an acute triangulation with size 18. ∎

Lemma 3.2. *Let ABE be a triangle with $AH \perp BE$ ($H \in$relintBE). Then for any two points $S \in$relintBH and $T \in$relintHE, ABE can be triangulated into at most 22 non-obtuse triangles such that the only side vertices on BE are S, H and T, and the angles at all vertices different from H are acute.*

Proof. Consider $S' \in \mathrm{relint}AB$, $T' \in \mathrm{relint}AE$ with $S'S \perp BE$, $T'T \perp BE$.

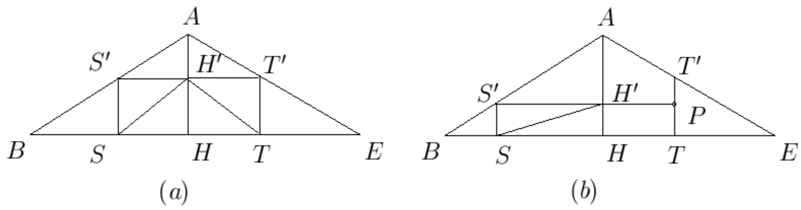

Figure 7: Three new vertices introduced on BE

Case 1. $S'T' \parallel BE$.

Let $H' = S'T' \cap AH$. Then ABE can be triangulated into 8 right triangles as shown in Figure 7(a). First we can slightly slide H' in direction $\overrightarrow{AH'}$ such that $AS'H'$, $S'SH'$, $AH'T'$, $H'TT'$ become acute. Second we slightly slide S'

in direction $\overrightarrow{AS'}$ and T' in direction $\overrightarrow{AT'}$ such that all the angles except for the angles SHH' and $H'HT$ become acute.

Case 2. $S'T' \nparallel BE$.

We may assume without loss of generality that $|S'S| < |T'T|$. Let $S'P \parallel BE$, as shown in Figure 7(b). Then P is a pivot of AHE. By Proposition 2.5, AHE can be triangulated into at most 18 acute triangles and therefore ABE can be triangulated into 22 non-obtuse triangles such that the only side vertices on BE are S, H and T. Now we slightly slide H' in direction $\overrightarrow{AH'}$, and after that slightly slide S' in direction $\overrightarrow{AS'}$ such that only the angle SHH' becomes a right angle. □

Lemma 3.3. *Let ABE be a triangle with $AH \perp BE$ ($H \in$relintBE). Then for any three points $S_1, S_2 \in$relintBH and $T \in$relintHE, ABE can be triangulated into at most 42 non-obtuse triangles such that the only side vertices on BE are S_1, S_2, H and T, and the angles at all vertices different from H are acute.*

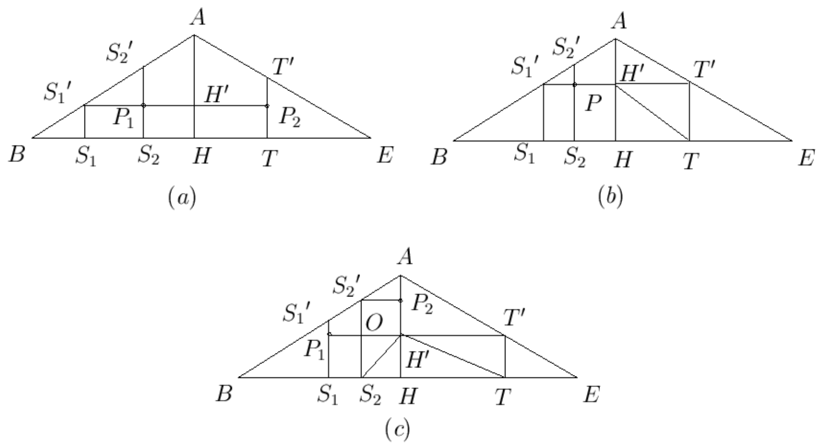

Figure 8: Four new vertices introduced on BE

Proof. Consider S_1', $S_2' \in$ relintAB, $T' \in$ relintAE with $S_1'S_1 \perp BE$, $S_2'S_2 \perp BE$, $T'T \perp BE$.

Case 1. $|S_1'S_1| < |T'T|$.

Let $S_1'P_2 \parallel BE$, as shown in Figure 8 (a). Then P_1 (resp. P_2) is a pivot of $AS_1'S_1H$ (resp. AHE). By Proposition 2.5, $AS_1'S_1H$ (resp. AHE) can be triangulated into at most 21 (resp. 18) acute triangles and therefore ABE admits a non-obtuse triangulation with size at most 40, where only $S_1'BS_1$ is a right triangle. Now we slightly slide ${S_1}'$ in direction $\overrightarrow{A{S_1}'}$, then we obtain an acute triangulation of ABE.

Case 2. $|S_1{S_1}'| = |TT'|$.

See Figure 8 (b), then P is a pivot of $A{S_1}'S_1H$ and therefore ABE can be triangulated into at most 26 non-obtuse triangles. Next we slightly slide H' in direction $\overrightarrow{AH'}$, and after that slightly slide ${S_1}'$ in direction $\overrightarrow{A{S_1}'}$, T' in direction $\overrightarrow{AT'}$ such that only the angle $H'HT$ remains a right angle.

Case 3. $|S_1{S_1}'| > |TT'|$.

Let $P_1T' \parallel BE$, ${S_2}'P_2 \perp AH$, as shown in Figure 8 (c). Then P_1 (resp. P_2) is a pivot of ${S_2}'BS_2$ (resp. $A{S_2}'OT'$). and therefore by Proposition 2.5, ${S_2}'BS_2$ (resp. $A{S_2}'OT'$) can be triangulated into 18 (resp. 19) acute triangles. Thus ABE admits a non-obtuse triangulation with size at most 42. Next we slightly slide H' in direction $\overrightarrow{AH'}$, and after that slightly slide T' in direction $\overrightarrow{AT'}$ such that only the angles S_2HH', $H'HT$ remain right angles. □

Lemma 3.4. *Every pentagon without acute corners can be triangulated into at most* 54 *acute triangles.*

Proof. If the pentagon Γ has no acute corner, then it is convex. Let $\Gamma = ABCDE$ be such a pentagon; we may assume, without loss of generality, that BE is the longest diagonal, which implies that $\angle EBC < \frac{\pi}{2}$, $\angle BED < \frac{\pi}{2}$. Let $AH \perp BE$ with $H \in \text{relint} BE$.

Case 1. Both $\angle BCH$ and $\angle EDH$ are less than $\frac{\pi}{2}$.

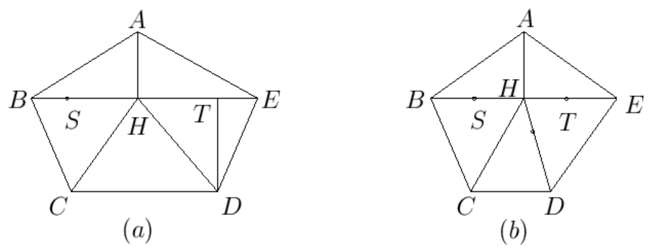

Figure 9: Both $\angle BCH$ and $\angle EDH$ are less than $\frac{\pi}{2}$

Since at least one of $\angle EHD$ and $\angle BHC$ is less than $\frac{\pi}{2}$, we may assume without loss of generality that $\angle EHD < \frac{\pi}{2}$.

Claim. The quadrilateral $BCDH$ can be triangulated into at most 10 acute triangles such that there is exactly one new vertex introduced on BH and at most one new vertex introduced on DH.

Proof of the Claim. Note that $\angle CBH < \frac{\pi}{2}$, $\angle BCH < \frac{\pi}{2}$. If $\angle HDC \geq \frac{\pi}{2}$, by Proposition 2.2, $HBCD$ admits an acute triangulation with size 9 such that there is no new vertex introduced on DH and exactly one new vertex introduced on BH. If $\angle HDC < \frac{\pi}{2}$ and $|DH| < |CD|$, then let $M \in CD$ such that $|DM| = |DH|$. Apply Proposition 2.2 to $HBCM$, and then we obtain an acute triangulation of $HBCD$ with size 10 such that there is no new vertex introduced on DH and exactly one new vertex introduced on BH. If $\angle HDC < \frac{\pi}{2}$ and $|DH| \geq |CD|$, then let $M \in DH$ such that $|DM| = |CD| - \epsilon$, where ϵ is a small positive number. Similarly, $HBCD$ can be triangulated into 10 acute triangles such that there is exactly one new vertex introduced on DH and exactly one new vertex introduced on BH. The proof of the Claim is complete.

If there is no new vertex on DH, then let $DT \perp HE$, as shown in Figure 9 (a). By Lemma 3.2, ABE can be triangulated into at most 22 non-obtuse triangles such that the only side vertices on BE are S, H and T. Therefore Γ can be triangulated into at most $10 + 2 + 22 = 34$ non-obtuse triangles. If there is a vertex on DH then, by Proposition 2.1 HDE can be triangulated into 4 acute triangles such that there is exactly one new vertex introduced on HE. Similarly, we can triangulate Γ into at most $10+4+22 = 36$ non-obtuse triangles. Finally, in both triangulations we slightly slide H in direction $\overrightarrow{AH}$ at first and then slightly slide T in direction $\overrightarrow{ET}$, and obtain the desired acute triangulations.

Case 2. Both $\angle BCH$ and $\angle EDH$ are greater than or equal to $\frac{\pi}{2}$.

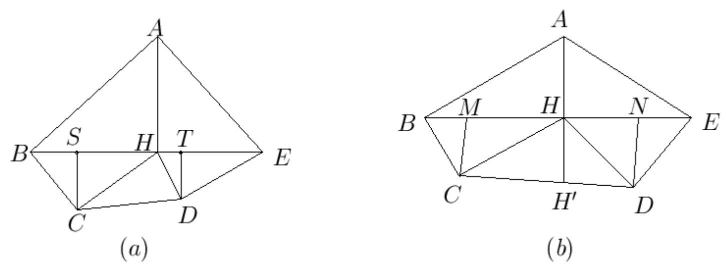

Figure 10: Both $\angle BCH$ and $\angle EDH$ are greater than or equal to $\frac{\pi}{2}$

If $\angle CHD < \frac{\pi}{2}$, then CHD is an acute triangle. Let $CS \perp BH$, $DT \perp HE$, as shown in Figure 10 (a). By Lemma 3.2, ABE can be triangulated into at most 22 non-obtuse triangles and therefore Γ can be triangulated into at most $5 + 22 = 27$ non-obtuse triangles, which can be converted into acute by slidings similar to those used in Case 1.

If $\angle CHD \geq \frac{\pi}{2}$ then the supporting line of AH intersects the relative interior of CD at a point H', as shown in Figure 10 (b). Since $\angle HCH' < \frac{\pi}{2}$ and $\angle BCH' > \frac{\pi}{2}$, there is a point $M \in \text{relint}BH$ such that $\angle MCH' = \frac{\pi}{2}$. Similarly, there is a point $N \in \text{relint}HE$ such that $\angle NDH' = \frac{\pi}{2}$. Because $\angle MCB < \pi/2$ and $\angle CBM < \pi/2$, and because $\angle AHB = \pi/2$, M is a pivot of $ABCH'$, and similarly N is a pivot of $AH'DE$. By the use of Proposition 2.5, both $ABCH'$ and $AH'DE$

can be triangulated into at most $4 \times 4 + 2 \times 3 - 1 = 21$ acute triangles such that H is the only new vertex introduced on AH'. Hence $ABCDE$ can be triangulated into at most $21 \times 2 = 42$ acute triangles.

Case 3. One of $\angle BCH$ and $\angle EDH$ is less than $\frac{\pi}{2}$ while the other is not.

We may assume without loss of generality that $\angle EDH < \frac{\pi}{2}$, $\angle BCH \geq \frac{\pi}{2}$.

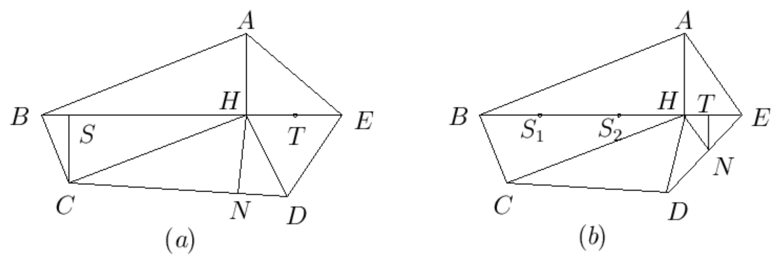

Figure 11: $\angle EDH < \frac{\pi}{2}$, $\angle BCH \geq \frac{\pi}{2}$.

If $\angle HDC$ is acute, then let $CS \perp BH$, $HN \perp CD$, as shown in Figure 11 (a). Since $\angle HED < \frac{\pi}{2}$, by Proposition 2.2, $HNDE$ can be triangulated into at most 9 acute triangles such that there is no new vertex introduced on $HN \cup ND$. Further, since $\angle HDE < \frac{\pi}{2}$, there is exactly one new vertex introduced on HE. Now we slightly slide N in direction $\overrightarrow{CD}$ such that the angle HNC becomes acute. By Lemma 3.2, ABE can be triangulated into at most 22 non-obtuse triangles and therefore Γ can be triangulated into at most $3 + 9 + 22 = 34$ non-obtuse triangles, and all of them can be converted into acute by properly slidings of H and S.

If $\angle HDC \geq \frac{\pi}{2}$ then, by Proposition 2.2, the quadrilateral $BCDH$ can be triangulated into at most 9 acute triangles such that there is no new vertex introduced on DH. Further, since $\angle BCH \geq \frac{\pi}{2}$, there are exactly two new vertices introduced on BH, as shown in Figure 11 (b). Let $HN \perp DE$ and slightly slide N in direction $\overrightarrow{DE}$ such that the angle HND becomes acute. Let $NT \perp HE$. By Lemma 3.3, ABE can be triangulated into at most 42 non-obtuse triangles such that the only side vertices on BE are S_1, S_2, H and T, and the angles at all vertices different from H are acute. Thus Γ can be triangulated into at most $9 + 3 + 42 = 54$ non-obtuse triangles. Now we slightly slide H in direction $\overrightarrow{AH}$ and after that slightly slide T in direction $\overrightarrow{ET}$, and obtain an acute triangulation of Γ. □

Combining Lemma 3.1 and Lemma 3.4, we have the following theorem.

Theorem 3.5. *Every planar pentagon can be triangulated into at most* 54 *acute triangles.*

4 Acute triangulations of double pentagons

Let Γ_d denote the double pentagon formed from a given convex pentagon Γ and its congruent copy Γ'. For any point P in Γ, let P' denote the corresponding point in Γ'.

From Section 3 we know that Γ admits an acute triangulation $\mathcal{T}$ with $|\mathcal{T}| \leq 54$. But, if $\mathcal{T}$ has a vertex with degree 2 (here we regard $\mathcal{T}$ as a plane graph), then $\mathcal{T} \cup \mathcal{T}'$ can not form a triangulation of Γ_d, and the details can be seen in the proof of Lemma 4.1. If $\mathcal{T}$ has no vertex with degree 2, then $\mathcal{T} \cup \mathcal{T}'$ obviously forms an acute triangulation of Γ_d. However, since edges of an acute triangulation of Γ_d are allowed to cross the common boundary of Γ and Γ', it is motivated to triangulate Γ_d in a different way, to obtain a size less than $2|\mathcal{T}|$, as shown in the proof of Lemma 4.3.

Lemma 4.1. *If the pentagon* Γ *has at least one acute angle, then* Γ_d *can be triangulated into at most* 68 *acute triangles.*

Proof. By Lemma 3.1, Γ admits an acute triangulation $\mathcal{T}$ of size at most 32. Furthermore, there are at most 2 vertices in $\mathcal{T}$ with degree 2. Obviously Γ_d can be divided into at most 42 acute triangles by $\mathcal{T} \cup \mathcal{T}'$. Now let A be a vertex with degree 2 in $\mathcal{T}$. This vertex belongs to two congruent triangles T, T', one on each face of Γ_d. Since the triangles T and T' have two sides in common, by the definition we know that the division obtained does not form a proper triangulation of Γ_d. Now suppose that $T = T' = \triangle EAF$, and $G \in \mathrm{bd}\Gamma$ is the other adjacent vertex of F. Now we slide F slightly into the interior of Γ in direction perpendicular to AF such that all the triangles having F as a vertex in Γ remain acute and both of AFF' and GFF' are acute as well. Recalling that there are at most 2 vertices in $\mathcal{T}$ with degree 2, we can conclude that Γ_d can be triangulated into at most 68 acute triangles. □

Lemma 4.2. *Consider the side* AB *of* Γ *and* $H \in int\Gamma$ *satisfying* $\angle AHB > \frac{\pi}{2}$. *Let* $\mathcal{D}_{AB} = ABH \cup ABH'$. *If* $M \in$relint$AH\cup$relintBH, *then* $\mathcal{D}_{AB}$ *admits a triangulation with precisely the points* M, M' *as side vertices and at most*

(*i*) 20 *non-obtuse triangles if* ABH *has two angles smaller than* $\frac{\pi}{4}$;

(*ii*) 30 *non-obtuse triangles otherwise,*

such that all triangles are acute excepting those at M, M'.

Proof. By unfolding $\mathcal{D}_{AB}$ in the plane, we obtain a quadrilateral $HAH'B$ with $AB \cap HH' = O$. We may assume without loss of generality that $M \in$relintAH. Since $\angle AHB$ is obtuse, there is a point $U \in$relint AO such that $UH \perp HB$. Let l denote the line passing through M and perpendicular to AH.

Case 1. $l \cap AU = \{X\}$.

Then X is a pivot of $AH'H$. By Proposition 2.5, $AH'H$ can be triangulated into at most 18 acute triangles such that only one new vertex O is introduced on

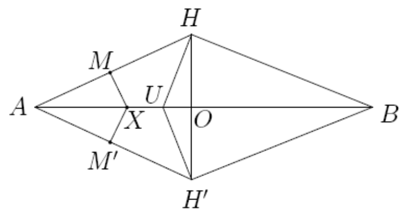

Figure 12: $l \cap AU = \{X\}$

HH'. Now we slightly slide O in direction $\overrightarrow{BO}$. So $\mathcal{D}_{AB}$ admits a triangulation with at most 20 acute triangles in which only M, M' are side vertices.

Case 2. $l \cap AU = \emptyset$.

We suppose that $l \cap HU = \{Y\}$.

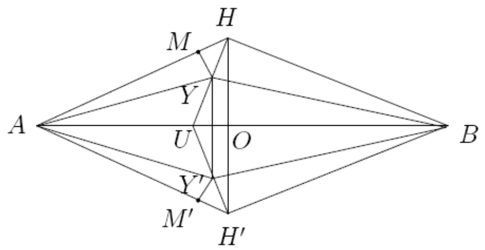

Figure 13: $l \cap AU = \emptyset$

(i) Assume first that both acute angles of ABH are less than $\frac{\pi}{4}$; then $\mathcal{D}_{AB}$ can be triangulated into 8 non-obtuse triangles AYM, $AY'M'$, HMY, $H'M'Y'$, HYB, $H'Y'B$, BYY' and AYY', as shown in Figure 13. Now we slightly slide Y in direction $\overrightarrow{MY}$ (and Y' in direction $\overrightarrow{M'Y'}$) such that only the four triangles adjacent to M or M' are right triangles.

(ii) Otherwise, we use the fact (easy to check) that Y is a pivot of ABH. By Proposition 2.5, ABH can be triangulated into at most 15 acute triangles and therefore $\mathcal{D}_{AB}$ can be triangulated into at most 30 acute triangles such that M, M' are the only side vertices. $\square$

Lemma 4.3. *If a convex pentagon* Γ *has no acute corner, then* Γ_d *can be triangulated into at most* 76 *acute triangles.*

Proof. If Γ has no acute corner, then it has at most two angles which are greater than or equal to $\frac{3\pi}{4}$.

Case 1. Two angles of Γ are greater than or equal to $\frac{3\pi}{4}$.

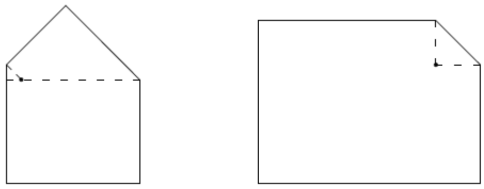

Figure 14: Two angles of Γ are greater than or equal to $\frac{3\pi}{4}$

Then Γ has two angles equal to $\frac{3\pi}{4}$ and three angles equal to $\frac{\pi}{2}$. So Γ has two possible non-isomorphic configurations as shown in Figure 14, and there is a pivot in intΓ for each of them. By Proposition 2.5 Γ can be triangulated into at most 26 acute triangles and therefore Γ_d can be triangulated into at most 52 acute triangles.

Case 2. At most one angle of Γ is greater than or equal to $\frac{3\pi}{4}$.

Subcase 2.1. Γ has a pivot in its interior.

Then as in Case 1, Γ_d can be triangulated into at most 52 acute triangles.

Subcase 2.2. Γ has no pivot in its interior.

(a) All of the five angles of Γ are obtuse.

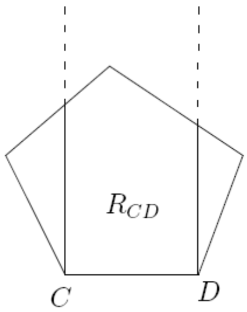

Figure 15: Region R_{CD}

For each side of $\Gamma = ABCDE$, say side CD, let $\overline{R}_{CD}$ denote the strip between the perpendicular lines to CD at C and D. Let $R_{CD} = \text{int}\Gamma \cap \overline{R}_{CD}$, as shown in Figure 15. Then any point $P \in R_{CD}$ is facing CD in Γ. With $\mathcal{F} = \{\mathcal{R}_{\mathcal{AB}}, \mathcal{R}_{\mathcal{BC}}, \mathcal{R}_{\mathcal{CD}}, \mathcal{R}_{\mathcal{DE}}, \mathcal{R}_{\mathcal{EA}}\}$, P is a pivot of Γ if and only if $P \in \cap_{\mathcal{S}\in\mathcal{F}}\mathcal{S}$. Consequently, Γ has no pivot in its interior means that $\cap_{\mathcal{S}\in\mathcal{F}}\mathcal{S} = \emptyset$. Notice that each member of $\mathcal{F}$ is a convex set, so by Helly's Theorem there are three sides e, f, g of Γ such that $R_e \cap R_f \cap R_g = \emptyset$. Furthermore, it is easy to check that e, f and g are not consecutive. Thus we may assume without loss of generality that $e = AE$, $f = BC$, $g = CD$ and the parallelogram $CRST = R_{BC} \cap R_{CD}$

lies to the right of R_{AE} (here we define the direction $\overrightarrow{BC}$ as the right direction), as shown in Figure 16. Recall that the angle at A in Γ is obtuse, so B and T are separated by R_{AE}. Let $HF = R_{AE} \cap l_{BT}$ (here l_{BT} denotes the line passing through the points B and T), hence $HF \subset \text{relint}BT$.

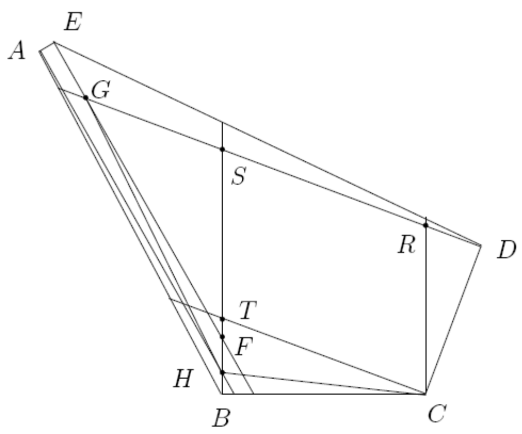

Figure 16: Γ has no pivot in its interior

We establish a Cartesian coordinate system with B as origin, BC as $x-$axis and BT as $y-$axis. Let $\{G\} = l_{EF} \cap l_{DS}$. The angles of Γ being obtuse, $\angle GFH > \frac{\pi}{2}$ and therefore $\angle AHG = \angle HGF < \frac{\pi}{2}$. So $EAHG$ is a right trapezoid with $\angle EGH > \frac{\pi}{2}$. Notice that $\angle GHC > \angle GBC > \frac{\pi}{2}$, thus $GHCD$ is a quadrilateral with $\angle GHC > \frac{\pi}{2}$, $\angle GDC = \frac{\pi}{2}$. Furthermore, $k_{AH} < 0$ (here k_{AH} denotes the slope of l_{AH}), $k_{EG} < 0$ and $k_{GD} < 0$ implies that both $\angle AHB$ and $\angle DGE$ are greater than $\frac{\pi}{2}$.

Now we slightly slide H away from AB in direction perpendicular to AB and slightly slide G in direction $\overrightarrow{EG}$ such that $\angle HAE$, $\angle HBC$, $\angle GDC$ are less than $\frac{\pi}{2}$ while the properties of the triangles ABH and DEG are not changed (here, the property of an obtuse triangle means that both of its acute angles are less than $\frac{\pi}{4}$ or not). Now we consider an acute triangulation of $EAHG$. Let Z be the orthogonal projection of G on EH. Clearly, $Z \in \text{relin}EH$. Since EAH is an acute triangle, by Proposition 2.1, EAH can be triangulated into 4 acute triangles such that Z is the only side vertex on EH, and there is exactly one new vertex introduced on AH. Slightly slide Z in direction $\overrightarrow{GZ}$, hence the quadrilateral $EAHG$ can be triangulated into 6 acute triangles such that there is no new vertex introduced on $EG \cup GH$ while there is exactly one new vertex introduced on AH, say, M. Similarly, $GHCD$ can be triangulated into 6 acute triangles such that there is no new vertex introduced on $GH \cup CH$ while there is precisely one new vertex introduced on DG, say, N. Recall that at most one angle of Γ is greater than or equal to $\frac{3\pi}{4}$, so at most one of the triangles AHB

and DEG has an acute angle which is greater than or equal to $\frac{\pi}{4}$. Then by Lemma 4.2, at most one of $AHB \cup AH'B$ and $DEG \cup DEG'$ admits a non-obtuse triangulation with size at most 30, while the other admits one with size at most 20. Further, M, M' (resp. N, N') are the only side vertices lying on $AHB \cup AH'B$ (resp. $DEG \cup DEG'$). Notice that the polygon $AHBCDGE$ admits an acute triangulation with size $6+1+6 = 13$ such that there are exactly two new vertices M, N introduced on its boundary. Thus Γ_d can be triangulated into at most $13 \times 2 + 20 + 30 = 76$ non-obtuse triangles, which can be converted into acute triangles by slightly sliding M, M', N, N' if necessary.

(b) Γ has at least one right angle.

Similarly to the discussion at (a), we may assume that $R_{BC} \cap R_{CD} \cap R_{AE} = \emptyset$ and the parallelogram $CRST = R_{BC} \cap R_{CD}$ lies to the right of R_{AE} (here we define the direction $\overrightarrow{BC}$ as the right direction). Then it is easy to deduce that $\angle ABC$, $\angle BCD$ and $\angle DEA$ must be greater than $\frac{\pi}{2}$. Now if $\angle EAB = \frac{\pi}{2}$ (or $\angle CDE = \frac{\pi}{2}$), then we chose a point on l_{BS} (or l_{EF}) which is very close to B (or E) on the role of the point H (or G) at (a). The configuration obtained has the same property as that described in Figure 16 except that $\angle EAH$ (or $\angle GDC$) is less than $\frac{\pi}{2}$ instead of being equal to $\frac{\pi}{2}$. By a method similar to the one used in (a) we can also triangulate Γ_d into at most 76 acute triangles. □

Combining Lemma 4.1 and Lemma 4.3, we obtain the following theorem.

Theorem 4.4. *Every double pentagon can be triangulated into at most* 76 *acute triangles.*

Remark.

– the results in this paper are based on inductive constructions;

– if a new point is used to triangulate a double pentagon, its correspondent on the opposite face is also used. This symmetry seems a strong restriction, and removing it could improve the upper bound.

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College of Mathematics and Information Science,
Hebei Normal University,
050016 Shijiazhuang, China
and
Mathematics Research Center of Hebei Province,
050016 Shijiazhuang, China.
E-mail: lpyuan@hebtu.edu.cn
lpyuan88@yahoo.com