On star partition dimension of the generalized gear graph

by

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Dedicated to the memory of Laurenţiu Panaitopol (1940-2008)
on the occasion of his 70th anniversary

Abstract

For a connected graph $G$ and any two vertices $u$ and $v$ in $G$, let $d(u, v)$ denote the distance between $u$ and $v$. For a subset $S$ of $V(G)$, the distance between a vertex $v$ and $S$ is $d(v, S) = \min\{d(v, x) \mid x \in S\}$. For an ordered $k$-partition of $V(G)$ $\Pi = \{S_1, S_2, \ldots, S_k\}$ and a vertex $v$, the representation of $v$ with respect to $\Pi$ is the $k$-vector $r(v \mid \Pi) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. $\Pi$ is a resolving partition for $G$ if the $k$-vectors $r(v \mid \Pi), v \in V(G)$ are distinct. The minimum $k$ for which there exists a resolving $k$-partition of $V(G)$ is the partition dimension of $G$, denoted by $pd(G)$. $\Pi = \{S_1, S_2, \ldots, S_k\}$ is a star resolving $k$-partition for $G$ if it is a resolving partition and each subgraph induced by $S_i$, $1 \leq i \leq k$, is a star. The minimum $k$ for which there exists a star resolving $k$-partition of $V(G)$ is the star partition dimension of $G$, denoted by $spd(G)$.

Let $J_{k,n}$ be the graph obtained from the wheel $W_{kn}$ by keeping spokes with step $k$, for $k \geq 2$ and $n \geq 2$. In this paper the star partition dimension for this family of graphs is determined.

Key Words: Distance, resolving partition, star resolving partition, partition dimension, star partition dimension, gear graph.

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1 Introduction

As described in [1] and [5], dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. Perhaps the best known example of this process is graph coloring. In [2], the vertices of a connected graph are represented by other criterion, namely through partitions of
Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices $u$ and $v$ in $G$, let $d(u, v)$ be the distance between $u$ and $v$. For a subset $S$ of $V(G)$ and a vertex $v$ of $G$, the distance $d(v, S)$ between $v$ and $S$ is defined as $d(v, S) = \min \{d(v, x) \mid x \in S\}$.

For an ordered $k$-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ and a vertex $v$ of $G$, the representation of $v$ with respect to $\Pi$ is the $k$-vector

$$r(v \mid \Pi) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k)).$$

$\Pi$ is called a resolving $k$-partition for $G$ if the $k$-vectors $r(v \mid \Pi)$, $v \in V(G)$ are distinct. The minimum $k$ for which there is a resolving $k$-partition of $V(G)$ is the partition dimension of $G$ and is denoted by $pd(G)$. A resolving partition of $V(G)$ containing $pd(G)$ classes is called a minimum resolving partition [2].

In [4] a particular case of resolving partitions - connected resolving partitions was considered. $\Pi = \{S_1, S_2, \ldots, S_k\}$ is said to be a connected resolving $k$-partition if it is a resolving partition and each subgraph induced by $S_i$, $1 \leq i \leq k$, is connected in $G$. The minimum $k$ for which there is a connected resolving $k$-partition of $V(G)$ is the connected partition dimension of $G$, denoted by $\text{cpd}(G)$.

In this paper we consider a particular case of resolving partitions - star resolving partitions, mentioned in [4] as topic for study. $\Pi = \{S_1, S_2, \ldots, S_k\}$ is called a star resolving $k$-partition if it is a resolving partition and each subgraph induced by $S_i$ is a star, for $1 \leq i \leq k$. The minimum $k$ for which there is a star resolving $k$-partition of $V(G)$ is the star partition dimension of $G$, denoted by $\text{spd}(G)$. A star resolving partition of $V(G)$ containing $\text{spd}(G)$ classes is called a minimum star resolving partition.

If $\Pi = \{S_1, S_2, \ldots, S_k\}$ is an ordered partition of $V(G)$ and $u_1, u_2, \ldots, u_r$ are $r$ distinct vertices, we say that $u_1, u_2, \ldots, u_r$ are separated by classes $S_{i_1}, \ldots, S_{i_q}$ of partition $\Pi$ if the $q$-vectors

$$(d(u_p, S_{i_1}), d(u_p, S_{i_2}), \ldots, d(u_p, S_{i_q})),$$ for $1 \leq p \leq r$

are distinct.

Let $J_{k,n}$ be the graph obtained from the wheel $W_{kn}$ by keeping spokes with step $k$, for $k \geq 2$ and $n \geq 2$. This is a generalization of gear graph [3], also known as Jahangir graph $J_{2n}$ [6]. More precisely, $J_{k,n}$ is obtained from the cycle $C_{kn}$ and a new vertex, as follows. Denote by $0, 1, \ldots, kn - 1$ the vertices of the cycle $C_{kn}$, and by $c$ the new vertex. We join by edges the vertex $c$ with vertices $0, k, 2k, \ldots, (n - 1)k$ of the cycle $C_{kn}$ (see fig. 1 for $k = 3$ and $n = 5$).

In the next section we will determine the star partition dimension of $J_{k,n}$.
2 Star partition dimension of $J_{k,n}$

We call a partition $\Pi = \{S_0, S_1, \ldots, S_q\}$ of $V(J_{k,n})$ a star partition of $J_{k,n}$ if each subgraph induced by $S_i$, $0 \leq i \leq q$, is a star. Hence, a star resolving partition is a star partition which is also a resolving partition.

Moreover, for a star partition of $J_{k,n}$ $\Pi = \{S_0, S_1, \ldots, S_q\}$, we assume that classes are numbered such that center $c$ is in $S_0$. We denote by $i_1, i_2, \ldots, i_r$ ($0 \leq i_1 < i_2 < \ldots < i_r \leq kn - 1$) the vertices of $S_0 \cap V(C_{kn})$. Hence we have

$$S_0 = \{c, i_1, i_2, \ldots, i_r\}.$$ 

Every two consecutive vertices from $S_0 \cap V(C_{kn})$ determine sets of vertices which induce paths on $C_{kn}$ called gaps [6]. More precisely, we say that the pair of vertices $\{i_t, i_{t+1}\}$, $1 \leq t \leq r - 1$ generates the gap $\{j| i_t < j < i_{t+1}\}$ and the pair of vertices $\{i_r, i_1\}$ generates the gap $\{j| i_r < j \leq kn - 1\} \cup \{j| 0 \leq j < i_1\}$. It is clear that some gaps may be empty.

**Theorem 2.1.** For any $k \geq 2$ and $n \geq 2$ we have

$$\text{spd}(J_{k,n}) = \begin{cases} 
3, & \text{if } k = 2 \text{ or } 3 \text{ and } n = 2; \\
\frac{k}{3}n, & \text{if } k \equiv 0 \text{ (mod 3)} \text{ and } (k, n) \neq (3, 2); \\
\left\lfloor \frac{k}{3} \right\rfloor n + 1, & \text{if } k \equiv 1 \text{ (mod 3)} \text{ and } (n \geq 3 \text{ or } (n = 2 \text{ and } k \geq 4)); \\
\left\lfloor \frac{k}{3} \right\rfloor n + 1 + \left\lceil \frac{n}{2} \right\rceil, & \text{if } k \equiv 2 \text{ (mod 3)} \text{ and } (n \geq 4 \text{ or } (n = 2 \text{ and } k \geq 5)); \\
3 \left\lceil \frac{k}{3} \right\rceil + 2, & \text{if } k \equiv 2 \text{ (mod 3)} \text{ and } n = 3.
\end{cases}$$

**Proof:** We will determine the minimum number of classes for a star partition of $J_{k,n}$, which is not necessarily a resolving partition. Let denote this number by $N_s$.

We consider three cases, in accordance with the residue class modulo 3 to which $k$ belongs.
Case 1: \( k \equiv 1(\mod 3) \).

Let \( k = 3p + 1, \ p \geq 1 \). It is easy to see that if a class of a star partition of \( J_{k,n} \) contains only vertices of the cycle \( C_{kn} \), then this class has maximum three elements. 

It follows that for \( n \geq 3 \) or \( (n = 2 \text{ and } p \geq 2) \) there exists a unique star partition with minimum number of classes, and that partition is \( \Pi = \{S_0, S_1, \ldots, S_{pn}\} \) where

\[
S_0 = \{c\} \cup \{kt|t \in \{0, 1, 2, \ldots, n - 1\}\}
\]

\[
S_{tp+1} = \{kt + 1, kt + 2, kt + 3\}, S_{tp+2} = \{kt + 4, kt + 5, kt + 6\}, \ldots,
\]

\[
S_{(t+1)p} = \{k(t + 1) - 3, k(t + 1) - 2, k(t + 1) - 1\}, \text{ for every } 0 \leq t \leq n - 1
\]

(see fig. 2 for \( k = 4 \) and \( n = 5 \)).

![Figure 2: \( J_{k,5} \)](image)

\( \Pi \) has \( N_s = pn + 1 = \left\lfloor \frac{k}{3} \right\rfloor n + 1 \) classes, hence

\[
\text{spd}(J_{k,n}) \geq N_s = \left\lfloor \frac{k}{3} \right\rfloor n + 1.
\]

But we can easily prove that \( \Pi \) is a resolving partition, which implies

\[
\text{spd}(J_{k,n}) = \left\lfloor \frac{k}{3} \right\rfloor n + 1.
\]

Indeed, the vertices from class \( S_i \), for \( 1 \leq i \leq pn \) are separated by classes \( S_{i-1} \) or \( S_{i+1} \).

For \( n \geq 3 \), vertices from \( S_0 \) are separated by classes \( S_1, S_{p+1}, S_{2p+1}, \ldots, S_{(n-1)p+1} \) because we have

\[
d(kt, S_{tp+1}) = 1, \text{ for every } 0 \leq t \leq n - 1
\]

\[
d(kt, S_{jp+1}) = 3, \text{ for every } 0 \leq t < j \leq n - 2
\]

\[
d(k(n - 1), S_1) = 3
\]

\[
d(c, S_{tp+1}) = 2, \text{ for every } 0 \leq t \leq n - 1.
\]

For \( n = 2 \) and \( p \geq 2 \) (or, equivalently, \( k \geq 7 \)) we have \( S_0 = \{c, 0, k\} \) and vertices \( c, 0 \) and \( k \) are separated by classes \( S_1 \) and \( S_{(n-1)p+1} \).
If \( n = 2 \) and \( p = 1 \) (\( k = 4 \)), \( kn = 8 \) there exists a star resolving partition with three classes
\[
\Pi = \{ \{c, 0, 1, 7\}, \{2, 3, 4\}, \{5, 6\} \}
\]
hence \( spd(J_{4,2}) = 3 \).

**Case 2:** \( k \equiv 0 \pmod{3} \).
Let \( k = 3p, \ p \geq 1 \). Let \( \Pi = \{S_0, S_1, \ldots, S_{N_s-1}\} \) be a star partition of \( J_{k,n} \) with minimum number of classes. We assume \( c \in S_0 \).

If \( |S_0| = 1 \), then the minimum number of classes in a star partition of \( J_{k,n} \) is equal to \( pn + 1 \), each of the classes \( S_1, \ldots, S_{pn} \) containing exactly three consecutive vertices of the cycle \( C_{kn} \).

Assume \( |S_0| > 1 \).

If \( \Pi \) induces on \( C_{kn} \) exactly one nonempty gap, we have \( 2 \leq |S_0| \leq 4 \), which implies that the minimum number of classes in a star partition of \( J_{k,n} \) is at least
\[
1 + \left\lceil \frac{kn - 3}{3} \right\rceil = \frac{kn}{3} = pn.
\]

If \( \Pi \) induces at least 2 nonempty gaps, denote by \( w = |S_0| - 1 \) the number of these gaps.

It is not difficult to see that in this case the number of vertices from any gap is congruent with 2 modulo 3. It follows that in a minimum star partition of \( J_{k,n} \) the vertices of every gap must be partitioned in paths with 3 vertices and one path with 2 vertices, hence the number of classes with 2 vertices is \( w \).

Consequently, the number of classes in a minimum star partition equals
\[
1 + \left\lceil \frac{kn - w - 2w}{3} \right\rceil + w = 1 + \frac{kn}{3}
\]

We have obtained that \( spd(J_{k,n}) \geq N_s \geq \frac{kn}{3} \).

Moreover, for \( n \geq 3 \) or \( (n = 2 \) and \( p \geq 2 \), partition \( \Pi = \{S_0, S_1, \ldots, S_{pn-1}\} \) having classes \( S_0 = \{c, 0, 1, kn - 1\} \) and \( S_t = \{3t-1, 3t, 3t+1\} \), for \( 1 \leq t \leq pn - 1 \) is a star resolving partition with \( pm \) classes of \( J_{k,n} \) (see fig. 3 for \( k = 3 \) and \( n = 5 \)).

\[\text{Figure 3: } J_{3,5}\]
For $n = 2$ and $p = 1$ partition $\Pi = \{\{c, 3\}, \{0, 1, 2\}, \{4, 5\}\}$ is a minimum star resolving partition, so in this case $spd(J_{3, 2}) = 3$.

It follows that $spd(J_{k, n}) = \frac{kn}{3}$ for every $(k, n) \neq (3, 2)$ and $spd(J_{3, 2}) = 3$.

**Case 3:** $k \equiv 2 \mod 3$.

Let $k = 3p + 2$, $p \geq 0$. Let $\Pi = \{S_0, S_1, \ldots, S_{N_{s} - 1}\}$ be a star partition of $J_{k, n}$ having minimum number of classes such that $c \in S_0$.

If the number of nonempty gaps induced on $C_{kn}$ by vertices from $S_0 - \{c\}$ is at least 2, then we can suppose that $\Pi$ has no gap containing exactly $k - 1 = 3p + 1$ vertices, bounded by 2 consecutive vertices of degree 3, $kt$ and $k(t + 1)$ of $C_{kn}$.

Indeed, if $\Pi$ has such a gap, let $W$, then $\Pi$ induces a minimum star partition $\Pi|_W$ of $W$ having $p + 1$ classes (one class being a singleton or 2 classes inducing each a path with 2 vertices and the remaining classes induce each a path with 3 vertices).

We shall delete $kt$ or $k(t + 1)$ from $S_0$; suppose that this choice is $kt$, hence we define $S'_0 = S_0 - \{kt\}$. Then we shall add vertex $kt$ to $W$ and we shall consider a star partition of $W \cup \{kt\}$ into $p + 1$ classes.

In this way a new star partition of $J_{k, n}$ having minimum number of classes has been produced with no induced gap of cardinality equal to $3p + 1$.

Consider now a minimum star partition $\Pi = \{S_0, S_1, \ldots, S_{N_{s} - 1}\}$ of $J_{k, n}$ such that $c \in S_0$ having no gap of cardinality $3p + 1$.

If $|S_0| = 1$, then the minimum number of classes in a star partition of $J_{k, n}$ is equal to

$$1 + \left\lceil \frac{kn}{3} \right\rceil = 1 + pm + \left\lceil \frac{2n}{3} \right\rceil \geq 1 + pm + \left\lceil \frac{n}{2} \right\rceil.$$

If $\Pi$ induces on $C_{kn}$ exactly one gap, then $2 \leq |S_0| \leq 4$.

It follows that this minimum number of classes in a star partition is at least

$$1 + \left\lceil \frac{kn - 3}{3} \right\rceil = 1 + pm + \left\lceil \frac{2n - 3}{3} \right\rceil.$$

For $n = 2$ or $n \geq 4$ we have

$$1 + pm + \left\lceil \frac{2n - 3}{3} \right\rceil \geq 1 + pm + \left\lceil \frac{n}{2} \right\rceil,$$

but for $n = 3$ we have $\left\lfloor \frac{2n - 3}{3} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor - 1$, so this lower bound becomes

$$3p + 2 = 3 \left\lceil \frac{k}{3} \right\rceil + 2 = pm + \left\lceil \frac{n}{2} \right\rceil.$$

If $\Pi$ induces at least 2 nonempty gaps, let $w = |S_0| - 1$ denote the number of these gaps. Since every gap has cardinality at least $6p + 3$, one deduces

$$w \leq \left\lceil \frac{n}{2} \right\rceil.$$
It follows that in this case the minimum number of classes in a star partition equals
\[ 1 + \left\lceil \frac{kn - w}{3} \right\rceil = 1 + pm + \left\lceil \frac{2n - w}{3} \right\rceil \geq 1 + pm + \left\lceil \frac{n}{2} \right\rceil. \]

If \( n = 3 \) in all cases we deduced
\[ \text{spd}(J_{k,n}) \geq N_s \geq 1 + pn + \left\lceil \frac{n}{2} \right\rceil = 3p + 2. \]

But for \( n = 3 \) a resolving star partition with \( 3p + 2 \) classes is built by choosing \( S_0 = \{c, 0, 1, kn - 1\} \), and other \( 3p + 1 \) classes are obtained by partitioning the vertices \( \{2, 3, \ldots, kn - 2 = 9p + 4\} \) into classes containing 3 consecutive vertices each: \( \{2, 3, 4\}, \{5, 6, 7\}, \ldots, \{9p + 2, 9p + 3, 9p + 4\} \) (see fig. 4 for \( k = 5 \) and \( n = 3 \)). Consequently, \( \text{spd}(J_{k,3}) = 3p + 2 \).

![Figure 4: J_{5,3}](image)

If \( n \neq 3 \) in all cases we have
\[ \text{spd}(J_{k,n}) \geq N_s \geq 1 + pm + \left\lceil \frac{n}{2} \right\rceil. \]

We consider a star partition \( \Pi = \{S_0, S_1, \ldots, S_r\} \) defined as follows:
\[ S_0 = \{c\} \cup \{2kt|0 \leq t \leq \left\lceil \frac{n}{2} \right\rceil - 1\} \]

and every gap bounded by vertices from \( S_0 \) is divided into \( 2p + 1 \) classes with 3 consecutive vertices each, with at most one exception (for \( n \) odd) - the last gap (bounded by \( 2k \left\lceil \frac{n}{2} \right\rceil - 1 \) and 0) has a class with one element (see fig. 5 for \( k = 5 \) and \( n = 5 \)). We denote the classes obtained, in the clockwise sense, by \( S_1, \ldots, S_r \). The number of classes in this partition is
\[ r = 1 + \left\lceil \frac{kn - \left\lceil \frac{n}{2} \right\rceil}{3} \right\rceil = 1 + pm + \left\lceil \frac{2n - \left\lceil \frac{n}{2} \right\rceil}{3} \right\rceil = 1 + pm + \left\lceil \frac{n}{2} \right\rceil. \]
It is easy to see that for \( n \geq 3 \) or \((n = 2 \text{ and } p \geq 1)\) \(\Pi\) is a resolving partition, this implying that
\[
spd(J_{k,n}) = 1 + pn + \left\lceil \frac{n}{2} \right\rceil.
\]

For \( n = 2 \) and \( p = 0 \) \((k = 2)\) we have \(spd(J_{k,n}) = 3\), a minimum star resolving partition being \(\{\{c\}, \{0,1\}, \{2,3\}\}\).

References


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