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Some remarks on Ramanujan sums and cyclotomic polynomials

by
László Tóth

Dedicated to the memory of Laurenţiu Panaitopol (1940-2008) on the occasion of his 70th anniversary

Abstract

We investigate the polynomials $\sum_{k=0}^{n-1} c_n(k)x^k$ and $\sum_{k=0}^{n-1} |c_n(k)|x^k$, where $c_n(k)$ denote the Ramanujan sums. We point out connections and analogies to the cyclotomic polynomials.

Key Words: Ramanujan sum, cyclotomic polynomial, Euler's function, Möbius function, divisibility of polynomials.
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1 Introduction

The Ramanujan sum $c_n(k)$ is defined as the sum of kth powers ($k \in \mathbb{Z}$) of the primitive nth roots of unity, that is,

$$c_n(k) := \sum_{j \in A_n} \eta_j^k, \tag{1}$$

where $\eta_j = \exp(2\pi i j/n)$ and $A_n = \{j \in \mathbb{N} : 1 \le j \le n, (j,n) = 1\}$. Here $c_n(k)$ is an n-periodic function of k, i.e., $c_n(k) = c_n(\ell)$ for any $k \equiv \ell \pmod{n}$. Note that for $n \mid k$, $c_n(k) = c_n(0) = \varphi(n)$ is Euler's function and for $(k,n) = 1$, $c_n(k) = c_n(1) = \mu(n)$ is the Möbius function.

The nth cyclotomic polynomial $\Phi_n(x)$ is the monic polynomial whose roots are the primitive nth roots of unity, i.e.,

$$\Phi_n(x) := \prod_{j \in A_n} (x - \eta_j). \tag{2}$$

The following representations are well known:

$$c_n(k) = \sum_{d|(n,k)} d\mu(n/d), \tag{3}$$

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}. \tag{4}$$

Cyclotomic polynomials and Ramanujan sums are closely related as it is shown by the following theorem.

Theorem 1. *i) For any $n \geq 1$,*

$$(x^n - 1)\frac{\Phi_n'(x)}{\Phi_n(x)} = \sum_{k=1}^{n} c_n(k) x^{k-1}, \tag{5}$$

where $\Phi_n'(x)$ is the derivative of $\Phi_n(x)$.

ii) For any $n > 1$ and $|x| < 1$,

$$\Phi_n(x) = \exp\left(-\sum_{k=1}^{\infty} \frac{c_n(k)}{k} x^k\right). \tag{6}$$

These formulae are not widely known and were first derived by Nicol [7, Th. 3.1, Cor. 3.2]. In that paper formula (5) was deduced by differentiating (4), which gives

$$\frac{\Phi_n'(x)}{\Phi_n(x)} = \sum_{d|n} \frac{d\mu(n/d)x^{d-1}}{x^d - 1} \quad (n \geq 1) \tag{7}$$

and then using (3), while (6) was obtained as a corollary of (5).

Formula (7) was given also by Motose [6, Th. 1], without referring to the paper of Nicol [7]. The following result was obtained by Motose [6, Lemma 1, Th. 1] in the same paper.

Theorem 2. *For any $n \geq 1$,*

$$\frac{\Phi_n'(1/x)}{x\Phi_n(1/x)} = \frac{1}{1 - x^n} \sum_{k=0}^{n-1} c_n(k) x^k = \sum_{d|n} \frac{d\mu(n/d)}{1 - x^d}. \tag{8}$$

Note that (8) is a simple consequence of formulae (5) and (7) by putting $x := 1/x$.

In this paper we first give new direct proofs of (5) and (6) which use only the definitions of the Ramanujan sums and of the cyclotomic polynomials (Section 2).

Then in Section 3 we investigate the polynomials with integer coefficients

$$R_n(x) := \sum_{k=0}^{n-1} c_n(k) x^k \tag{9}$$

appearing in (8). We deduce for $R_n(x)$ formulas which are similar to the following well known formulas valid for the cyclotomic polynomials: $\Phi_n(x) = \Phi_{\gamma(n)}(x^{n/\gamma(n)})$, where $\gamma(n) = \prod_{p|n} p$ is the squarefree kernel of n, $\Phi_{np}(x) = \Phi_n(x^p)$ for any prime $p \mid n$, $\Phi_{np}(x) = \Phi_n(x^p)/\Phi_n(x)$ for any prime $p \nmid n$, $\Phi_{2n}(x) = \Phi_n(-x)$ for any n odd, see for ex. [8]. We also derive certain divisibility properties of the polynomials $R_n(x)$.

In Section 4 we consider the polynomials

$$T_n(x) := \sum_{k=0}^{n-1} |c_n(k)| x^k \tag{10}$$

and compare their properties to those of the polynomials $R_n(x)$.

We show – among others – that $R_{2n}(x) = (1 - x^n)R_n(-x)$, $T_{2n}(x) = (1 + x^n)T_n(x)$ for any $n \geq 1$ odd, and that the cyclotomic polynomial $\Phi_n(x)$ divides $T_n(x)$ for any $n \geq 2$ even.

Section 5 contains tables of the polynomials $R_n(x)$ and $T_n(x)$ for $1 \leq n \leq 20$.

For material concerning Ramanujan sums we refer to the books [3, 4, 5].

2 Proof of Theorem 1

Proof: i) First note that for the generating function of the sequence $(c_n(k))_{k\geq 1}$ we have by using the periodicity of the Ramanujan sums,

$$\sum_{k=1}^{\infty} c_n(k)x^k = \sum_{\ell=0}^{\infty}\sum_{j=1}^{n} c_n(\ell n+j)x^{\ell n+j} = \sum_{\ell=0}^{\infty} x^{\ell n}\sum_{j=1}^{n} c_n(j)x^j = \frac{1}{1-x^n}\sum_{j=1}^{n} c_n(j)x^j.$$

Now let $|x| < 1$. Applying the power series $(1-t)^{-1} = 1 + t + t^2 + \ldots$ for $t = x/\eta_j$, where $|t| = |x| < 1$,

$$\begin{aligned}\frac{\Phi_n'(x)}{\Phi_n(x)} &= \sum_{j\in A_n} \frac{1}{x-\eta_j} = -\sum_{j\in A_n} \frac{1}{\eta_j}\cdot\frac{1}{1-x/\eta_j} = -\sum_{j\in A_n}\frac{1}{\eta_j}\sum_{k=0}^{\infty}\frac{x^k}{\eta_j^k} \\ &= -\sum_{k=0}^{\infty} x^k \sum_{j\in A_n} \eta_j^{-k-1} = -\sum_{k=0}^{\infty} x^k c_n(-k-1) = -\sum_{k=1}^{\infty} c_n(-k)x^{k-1} \\ &= -\sum_{k=1}^{\infty} c_n(k)x^{k-1} = \frac{1}{x^n-1}\sum_{j=1}^{n} c_n(j)x^{j-1},\end{aligned}$$

where we have used that $c_n(-k) = c_n(k)$ for any k. Justification: if in (1) j runs through a reduced residue system (mod n), then so does $-j$. Hence the given polynomial identity holds, which finishes the proof of i).

ii) We use that for $n > 1$,

$$\Phi_n(x) = \prod_{j\in A_n}\left(1 - \frac{x}{\eta_j}\right). \tag{11}$$

This follows from (2) by $\prod_{j\in A_n} \eta_j = 1$, valid for $n > 2$. Note that (11) holds also for $n = 2$. We have, using the power series $\log(1-t) = -t - t^2/2 - t^3/3 - \dots$ for $t = x/\eta_j$, where $|t| = |x| < 1$,

$$\log \Phi_n(x) = \sum_{j\in A_n} \log\left(1 - \frac{x}{\eta_j}\right) = -\sum_{j\in A_n}\sum_{k=1}^{\infty} \frac{x^k}{k\eta_j^k} = -\sum_{k=1}^{\infty} \frac{x^k}{k} \sum_{j\in A_n} \eta_j^{-k}$$

$$= -\sum_{k=1}^{\infty} \frac{x^k}{k} c_n(-k) = -\sum_{k=1}^{\infty} \frac{x^k}{k} c_n(k).$$

Alternatively, one can apply that for $n > 1$,

$$\Phi_n(x) = \prod_{d|n} (1 - x^d)^{\mu(n/d)}, \tag{12}$$

which follows at once by (4) and by $\sum_{d|n} \mu(n/d) = 0$ $(n > 1)$. We deduce

$$\log \Phi_n(x) = \sum_{d|n} \mu(n/d) \log(1 - x^d) = -\sum_{d|n} \mu(n/d) \sum_{j=1}^{\infty} \frac{x^{dj}}{j}$$

$$= -\sum_{m=1}^{\infty} \frac{x^m}{m} \sum_{d|(n,m)} d\mu(n/d) = -\sum_{m=1}^{\infty} \frac{x^m}{m} c_n(m),$$

using (3). This approach was given and applied by Erdős and Vaughan [2, Proof of Th. 1]. □

3 The polynomials $R_n(x)$

In this section we investigate properties of the polynomials $R_n(x)$ defined by (9). Note that the polynomials appearing in (5) are given by $P_n(x) := \sum_{k=1}^{n} c_n(k) x^{k-1}$. The connection between the polynomials $R_n(x)$ and $P_n(x)$ is given by $xP_n(x) = R_n(x) + \varphi(n)(x^n - 1)$. Hence it is sufficient to study the polynomials $R_n(x)$.

According to (8) for any $n \geq 1$,

$$R_n(x) = (1 - x^n) \sum_{d|n} \frac{d\mu(n/d)}{1 - x^d}. \tag{13}$$

Theorem 3. *Let $n \geq 1$.*

i) The number of nonzero coefficients of $R_n(x)$ is $\gamma(n)$.

ii) The degree of $R_n(x)$ is $n - n/\gamma(n)$.

iii) $R_n(x)$ has coefficients ± 1 if and only if n is squarefree and in this case the number of coefficients ± 1 of $R_n(x)$ is $\varphi(n)$ for n odd and is $2\varphi(n/2)$ for n even.

Proof: For $n = 1$ the assertions hold true. Let $n = p_1^{a_1} \cdots p_r^{a_r} > 1$.

i) We use that $c_n(k)$ is multiplicative in n and for any prime power p^a,

$$c_{p^a}(k) = \begin{cases} p^a - p^{a-1}, & \text{if } p^a \mid k, \\ -p^{a-1}, & \text{if } p^{a-1} \mid k, p^a \nmid k, \\ 0, & \text{if } p^{a-1} \nmid k. \end{cases}$$

Therefore, $c_n(k) \neq 0$ if and only if $p_1^{a_1-1} \mid k$, ..., $p_r^{a_r-1} \mid k$, i.e., $k = p_1^{a_1-1} \cdots p_r^{a_r-1} m$ with $0 \leq m < p_1 \cdots p_r = \gamma(n)$. Hence the number of nonzero values of $c_n(k)$ is $\gamma(n)$.

ii) By the proof of i) the largest k such that $c_n(k) \neq 0$ is

$$k = p_1^{a_1-1} \cdots p_r^{a_r-1}(p_1 \cdots p_r - 1) = n - n/\gamma(n),$$

and this is the degree of $R_n(x)$.

iii) $c_n(k) = \pm 1$ if and only if $c_{p_i^{a_i}}(k) = \pm 1$ for any $i \in \{1, \ldots, r\}$, that is $a_i = 1$ for any i (n is squarefree) and either $p_i \nmid k$ or $p_i = 2 \mid k$ for any i.

Suppose that $n = p_1 \cdots p_r$ (squarefree). If n is odd, then by condition $p_i \nmid k$ for any i we have $(n, k) = 1$, hence the number of such values of k is $\varphi(n)$. For n even either $(k, n) = 1$ or $k = 2\ell$ with $(\ell, n/2) = 1$. We obtain that the number of such values of k is $\varphi(n) + \varphi(n/2) = 2\varphi(n/2)$. □

We have for any $n > 1$, $R_n(0) = c_n(0) = \varphi(n)$ and $R_n(1) = \sum_{k=0}^{n-1} c_n(k) = 0$, as it is well known. Hence $1 - x$ divides $R_n(x)$ for any $n > 1$. Now a look at the polynomials $R_n(x)$, see Section 5, suggests that $1 + x$ divides $R_n(x)$ for any $n > 2$ even. This is confirmed by the next result.

Theorem 4. *We have $R_2(-1) = 2$ and*

i) $R_n(-1) = \varphi(n)$ for any $n \geq 1$ odd,

ii) $R_n(-1) = 0$ for any $n > 2$ even,

iii) the cyclotomic polynomial $\Phi_n(x)$ divides the polynomial $R_n(x) - n$ for any $n \geq 1$.

Proof: i) Consider also the polynomials $Q_n(x) = \sum_{k=0}^{n} c_n(k) x^k = R_n(x) + \varphi(n) x^n$, which are symmetric for any $n \geq 1$ since $c_n(k) = c_n(n-k)$ $(0 \leq k \leq n)$. Hence, for any n odd, $Q_n(-1) = 0$ and $R_n(-1) = \varphi(n)$.

ii) Now use (13), which can be written as

$$R_n(x) = \sum_{d \mid n} d\mu(n/d)(x^{n-d} + x^{n-2d} + \ldots + x^d + 1). \tag{14}$$

We obtain that for any $n = 2k > 2$ even,

$$R_n(-1) = \sum_{\substack{d \mid n \\ d \text{ even}}} d\mu(n/d)\frac{n}{d} = n \sum_{\substack{d \mid n \\ d \text{ even}}} \mu(n/d) = n \sum_{\delta \mid k} \mu(k/\delta) = 0.$$

iii) If η is any primitive nth root of unity, then $R_n(\eta) = n$. This follows from (13):

$$R_n(x) = n + (1 - x^n) \sum_{\substack{d|n \\ d<n}} \frac{d\mu(n/d)}{1 - x^d},$$

where $\eta^d \neq 1$ for any $d \mid n$, $d < n$. ∎

If p is a prime, then it follows from (13) that

$$R_p(x) = (p-1) - x - x^2 - \ldots - x^{p-1}. \tag{15}$$

Also, if p, q are distinct primes, then

$$R_{pq}(x) = (p-1)(q-1) + x + x^2 + \ldots + x^{pq-1} \tag{16}$$

$$-p(x^p + x^{2p} + \ldots + x^{(q-1)p}) - q(x^q + x^{2q} + \ldots + x^{(p-1)q}).$$

Next we show that $R_n(x)$ have some properties which are similar to those of the cyclotomic polynomials $\Phi_n(x)$.

Theorem 5. *i) If $n \geq 1$, then*

$$R_n(x) = \frac{n}{\gamma(n)} R_{\gamma(n)}(x^{n/\gamma(n)}). \tag{17}$$

ii) Let $n \geq 1$ and p be a prime. If $p \mid n$, then $R_{np}(x) = pR_n(x^p)$. If $p \nmid n$, then

$$R_{np}(x) = pR_n(x^p) - (1 + x^n + x^{2n} + \ldots + x^{(p-1)n})R_n(x). \tag{18}$$

iii) If $n > 1$, p is a prime and $p \nmid n$, then $(1 - x^p) \mid R_{np}(x)$.

Proof: i) From (13) again,

$$R_n(x) = (1 - x^n) \sum_{d|n} \frac{\mu(d)\frac{n}{d}}{1 - x^{n/d}} = (1 - x^n) \sum_{d|\gamma(n)} \frac{\mu(d)\frac{n}{d}}{1 - x^{n/d}},$$

and by this representation of $R_n(x)$,

$$R_{\gamma(n)}(x^{n/\gamma(n)}) = (1 - x^n) \sum_{d|\gamma(n)} \frac{\mu(d)\frac{\gamma(n)}{d}}{1 - x^{n/d}}$$

$$= (1 - x^n)\frac{\gamma(n)}{n} \sum_{d|\gamma(n)} \frac{\mu(d)\frac{n}{d}}{1 - x^{n/d}} = \frac{\gamma(n)}{n} R_n(x).$$

ii) For $p \mid n$ this follows at once from i) by $\gamma(np) = \gamma(n)$. Now let $p \nmid n$. Then by (13),

$$R_{np}(x) = (1-x^{np}) \sum_{d \mid np} \frac{d\mu(np/d)}{1-x^d}$$

$$= (1-x^{np}) \left(\sum_{d \mid n} \frac{d\mu(np/d)}{1-x^d} + \sum_{d=\delta p,\ \delta \mid n} \frac{\delta p \mu(n/\delta)}{1-x^{\delta p}} \right)$$

$$= (1-x^{np}) \left(-\sum_{d \mid n} \frac{d\mu(n/d)}{1-x^d} + p \sum_{\delta \mid n} \frac{\delta \mu(n/\delta)}{1-x^{\delta p}} \right)$$

$$= pR_n(x^p) - \frac{1-x^{np}}{1-x^n} R_n(x).$$

iii) Using that $x = 1$ is a root of $R_n(x)$ for $n > 1$ we deduce that $(1-x^p) \mid R_n(x^p)$ and by the formula (18) we obtain $(1-x^p) \mid R_{np}(x)$. □

In particular, for any prime power p^k $(k \geq 1)$,

$$R_{p^k}(x) = p^{k-1} R_p(x^{p^{k-1}}) = p^{k-1}(p - 1 - x^{p^{k-1}} - x^{2p^{k-1}} - \ldots - x^{(p-1)p^{k-1}}) \quad (19)$$

and for $p = 2$,

$$R_{2^k}(x) = 2^{k-1}(1 - x^{2^{k-1}}). \quad (20)$$

Theorem 6. *i) For any $n \geq 1$ odd,*

$$R_{2n}(x) = (1-x^n) R_n(-x), \quad (21)$$

ii) More generally, for any $n \geq 1$ odd and any $k \geq 1$,

$$R_{2^k n}(x) = 2^{k-1}(1 - x^{2^{k-1} n}) R_n(-x^{2^{k-1}}). \quad (22)$$

Proof: i) By (18) we have

$$R_{2n}(x) = 2R_n(x^2) - (1+x^n) R_n(x)$$

$$= 2(1-x^{2n}) \sum_{d \mid n} \frac{d\mu(n/d)}{1-x^{2d}} - (1+x^n)(1-x^n) \sum_{d \mid n} \frac{d\mu(n/d)}{1-x^d}$$

$$= (1-x^{2n}) \sum_{d \mid n} \frac{d\mu(n/d)}{1+x^d} = (1-x^n) R_n(-x),$$

hence (21) holds.

ii) By (17) and (21) we obtain

$$R_{2^k n}(x) = \frac{2^{k-1}n}{\gamma(n)} R_{2\gamma(n)}(x^{2^{k-1}n/\gamma(n)})$$

$$= \frac{2^{k-1}n}{\gamma(n)}(1 - x^{2^{k-1}n}) R_{\gamma(n)}(-x^{2^{k-1}n/\gamma(n)}).$$

Here by (17) again,

$$R_n(-x^{2^{k-1}}) = \frac{n}{\gamma(n)} R_{\gamma(n)}(-x^{2^{k-1}n/\gamma(n)}),$$

ending the proof. □

Theorem 7. *i) If $n = p^k$, p prime, $k \geq 1$, then*

$$(1 - x^{p^{k-1}}) \mid R_n(x). \tag{23}$$

ii) If $n = 2^k m$, $k \geq 1$, $m > 1$ odd, then

$$(1 - x^{n/2})(1 + x^{n/\gamma(n)}) \mid R_n(x). \tag{24}$$

iii) If $n = p^k m$, $p > 2$ prime, $k \geq 1$, $m > 1$ odd, $p \nmid m$, then

$$(1 - x^{pn/\gamma(n)}) \mid R_n(x). \tag{25}$$

iv) If $n = 2^k m$, $k \geq 1$, $m > 1$ odd, m has at least two prime divisors, p prime, $p \mid m$, then

$$(1 - x^{n/2})(1 + x^{pn/\gamma(n)}) \mid R_n(x). \tag{26}$$

Proof: i) $R_{p^k}(x) = p^{k-1} R_p(x^{p^{k-1}})$ by (19), and use that $x = 1$ is a root of $R_p(x)$.

ii) For $n = 2^k m$, $k \geq 1$, $m > 1$ odd we have

$$R_n(x) = \frac{2^{k-1}m}{\gamma(m)} \left(1 - x^{2^{k-1}m}\right) R_{\gamma(m)}(-x^{2^{k-1}m/\gamma(m)}),$$

see the proof of Theorem 6/ii), and use that $x = 1$ is a root of $R_{\gamma(m)}(x)$.

iii) Now

$$R_n(x) = \frac{n}{\gamma(n)} R_{p\gamma(m)}(x^{p^{k-1}m/\gamma(m)}),$$

where $(1 - x^p) \mid R_{p\gamma(m)}(x)$ with $\gamma(m) > 1$, cf. Theorem 5/iii).

iv) By combining the above results. □

As examples, Theorem 7 gives that $(1-x^9)(1+x^3) \mid R_{18}(x)$ and $(1-x^{15})(1+x^3) \mid R_{30}(x)$. It is possible to deduce from Theorem 7 other divisibility properties for the polynomials $R_n(x)$, e.g. the next one.

Theorem 8. *For any $k \geq 1$ and $m > 1$,*

$$\left(1 + x^{2^{k-1}}\right) \mid R_{2^k m}(x). \tag{27}$$

Proof: Follows from Theorem 7/ii). ∎

Another representation of the polynomials $R_n(x)$ is given by

Theorem 9. *For any $n \geq 1$,*

$$R_n(x) = \varphi(n)\left(1 - x^n + \sum_{d|n} \frac{\mu(d)}{\varphi(d)} \Psi_d(x^{n/d})\right), \tag{28}$$

where $\Psi_n(x) := \sum_{j \in A_n} x^j$.

Proof: We use Hölder's formula

$$c_n(k) = \frac{\varphi(n)\mu(n/(n,k))}{\varphi(n/(n,k))} \tag{29}$$

and by grouping the terms according to $(n,k) = d$, obtain

$$\sum_{k=1}^{n} c_n(k)x^k = \sum_{k=1}^{n} \frac{\varphi(n)\mu(n/(n,k))}{\varphi(n/(n,k))} x^k = \varphi(n) \sum_{d|n} \frac{\mu(n/d)}{\varphi(n/d)} \sum_{j \in A_{n/d}} x^{dj}$$

$$= \varphi(n) \sum_{d|n} \frac{\mu(d)}{\varphi(d)} \Psi_d(x^{n/d}).$$

∎

Remark 1. The polynomials $\Psi_n(x) = \sum_{j \in A_n} x^j$ are $\Psi_1(x) = x$, $\Psi_2(x) = x$, $\Psi_3(x) = x + x^2$, $\Psi_4(x) = x + x^3$, $\Psi_5(x) = x + x^2 + x^3 + x^4$, $\Psi_6(x) = x + x^5$, etc., having the representations

$$\Psi_n(x) = (1 - x^n) \sum_{d|n} \frac{\mu(d)x^d}{1 - x^d} = (1 - x^n) \sum_{d|n} \frac{\mu(d)}{1 - x^d}, \quad n > 1, \tag{30}$$

the first one being valid for $n \geq 1$.

If η is any primitive nth root of unity, then $\Psi_n(\eta) = \mu(n)$. Hence the cyclotomic polynomial $\Phi_n(x)$ divides the polynomial $\Psi_n(x) - \mu(n)$ for any $n \geq 1$. For these properties see [9, p. 71]. Furthermore, it is immediate from (30) that $\Psi_n(1) = \varphi(n)$ for any $n \geq 1$ and $\Psi_n(-1) = -\varphi(n)$ for any $n \geq 2$ even. Also, $\Psi_1(-1) = -1$ and $\Psi_n(-1) = 0$ for any $n > 1$ odd, since by (30), $\Psi_n(-1) = (1 - (-1)^n) \sum_{d|n} \frac{\mu(d)}{1-(-1)^d} = -\sum_{d|n} \mu(d) = 0$.

4 The polynomials $T_n(x)$

We consider in what follows the polynomials $T_n(x)$ given by (10).

Theorem 3 holds for the polynomials $T_n(x)$, as well. Also, for any prime p, $T_p(x) = p - 1 + x + x^2 + \ldots + x^{p-1}$, which follows at once from (15).

Theorem 10.

$$T_n(x) = \varphi(n) \left(1 - x^n + \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)} \Psi_d(x^{n/d}) \right). \tag{31}$$

Proof: Similar to the proof of Theorem 9, using Hölder's formula. □

Note that for every $n \geq 1$, $T_n(0) = |c_n(0)| = \varphi(n)$ is Euler's function. Also, $T_n(1) = \sum_{k=0}^{n-1} |c_n(k)| = \varphi(n) 2^{\omega(n)}$, where $\omega(n)$ denotes, as usual, the number of distinct prime factors of n. This identity follows at once by (31) and is given in [1].

Now we deduce for $T_n(x)$ a formula which is similar to (13).

Theorem 11. *For any $n \geq 1$,*

$$T_n(x) = (1 - x^n)\varphi(n) \sum_{d|n} \frac{\mu^2(d) f_d(n/d)}{\varphi(d)(1 - x^{n/d})}, \tag{32}$$

where $f_k(n)$ denotes the multiplicative function in n given by

$$f_k(n) = \prod_{\substack{p|n \\ p\nmid k}} \left(1 - \frac{1}{p-1} \right). \tag{33}$$

Note that $f_k(n) = 0$ for any n even and k odd.

Proof: Formula (3) can not be used in this case and we start with (31). Using also (30) we deduce

$$T_n(x) - (1 - x^n)\varphi(n) = \varphi(n) \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)} \left(1 - (x^{n/d})^d \right) \sum_{\delta|d} \frac{\mu(\delta) x^{n\delta/d}}{1 - x^{n\delta/d}}$$

$$= \varphi(n)(1 - x^n) \sum_{ab\delta = n} \frac{\mu^2(b\delta)\mu(\delta) x^{a\delta}}{\varphi(b\delta)(1 - x^{a\delta})}$$

$$= \varphi(n)(1 - x^n) \sum_{\substack{ab\delta = n \\ (b,\delta)=1}} \frac{\mu^2(b)\mu(\delta) x^{a\delta}}{\varphi(b)\varphi(\delta)(1 - x^{a\delta})}$$

$$= (1-x^n)\varphi(n)\sum_{bc=n}\frac{\mu^2(b)x^c}{\varphi(b)(1-x^c)}\sum_{\substack{a\delta=n\\(\delta,b)=1}}\frac{\mu(\delta)}{\varphi(\delta)},$$

where the inner sum is $f_b(c)$ and the given formula follows by writing $\frac{x^c}{1-x^c} = -1+\frac{1}{1-x^c}$ and using that $\sum_{bc=n}\frac{\mu^2(b)f_b(c)}{\varphi(b)} = 1$, which can be checked easily by the multiplicativity of the involved functions. ∎

In particular, if p, q are distinct primes, then by (32) we obtain

$$T_{pq}(x) = (p-1)(q-1) + x + x^2 + \ldots + x^{pq-1} \tag{34}$$

$$+(p-2)(x^p + x^{2p} + \ldots + x^{(q-1)p}) + (q-2)(x^q + x^{2q} + \ldots + x^{(p-1)q}),$$

which follows also from (16).

Theorem 12. *We have*

i) $T_n(-1) = \varphi(n)$ *for any* $n \geq 1$ *odd,*

ii) $T_n(-1) = 0$ *for any* $n = 4k+2$, $k \geq 0$,

iii) $T_n(-1) = \varphi(n)2^{\omega(n)}$ *for any* $n = 4k$, $k \geq 1$,

iv) $T_n(\eta) = n\prod_{p|n}(1-\frac{2}{p})$ *for any primitive* n*th root of unity* η*. The cyclotomic polynomial* $\Phi_n(x)$ *divides the polynomial* $T_n(x)$ *for any* $n \geq 2$ *even.*

Proof: For i)–iii) we use formula (31) and the properties of the polynomials $\Psi_n(x)$, mentioned in Remark 1.

i) For any $n \geq 1$ odd,

$$T_n(-1) = \varphi(n)\left(2 + \Psi_1(-1) + \varphi(n)\sum_{d|n,\, d>1}\frac{\mu^2(d)}{\varphi(d)}\Psi_d(-1)\right) = \varphi(n).$$

ii) For any $n = 4k+2$, $k \geq 0$,

$$T_n(-1) = \varphi(n)\sum_{d|n}\frac{\mu^2(d)}{\varphi(d)}\Psi_d((-1)^{n/d})$$

$$= \varphi(n)\sum_{d|2k+1}\frac{\mu^2(d)}{\varphi(d)}\Psi_d(1) + \varphi(n)\sum_{d=2\delta,\,\delta|2k+1}\frac{\mu^2(2\delta)}{\varphi(2\delta)}\Psi_{2\delta}(-1)$$

$$= \varphi(n)\sum_{d|2k+1}\mu^2(d) - \varphi(n)\sum_{\delta|2k+1}\mu^2(\delta) = 0.$$

iii) For any $n = 4k$, $k \geq 1$,

$$T_n(-1) = \varphi(n)\sum_{d|n}\frac{\mu^2(d)}{\varphi(d)}\Psi_d((-1)^{n/d}),$$

where for any d with $4 \mid d$, $\mu^2(d) = 0$. Hence

$$T_n(-1) = \varphi(n) \sum_{d \mid n,\, 4 \nmid d} \frac{\mu^2(d)}{\varphi(d)} \Psi_d(1) = \varphi(n) \sum_{d \mid n,\, 4 \nmid d} \mu^2(d)$$

$$= \varphi(n) \sum_{d \mid n} \mu^2(d) = \varphi(n) 2^{\omega(n)}.$$

iv) Now we use (32). The property follows from

$$T_n(x) = \varphi(n) f_1(n) + (1 - x^n) \sum_{d \mid n, d > 1} \frac{\mu^2(d) f_d(n/d)}{\varphi(d)(1 - x^{n/d})},$$

where $\eta^k \neq 1$ for any $k \mid n$, $k < n$ and $\varphi(n) f_1(n) = n \prod_{p \mid n} (1 - 2/p)$. ∎

Theorem 13. *i) If $n \geq 1$, then*

$$T_n(x) = \frac{n}{\gamma(n)} T_{\gamma(n)}(x^{n/\gamma(n)}). \tag{35}$$

ii) Let $n \geq 1$ and p be a prime. If $p \mid n$, then $T_{np}(x) = p T_n(x^p)$. If $p \nmid n$, then $T_{np}(x) = (p-2)\varphi(n) T_n(x^p) + (1 + x^n + x^{2n} + \ldots + x^{(p-1)n}) T_n(x)$.

Proof: i) This follows at once from Theorem 5/i) and from the definitions of the polynomials $T_n(x)$ and $R_n(x)$.

ii) For $p \mid n$ this follows from i) by $\gamma(np) = \gamma(n)$. For $p \nmid n$ by (32),

$$T_{np}(x) = (1 - x^{np}) \varphi(np) \sum_{d \mid np} \frac{\mu^2(d) f_d(np/d)}{\varphi(d)(1 - x^{np/d})}$$

$$= (1 - x^{np}) \varphi(n) \varphi(p) \left(\sum_{d \mid n} \frac{\mu^2(d) f_d(n/d) f_d(p)}{\varphi(d)(1 - x^{np/d})} + \sum_{d = \delta p,\ \delta \mid n} \frac{\mu^2(\delta p) f_{\delta p}(n/\delta)}{\varphi(\delta p)(1 - x^{n/\delta})} \right)$$

$$= (1 - x^{np}) \varphi(n) \left((p-2) \sum_{d \mid n} \frac{\mu^2(d) f_d(n/d)}{\varphi(d)(1 - x^{np/d})} + \sum_{\delta \mid n} \frac{\mu^2(\delta) f_{\delta p}(n/\delta)}{\varphi(\delta)(1 - x^{n/\delta})} \right)$$

$$= (p-2) \varphi(n) T_n(x^p) + \frac{1 - x^{np}}{1 - x^n} T_n(x),$$

where $f_{\delta p}(n/\delta) = f_\delta(n/d)$ for any $\delta \mid n$, $p \nmid n$. ∎

Theorem 14. *i) For any $n \geq 1$ odd, $T_{2n}(x) = (1 + x^n)T_n(x)$.*
ii) For any $n \geq 1$ odd and any $k \geq 1$,

$$T_{2^k n}(x) = 2^{k-1}\left(1 + x^{2^{k-1}n}\right) T_n(x^{2^{k-1}}). \tag{36}$$

iii) For any even n, $\left(1 + x^{n/2}\right) \mid T_n(x)$.

Proof: i) This follows at once from Theorem 13/ii) by $p = 2$.
ii), iii) The same proof as for the polynomials $R_n(x)$. □

Remark 2. Consider the polynomials

$$V_n(x) = \sum_{k=0}^{n-1} (c_n(k))^2 x^k. \tag{37}$$

For every $n \geq 1$, $V_n(0) = (c_n(0))^2 = (\varphi(n))^2$ and $V_n(1) = \sum_{k=0}^{n-1}(c_n(k))^2 = n\varphi(n)$, as it is known. For the polynomials $V_n(x)$ similar properties can be derived as for $T_n(x)$.

5 Tables of $R_n(x)$ and $T_n(x)$

The next two tables were produced using Maple. The polynomials $R_n(x)$ were generated by the following procedure (similar for $T_n(x)$):

```
with(numtheory): Ramanujanpol:= proc(n,x) local a, k: a:= 0: for k
from 0 to n-1 do a:=a+phi(n)*mobius(n/gcd(n,k))/phi(n/gcd(n,k))
*x^k: od: RETURN(R[n](x)=a) end;
```

Table of $R_n(x)$ for $1 \leq n \leq 20$

n	$R_n(x)$
1	1
2	$1 - x$
3	$2 - x - x^2 = (1 - x)(2 + x)$
4	$2 - 2x^2 = 2(1 - x)(1 + x)$
5	$4 - x - x^2 - x^3 - x^4 = (1 - x)(4 + 3x + 2x^2 + x^3)$
6	$2 + x - x^2 - 2x^3 - x^4 + x^5$ $= (1 - x)(2 - x)(1 + x)(1 + x + x^2)$
7	$6 - x - x^2 - x^3 - x^4 - x^5 - x^6$ $= (1 - x)(6 + 5x + 4x^2 + 3x^3 + 2x^4 + x^5)$
8	$4 - 4x^4 = 4(1 - x)(1 + x)(1 + x^2)$
9	$6 - 3x^3 - 3x^6 = 3(1 - x)(2 + x^3)(1 + x + x^2)$
10	$4 + x - x^2 + x^3 - x^4 - 4x^5 - x^6 + x^7 - x^8 + x^9$ $= (1 - x)(1 + x)(4 - 3x + 2x^2 - x^3)(1 + x + x^2 + x^3 + x^4)$

n	$R_n(x)$
11	$10-x-x^2-x^3-x^4-x^5-x^6-x^7-x^8-x^9-x^{10}$ $=(1-x)(10+9x+8x^2+7x^3+6x^4+5x^5+4x^6+3x^7+2x^8+x^9)$
12	$4+2x^2-2x^4-4x^6-2x^8+2x^{10}$ $=2(1-x)(1+x)(2-x^2)(1+x^2)(1-x+x^2)(1+x+x^2)$
13	$12-x-x^2-x^3-x^4-x^5-x^6-x^7-x^8-x^9-x^{10}-x^{11}-x^{12}$ $=(1-x)(12+11x+10x^2+9x^3+8x^4+7x^5+6x^6+5x^7+4x^8$ $+3x^9+2x^{10}+x^{11})$
14	$6+x-x^2+x^3-x^4+x^5-x^6-6x^7-x^8+x^9-x^{10}+x^{11}-x^{12}$ $+x^{13}$ $=(1-x)(1+x)(1+x+x^2+x^3+x^4+x^5+x^6)(6-5x+4x^2$ $-3x^3+2x^4-x^5)$
15	$8+x+x^2-2x^3+x^4-4x^5-2x^6+x^7+x^8-2x^9-4x^{10}+x^{11}$ $-2x^{12}+x^{13}+x^{14}$ $=(1-x)(1+x+x^2+x^3+x^4)(8-7x+5x^3-4x^4+3x^5-x^7)$ $(1+x+x^2)$
16	$8-8x^8=8(1-x)(1+x)(1+x^2)(1+x^4)$
17	$16-x-x^2-x^3-x^4-x^5-x^6-x^7-x^8-x^9-x^{10}-x^{11}-x^{12}$ $-x^{13}-x^{14}-x^{15}-x^{16}$ $=(1-x)(16+15x+14x^2+13x^3+12x^4+11x^5+10x^6+9x^7$ $+8x^8+7x^9+6x^{10}+5x^{11}+4x^{12}+3x^{13}+2x^{14}+x^{15})$
18	$6+3x^3-3x^6-6x^9-3x^{12}+3x^{15}$ $=3(1-x)(1+x)(1-x+x^2)(1+x+x^2)(1+x^3+x^6)(2-x^3)$
19	$18-x-x^2-x^3-x^4-x^5-x^6-x^7-x^8-x^9-x^{10}-x^{11}-x^{12}$ $-x^{13}-x^{14}-x^{15}-x^{16}-x^{17}-x^{18}$ $=(1-x)(18+17x+16x^2+15x^3+14x^4+13x^5+12x^6+11x^7$ $+10x^8+9x^9+8x^{10}+7x^{11}+6x^{12}+5x^{13}+4x^{14}+3x^{15}+2x^{16}+x^{17})$
20	$8+2x^2-2x^4+2x^6-2x^8-8x^{10}-2x^{12}+2x^{14}-2x^{16}+2x^{18}$ $=2(1-x)(1+x)(1+x^2)(4-3x^2+2x^4-x^6)(1+x+x^2+x^3$ $+x^4)(1-x+x^2-x^3+x^4)$

Table of $T_n(x)$ for $1 \le n \le 20$

n	$T_n(x)$
1	1
2	$1+x$
3	$2+x+x^2$
4	$2+2x^2=2(1+x^2)$
5	$4+x+x^2+x^3+x^4$
6	$2+x+x^2+2x^3+x^4+x^5$ $=(1+x)(1-x+x^2)(2+x+x^2)$
7	$6+x+x^2+x^3+x^4+x^5+x^6$
8	$4+4x^4=4(1+x^4)$
9	$6+3x^3+3x^6$
10	$4+x+x^2+x^3+x^4+4x^5+x^6+x^7+x^8+x^9$ $=(1+x)(4+x+x^2+x^3+x^4)(1-x+x^2-x^3+x^4)$
11	$10+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}$
12	$4+2x^2+2x^4+4x^6+2x^8+2x^{10}$ $=2(1+x^2)(2+x^2+x^4)(1-x^2+x^4)$
13	$12+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}$
14	$6+x+x^2+x^3+x^4+x^5+x^6+6x^7+x^8+x^9+x^{10}+x^{11}+x^{12}$ $+x^{13}$ $=(1+x)(1-x+x^2-x^3+x^4-x^5+x^6)(6+x+x^2+x^3+x^4$ $+x^5+x^6)$
15	$8+x+x^2+2x^3+x^4+4x^5+2x^6+x^7+x^8+2x^9+4x^{10}+x^{11}$ $+2x^{12}+x^{13}+x^{14}$
16	$8+8x^8=8(1+x^8)$
17	$16+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}$ $+x^{13}+x^{14}+x^{15}+x^{16}$
18	$6+3x^3+3x^6+6x^9+3x^{12}+3x^{15}$ $=3(1+x)(1-x+x^2)(2+x^3+x^6)(1-x^3+x^6)$
19	$18+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}$ $+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}+x^{18}$
20	$8+2x^2+2x^4+2x^6+2x^8+8x^{10}+2x^{12}+2x^{14}+2x^{16}+2x^{18}$ $=2(1+x^2)(4+x^2+x^4+x^6+x^8)(1-x^2+x^4-x^6+x^8)$

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Department of Mathematics, University of Pécs
7624 Pécs, Ifjúság u. 6., Hungary
E-mail: `ltoth@gamma.ttk.pte.hu`