When is a Fully Idempotent Module a V-Module?

by

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Abstract

Let \( R \) be a P.I.-ring and \( M \) any \( R \)-module. If \( M \) is fully idempotent, then \( M \) is a \( V \)-module.

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1 Introduction

Throughout this paper all rings are associative with identity element and all modules are unitary right \( R \)-modules. \( \text{Ann}_R(M) \) will denote the annihilator ideal of \( M \) in \( R \), i.e. the ideal consisting of all elements \( r \) of \( R \) such that \( mr = 0 \) for all \( m \in M \).

A submodule \( N \) of a module \( M \) is called idempotent if \( N = \text{Hom}(M, N)N = \sum\{\varphi(N) \mid \varphi : M \to N\} \) (see [1], page 32). Note that if \( A \) is a right ideal of \( R \), then \( A \) is an idempotent submodule of the module \( R_R \) if and only if \( A = A^2 \), i.e. \( A \) is an idempotent right ideal of \( R \). The module \( M \) is called fully idempotent if every submodule of \( M \) is idempotent. It is easy to see that any sum of idempotent submodules of any module \( M \) is again an idempotent submodule of \( M \). Therefore as an easy observation, if \( R \) is a von Neumann regular ring, then \( R_R \) is fully idempotent since every direct summand is idempotent in any module. Note that any idempotent submodule need not be a direct summand as we see in the following:

Example 1.1. (i) Let \( \mathbb{Z} \) denote the ring of integers and \( M = \mathbb{Z} \oplus \mathbb{Z} \) the free \( \mathbb{Z} \)-module of rank 2. Let \( N = \mathbb{Z}(2, 3) + \mathbb{Z}(5, 0) \). Suppose \( N \) is a direct summand of \( M \). Then \( M/N \) is torsion-free. But \( (0, 15) = 5(2, 3) - 2(5, 0) \) so that \( 15(0, 1) \in N \) but \( (0, 1) \notin N \). Thus \( M/N \) is not torsion-free and hence \( N \) is not a direct summand of \( M \).
Define $\alpha_1 : M \longrightarrow N$ by $\alpha_1(u,v) = (2u-v)(2,3)$ $(u,v \in \mathbb{Z})$. Then $\alpha_1$ is a $\mathbb{Z}$-homomorphism such that $\alpha_1(2,3) = (2,3)$. Also define $\alpha_2 : M \longrightarrow N$ by $\alpha_2(u,v) = (2u-v)(5,0)$ $(u,v \in \mathbb{Z})$. Then $\alpha_2$ is a $\mathbb{Z}$-homomorphism such that $\alpha_2(2,3) = (5,0)$. It follows that $N$ is an idempotent submodule of $M$.

(ii) Let $\mathbb{C}$ denote the field of complex numbers (in fact any field of characteristic 0 would do). Let $R$ denote the first Weyl algebra. Then $R$ is the ring of polynomials in indeterminates $x$ and $y$ subject to the relation $xy - yx = 1$. Note that $xy^n - y^n x = n y^{n-1}$ for every positive integer $n$. Now let $f(y)$ be any non-zero polynomial in $\mathbb{C}[y]$. Then $f(y) = c_0 + c_1 y + \cdots + c_t y^t$ for some non-negative integer $t$ and elements $c_i (1 \leq i \leq t)$ in $\mathbb{C}$ with $c_t$ non-zero. We call $t$ the degree of $f(y)$ as usual. Now $xf(y) - f(y)x = c_0(x-x) + c_1(xy-xy) + \cdots + c_t(xy^t - y^t x) = c_1 + \cdots + c_t y^{t-1}$. Because $\mathbb{C}$ has characteristic zero, $tc_t$ is non-zero if $t$ is non-zero. Let $f'(y)$ denote the polynomial $xf(y) - f(y)x$ above (Note that $f'(y)$ is called the formal derivative of $f(y)$).

Now consider the right ideal $xR$ of $R$. Note that $y$ does not belong to $xR$ so that $xR$ is a proper right ideal of $R$. Let $g(x,y)$ belong to $R$ but not $xR$. Because $yx = xy - 1$ it follows that $g(x,y) = g_0(y) + xg_1(y) + \cdots + x^s g_s(y)$ for some non-negative integer $s$ and polynomials $g_i(y)$ in $\mathbb{C}[y]$. Clearly, $g_0(y)$ is non-zero and $g_0(y)$ belongs to $xR + g(x,y)R$. Let the non-negative integer $m$ be the least integer such that $m$ is the degree of a non-zero polynomial $h(y)$ in the right ideal $xR + g(x,y)R$. Suppose that $m$ is at least 1. Then $xh(y) - h(y)x = h'(y) \in xR + g(x,y)R$ and $h'(y)$ is a non-zero polynomial of degree $m - 1$, a contradiction. Thus $m = 0$ and $h(y)$ is a non-zero element of $\mathbb{C}$ and thus a unit in $R$. It follows that $R = xR + g(x,y)R$. Hence $xR$ is a maximal right ideal of $R$. Because $yx$ does not belong to $xR$ it follows that $xR$ is an idempotent submodule of $R_R$ and $xR$ is not a direct summand of $R_R$ because the ring $R$ is a domain ([3, Examples 2.32(h)]). (In fact, $R$ is a simple ring so that every right (or left) ideal is idempotent).

Let $M$ be any module. $M$ is called a $V$-module if every simple $R$-module is $M$-injective. Any ring $R$ is a right $V$-ring iff $R_R$ is a $V$-module.

In this work, firstly we give an example of fully idempotent modules which are not $V$-modules (Example 2.1). Then we prove that if $M$ is a fully idempotent module such that $M/MP$ is a $V$-module for every right primitive ideal $P$ of $R$, then $M$ is a $V$-module (Theorem 2.6). As a corollary we prove that if $M$ is a fully idempotent module over a $P.I.$-ring, then $M$ is a $V$-module (Corollary 2.7).

2 Results

The following example shows that a fully idempotent module need not be a $V$-module.

Example 2.1. (i) Let $R$ be a simple ring (with identity). Then every right ideal of $R$ is idempotent but $R$ need not be a right $V$-ring. For, if $A$ is any non-zero right ideal of $R$ then $A^2 = AA = (AR)A = A(AR) = AR = A$. 

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(ii) Let \( R \) be the endomorphism ring of an infinite dimensional vector space. By [4, 23.6], \( R \) is a von Neumann regular ring but not a \( V \)-ring. Therefore \( R_R \) is a fully idempotent projective module which is not a \( V \)-module.

**Lemma 2.2.** Let \( M \) be a fully idempotent module. Let \( N \leq M \) and \( I \) an ideal of \( R \). Then \( N \cap MI = NI \).

**Proof:** Let \( x \in N \cap MI \). Since \( N \cap MI \) is an idempotent submodule of \( M \), there exist the homomorphisms \( \varphi_i : M \to N \cap MI \) and the elements \( x_i \in N \cap MI \) for some \( k \geq 1 \) and \( 1 \leq i \leq k \) such that \( x = \varphi_1(x_1) + \ldots + \varphi_k(x_k) \). Let \( 1 \leq i \leq k \). Then \( x_i = m_1a_1 + \ldots + m_ta_t \) for some \( t \geq 1 \), \( m_j \in M \), \( a_j \in I \) \( (1 \leq j \leq t) \). Therefore \( \varphi_i(x_i) = \varphi_i(m_1)a_1 + \ldots + \varphi_i(m_t)a_t \in NI \). Hence \( x \in NI \), and so \( NI = N \cap MI \).

**Remark** We do not know if the converse of Lemma 2.2 is true or not. But the converse is true if \( M = R_R \): Let \( A \leq R_R \) and \( I = RA \). Then \( A \cap RA = ARA \) gives \( A = A^2 \).

**Lemma 2.3.** Let \( M \) be a module. Then \( M \) is a \( V \)-module if and only if for all \( B < A \leq M \) with \( A/B \) simple, \( A/B \) is a direct summand of \( M/B \).

**Proof:** \((\Rightarrow): \) Let \( M \) be a \( V \)-module and let \( B < A \leq M \) with \( A/B \) simple. Since \( M \) is a \( V \)-module, \( A/B \) is \( M \)-injective. Then \( A/B \) is \( M/B \)-injective. Therefore \( A/B \) is a direct summand of \( M/B \).

\((\Leftarrow): \) Let \( S \) be a simple module. Let \( X \leq M \), \( i : X \to M \) be the inclusion map and \( f : X \to S \) be a nonzero homomorphism. Then \( X/Ker f \cong S \). By hypothesis, \( M/Ker f = X/Ker f \oplus Y/Ker f \) for some submodule \( Y \) of \( M \) with \( Ker f \subseteq Y \). Now \( M = X + Y \) and \( Ker f = X \cap Y \). Therefore the homomorphism \( g : M \to S \) defined by \( x + y \mapsto f(x) \) \( (x \in X, y \in Y) \) is well-defined. Clearly, \( gi = f \). Thus \( S \) is \( M \)-injective.

**Lemma 2.4.** Let \( M \) be a \( V \)-module. Then for all \( B < A \leq M \), there exists a submodule \( C \) with \( B \leq C < A \) such that \( A/C \) is simple.

**Proof:** By Lemma 2.3.

**Lemma 2.5.** Let \( M \) be a module such that \( M/MP \) is a \( V \)-module for each right primitive ideal \( P \). Then \( M \) is a \( V \)-module if and only if \( A \cap MP = AP \) for all \( A \leq M \) and right primitive ideals \( P \).

**Proof:** Assume \( M \) is a \( V \)-module. Let \( A \leq M \) and \( P \) a right primitive ideal. Suppose \( AP \nsubseteq A \cap MP \). Let \( AP \leq B \nsubseteq A \cap MP \) such that \( (A \cap MP)/B \) is simple (by Lemma 2.4), so \( M \)-injective. Now there exists a submodule \( C \) of \( M \) containing \( B \) such that \( M/B = (C/B) \oplus ((A \cap MP)/B) \). Hence \( (M/C)P = 0 \).
and so $MP \leq C$. Hence $A \cap MP = A \cap C \cap MP = B$, a contradiction. Thus $AP = A \cap MP$.

Conversely, we prove that $M$ is a $V$-module. Let $B \leq A$ be submodules of $M$ such that $A/B$ is simple. Let $P = \text{Ann}_R(A/B)$. Then $A \cap MP = AP$ by hypothesis. $(A + MP)/(B + MP) \cong A/B$ and so is simple. But $M/MP$ is a $V$-module. Therefore there exists a submodule $C$ of $M$ such that $M/B = C/B \oplus A/B$. It follows that $M$ is a $V$-module by Lemma 2.3.

**Theorem 2.6.** Let $M$ be a fully idempotent module such that $M/MP$ is a $V$-module for every right primitive ideal $P$ of $R$. Then $M$ is a $V$-module.

**Proof:** By Lemmas 2.5 and 2.2.

**Corollary 2.7.** Let $R$ be a P.I.-ring. If $M$ is fully idempotent, then $M$ is a $V$-module.

**Proof:** Since every primitive factor ring of a P.I.-ring $R$ is simple artinian by Kaplansky [2].

Here we are giving an application of Corollary 2.7:

**Example 2.8.** Let $K$ be a field. If we set $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ and $I = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$, then $R$ is a P.I.-ring and $I$ is a minimal right ideal of $R_R$. Assume that the module $R_R$ is fully idempotent. Then by Corollary 2.7, $R$ is a right $V$-ring, which is a contradiction since $I$ is not injective (the homomorphism $f: \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} \rightarrow I_R$ defined by $f(\begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}$ cannot be extended to a homomorphism of $R_R$ into $I_R$).

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References

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