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When is a Fully Idempotent Module a V-Module?

by
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Abstract

Let R be a $P.I.$-ring and M any R-module. If M is fully idempotent, then M is a V-module.

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1 Introduction

Throughout this paper all rings are associative with identity element and all modules are unitary right R-modules. $Ann_R(M)$ will denote the annihilator ideal of M in R, i.e. the ideal consisting of all elements r of R such that $mr = 0$ for all $m \in M$.

A submodule N of a module M is called *idempotent* if $N = Hom(M,N)N = \sum\{\varphi(N) \mid \varphi : M \to N\}$ (see [1], page 32). Note that if A is a right ideal of R, then A is an idempotent submodule of the module R_R if and only if $A = A^2$, i.e. A is an idempotent right ideal of R. The module M is called *fully idempotent* if every submodule of M is idempotent. It is easy to see that any sum of idempotent submodules of any module M is again an idempotent submodule of M. Therefore as an easy observation, if R is a von Neumann regular ring, then R_R is fully idempotent since every direct summand is idempotent in any module. Note that any idempotent submodule need not be a direct summand as we see in the following:

Example 1.1. (i) Let $\mathbb{Z}$ denote the ring of integers and $M = \mathbb{Z} \oplus \mathbb{Z}$ the free $\mathbb{Z}$-module of rank 2. Let $N = \mathbb{Z}(2,3) + \mathbb{Z}(5,0)$. Suppose N is a direct summand of M. Then M/N is torsion-free. But $(0,15) = 5(2,3) - 2(5,0)$ so that $15(0,1) \in N$ but $(0,1) \notin N$. Thus M/N is not torsion-free and hence N is not a direct summand of M.

Define $\alpha_1 : M \longrightarrow N$ by $\alpha_1(u,v) = (2u - v)(2,3)$ $(u, v \in \mathbb{Z})$. Then α_1 is a $\mathbb{Z}$-homomorphism such that $\alpha_1(2,3) = (2,3)$. Also define $\alpha_2 : M \longrightarrow N$ by $\alpha_2(u,v) = (2u - v)(5,0)$ $(u, v \in \mathbb{Z})$. Then α_2 is a $\mathbb{Z}$-homomorphism such that $\alpha_2(2,3) = (5,0)$. It follows that N is an idempotent submodule of M.

(ii) Let $\mathbb{C}$ denote the field of complex numbers (in fact any field of characteristic 0 would do). Let R denote the first Weyl algebra. Then R is the ring of polynomials in indeterminates x and y subject to the relation $xy - yx = 1$. Note that $xy^n - y^n x = ny^{n-1}$ for every positive integer n. Now let $f(y)$ be any non-zero polynomial in $\mathbb{C}[y]$. Then $f(y) = c_0 + c_1 y + \cdots + c_t y^t$ for some non-negative integer t and elements $c_i (1 \leq i \leq t)$ in $\mathbb{C}$ with c_t non-zero. We call t the degree of $f(y)$ as usual. Now $xf(y) - f(y)x = c_0(x - x) + c_1(xy - yx) + \cdots + c_t(xy^t - y^t x) = c_1 + \ldots tc_t y^{t-1}$. Because $\mathbb{C}$ has characteristic zero, tc_t is non-zero if t is non-zero. Let $f'(y)$ denote the polynomial $xf(y) - f(y)x$ above (Note that $f'(y)$ is called the formal derivative of $f(y)$).

Now consider the right ideal xR of R. Note that y does not belong to xR so that xR is a proper right ideal of R. Let $g(x,y)$ belong to R but not xR. Because $yx = xy - 1$ it follows that $g(x,y) = g_0(y) + xg_1(y) + \cdots + x^s g_s(y)$ for some non-negative integer s and polynomials $g_i(y)$ in $\mathbb{C}[y]$. Clearly, $g_0(y)$ is non-zero and $g_0(y)$ belongs to $xR + g(x,y)R$. Let the non-negative integer m be the least integer such that m is the degree of a non-zero polynomial $h(y)$ in the right ideal $xR + g(x,y)R$. Suppose that m is at least 1. Then $xh(y) - h(y)x = h'(y) \in xR + g(x,y)R$ and $h'(y)$ is a non-zero polynomial of degree $m - 1$, a contradiction. Thus $m = 0$ and $h(y)$ is a non-zero element of $\mathbb{C}$ and thus a unit in R. It follows that $R = xR + g(x,y)R$. Hence xR is a maximal right ideal of R. Because yx does not belong to xR it follows that xR is an idempotent submodule of R_R and xR is not a direct summand of R_R because the ring R is a domain ([3, Examples 2.32(h)]). (In fact, R is a simple ring so that every right (or left) ideal is idempotent).

Let M be any module. M is called a *V-module* if every simple R-module is M-injective. Any ring R is a right V-ring iff R_R is a V-module.

In this work, firstly we give an example of fully idempotent modules which are not V-modules (Example 2.1). Then we prove that if M is a fully idempotent module such that M/MP is a V-module for every right primitive ideal P of R, then M is a V-module (Theorem 2.6). As a corollary we prove that if M is a fully idempotent module over a $P.I.$-ring, then M is a V-module (Corollary 2.7).

2 Results

The following example shows that a fully idempotent module need not be a V-module.

Example 2.1. (i) Let R be a simple ring (with identity). Then every right ideal of R is idempotent but R need not be a right V-ring. For, if A is any non-zero right ideal of R then $A^2 = AA = (AR)A = A(RA) = AR = A$.

(ii) Let R be the endomorphism ring of an infinite dimensional vector space. By [4, 23.6], R is a von Neumann regular ring but not a V-ring. Therefore R_R is a fully idempotent projective module which is not a V-module.

Lemma 2.2. *Let M be a fully idempotent module. Let $N \leq M$ and I an ideal of R. Then $N \cap MI = NI$.*

Proof: Let $x \in N \cap MI$. Since $N \cap MI$ is an idempotent submodule of M, there exist the homomorphisms $\varphi_i : M \to N \cap MI$ and the elements $x_i \in N \cap MI$ for some $k \geq 1$ and $1 \leq i \leq k$ such that $x = \varphi_1(x_1) + \ldots + \varphi_k(x_k)$. Let $1 \leq i \leq k$. Then $x_i = m_1 a_1 + \ldots + m_t a_t$ for some $t \geq 1$, $m_j \in M$, $a_j \in I$ $(1 \leq j \leq t)$. Therefore $\varphi_i(x_i) = \varphi_i(m_1)a_1 + \ldots + \varphi_i(m_t)a_t \in NI$. Hence $x \in NI$, and so $NI = N \cap MI$. □

Remark We do not know if the converse of Lemma 2.2 is true or not. But the converse is true if $M = R_R$: Let $A \leq R_R$ and $I = RA$. Then $A \cap RA = ARA$ gives $A = A^2$.

Lemma 2.3. *Let M be a module. Then M is a V-module if and only if for all $B < A \leq M$ with A/B simple, A/B is a direct summand of M/B.*

Proof: ($\Rightarrow$): Let M be a V-module and let $B < A \leq M$ with A/B simple. Since M is a V-module, A/B is M-injective. Then A/B is M/B-injective. Therefore A/B is a direct summand of M/B.

($\Leftarrow$): Let S be a simple module. Let $X \leq M$, $i : X \to M$ be the inclusion map and $f : X \to S$ be a nonzero homomorphism. Then $X/Kerf \cong S$. By hypothesis, $M/Kerf = X/Kerf \oplus Y/Kerf$ for some submodule Y of M with $Kerf \subseteq Y$. Now $M = X + Y$ and $Kerf = X \cap Y$. Therefore the homomorphism $g : M \to S$ defined by $x + y \mapsto f(x)$ $(x \in X, y \in Y)$ is well-defined. Clearly, $gi = f$. Thus S is M-injective. □

Lemma 2.4. *Let M be a V-module. Then for all $B < A \leq M$, there exists a submodule C with $B \leq C < A$ such that A/C is simple.*

Proof: By Lemma 2.3. □

Lemma 2.5. *Let M be a module such that M/MP is a V-module for each right primitive ideal P. Then M is a V-module if and only if $A \cap MP = AP$ for all $A \leq M$ and right primitive ideals P.*

Proof: Assume M is a V-module. Let $A \leq M$ and P a right primitive ideal. Suppose $AP \lneqq A \cap MP$. Let $AP \leq B \lneqq A \cap MP$ such that $(A \cap MP)/B$ is simple (by Lemma 2.4), so M-injective. Now there exists a submodule C of M containing B such that $M/B = (C/B) \oplus ((A \cap MP)/B)$. Hence $(M/C)P = 0$

and so $MP \leq C$. Hence $A \cap MP = A \cap C \cap MP = B$, a contradiction. Thus $AP = A \cap MP$.

Conversely, we prove that M is a V-module. Let $B \leq A$ be submodules of M such that A/B is simple. Let $P = Ann_R(A/B)$. Then $A \cap MP = AP$ by hypothesis. $(A + MP)/(B + MP) \cong A/B$ and so is simple. But M/MP is a V-module. Therefore there exists a submodule C of M such that $M/B = C/B \oplus A/B$. It follows that M is a V-module by Lemma 2.3. □

Theorem 2.6. *Let M be a fully idempotent module such that M/MP is a V-module for every right primitive ideal P of R. Then M is a V-module.*

Proof: By Lemmas 2.5 and 2.2. □

Corollary 2.7. *Let R be a P.I.-ring. If M is fully idempotent, then M is a V-module.*

Proof: Since every primitive factor ring of a $P.I.$-ring R is simple artinian by Kaplansky [2]. □

Here we are giving an application of Corollary 2.7:

Example 2.8. Let K be a field. If we set $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ and $I = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$, then R is a P.I.-ring and I is a minimal right ideal of R_R. Assume that the module R_R is fully idempotent. Then by Corollary 2.7, R is a right V-ring, which is a contradiction since I is not injective (the homomorphism $f : \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}_R \longrightarrow I_R$ defined by $f(\begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}$ cannot be extended to a homomorphism of R_R into I_R).

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