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On some absolute positiveness bounds

by
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Dedicated to the memory of Laurenţiu Panaitopol (1940-2008) on the occasion of his 70th anniversary

Abstract

We prove that some bounds for positive roots of univariate polynomials with real coefficients are absolute positiveness bounds. It is also proved that there exist positiveness bounds which are not absolute.

Key Words: Bound for absolute positiveness, Polynomial zeros.
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1 Introduction

The computation of bounds for positive roots of univariate polynomials with real coefficients is connected to many other problems in mathematics and related fields. For example it is important for some algorithms for real root isolation and for the problem of testing positiveness of polynomials. This last problem is decidable (see A. Tarski [14]).

H. Hoon and D. Jakus introduced [5] the bound for absolute positiveness of a polynomial with real coefficients. They proved that some bounds for positive roots of univariate polynomials are also bounds for absolute positiveness. However, absolute positiveness needs to be checked for any particular bound.

In this paper we observe that there exist upper bounds for positive roots that are not absolute positiveness bounds. Than we prove that some known bounds for positive roots of univariate polynomials are also bounds for absolute positiveness. We also discuss some cases where extensions of these results can be applied.

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2 Bounds for Positive Roots

Several bounds are known for the absolute values of the roots of univariate polynomials with complex coefficients (see M. Marden [7] and M. Mignotte–D. Ştefănescu [8]). They are expressed as functions of the degree and of he coefficients, and naturally they can be used also for the roots (real or complex) of polynomials with real coefficients. In the case of univariate polynomials with real coefficients these bounds are also upper bounds for the real roots, if any.

On the other hand, some specific bounds for real positive roots also exist (see. Akritas [2], D. Ştefănescu [13]). They are more accurate than those obtained for the absolute values of complex roots, but despite the case of complex roots there is no general result that assure the absolute positiveness. It is desirable to prove that a bound for positive roots is also a bound for the absolute positiveness.

3 Bounds for Absolute Positiveness

We remind that a number $B > 0$ is an *absolute positiveness bound* of the univariate polynomial $P \in \mathbb{R}[X]$ if, for any $t \in \mathbb{N}$, we have

$$P^{(t)}(x) > 0 \quad \text{for all} \quad x \geq B\,.$$

That means that B is an upper bound for the positive roots of P and for the positive roots of all its derivatives.

Remark: As H. Hoon noticed [6], the bounds for complex roots of univariate polynomials over the reals are also bounds for absolute positiveness, due to the theorem of Guaß–Lucas (s. M. Marden [7]). In fact, if P is univariate with real coefficients, the convex hull $K(P)$ of its zeros contains also the zeros of its derivative P'. Its trace on the real line contains the real zeros of P and also all zeros of P'.

Remark: A natural question is whether an arbitrary positive bound of a univariate polynomial over the real numbers is also a positive absoluteness bound. As noticed by H. Hoon [6], almost all known upper bounds for positive roots are bounds for absolute positiveness. The next example shows that a bound for the positive roots is not necessarily an absolute positiveness bound.

Example 1. Let $P(X) = X^5 - 10X^4 + 40X^3 - 80X^2 + 80X - 31\,.$

We observe that 1 is the largest (in fact the unique) positive root of P, so it is a bound for the positive roots.

However, the derivative $P'(X) = 5(X^4 - 8X^3 + 24X^2 - 32X + 16) = 5(X-2)^4$ has the positive root 2.

So $B(P) = 1$ is an upper bound for the positive roots which is not a bound for the roots of the derivative P', hence it is not an absolute positiveness bound.

We remind that a hyperbolic polynomial is a polynomial from $\mathbb{R}[X]$ which has only real roots. The following result is a corollary to known facts.

Proposition 1. *Let $P \in \mathbb{R}[X]$ be a hyperbolic polynomial and let (α, β) be an interval including its roots. The roots of the derivative are also included in (α, β).*

This can be proved in at least two ways, using as previously the Gauß–Lucas theorem or through the interlacing of roots of the polynomial and of its derivative.

Notation: Let $P \in \mathbb{R}[X] \backslash \mathbb{R}$ be such that it has an even number of sign variations and can be represented as

$$P(X) = c_1 X^{d_1} - b_1 X^{m_1} + c_2 X^{d_2} - b_2 X^{m_2} + \cdots + c_k X^{d_k} - b_k X^{m_k} + g(X),$$

with $g \in \mathbb{R}_+$, $d_1 > d_2, \ldots, d_k$, $c_i > 0$, $b_i > 0$ and $d_i > m_i$ for all i.
Denote

$$S_1(P) = \max\left\{ \left(\frac{b_1}{c_1}\right)^{1/(d_1 - m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k - m_k)} \right\}.$$

Proposition 2. *Let $P \in \mathbb{R}[X] \setminus \mathbb{R}$ be such that it has an even number of sign variations and can be represented as*

$$P(X) = c_1 X^{d_1} - b_1 X^{m_1} + c_2 X^{d_2} - b_2 X^{m_2} + \cdots + c_k X^{d_k} - b_k X^{m_k} + g(X),$$

with $g \in \mathbb{R}_+$, $d_1 > d_2, \ldots, d_k$, $c_i > 0$, $b_i > 0$ and $d_i > m_i$ for all i.
We have

$$S_1(P) > S_1(P').$$

Proof: By Theorem 2 in [11] we know that $S_1(P)$ is a bound for the positive roots of P, so we have

$$P(X) > 0 \quad \text{for all} \quad x \geq B_1.$$

Let's look to the corresponding bound for the derivative of P. We have

$$\begin{aligned} P'(X) &= d_1 c_1 X^{d_1 - 1} - m_1 b_1 X^{m_1 - 1} + \cdots + d_k c_k X^{d_k - 1} - m_k b_k X^{m_k - 1} + g(X) \\ &= c_1' X^{d_1'} - b_1' X^{m_1'} + \cdots + a_k' X^{d_k'} - b_k' X^{m_k'}. \end{aligned}$$

We have

$$a_i' = d_1 a_i, \quad b_i' = m_i' b_i, \quad d_i' = d_i - 1, \quad e_i' = e_i - 1.$$

Therefore $d_i' - e_i' = d_i - e_i$ and

$$\frac{b_i'}{e_i'} = \frac{m_i b_i}{d_i a_i} < \frac{b_i}{a_i},$$

since $m_i < d_i$.

We obtain that

$$\begin{aligned} S_1(P') &\leq \max\left\{\left(\frac{b'_1}{c'_1}\right)^{1/(d_1-m_1)}, \ldots, \left(\frac{b,_k}{c'_k}\right)^{1/(d_k-m_k)}\right\} \\ &< \max\left\{\left(\frac{b_1}{c_1}\right)^{1/(d_1-m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k-m_k)}\right\} \\ &= S_1(P)\,. \end{aligned}$$

□

Corollary 3. *The following inequalities hold*

$$S_1(P) > S_1(P') > \ldots > S_1(P^{d_1-1}) > 0\,.$$

Theorem 4. *The number $S_1(P)$ is an absolute positiveness bound for the polynomial P .*

Proposition 5. *Let*

$$P(X) = a_0X^d + a_1X^{d-1} + \cdots + a_mX^{d-m} - a_{m+1}X^{d-m-1} \pm \cdots \pm a_d \in \mathbb{R}[X]$$

with all $a_i \geq 0$, $a_0 > 0$,, $a_{m+1} > 0$. Let A be the largest absoulute value of the negative coefficients of P . The bound of Lagrange

$$L(P) = 1 + \left(\frac{A}{a_0}\right)^{\frac{1}{m+1}}$$

is an absolute positiveness bound for P .

Proof: We remind that $L(P)$ is an upper bound for the positive roots by a known theorem of Lagrange ...

We have

$$P'(X) = da_0X^{d-1}+\cdots+(d-m)a_mX^{d-m}-(d-m-i)a_{m+1}X^{d-m-2}\pm\cdots\pm a_{d-1}\,.$$

If $m = d-1$ we have no negative coefficient in P', so P' has no positive roots because all its coefficients are positive. So $L(P)$ is an absolute positiveness bound. If $m < d-1$ we have $d-m-2 \geq 0$, so the derivative P' has also negative coefficients. We denote by A' the absolute value of the largest negative coefficient of P'. We observe that

$$\begin{aligned} A' &= \max_{0\leq j\leq m+1}\{(d-m-j-1)a_j\,;\,\mathrm{coeff}(X^{d-m-j}) < 0\} \\ &\leq A \cdot \max_{0\leq j\leq m+1}\{(d-m-j-1)\} \\ &= A(d-m-1)\,. \end{aligned}$$

It follows that

$$L(P') \;=\; 1+\left(\frac{d-m-1}{d}\cdot\frac{A}{d}\right)^{1/(m+1)} \;\le\; 1+\left(\frac{A}{d}\right)^{1/(m+1)} \;=\; L(P)\,.$$

From this we deduce that $L(P)$ is a bound for absolute positiveness. ∎

Theorem 6. *Let*

$$P(X) \;=\; X^d - b_1X^{d-m_1} - \cdots - b_kX^{d-m_k} + g(X)\,,$$

where $b_1,\ldots,b_k>0$ *and* $g\in\mathbb{R}_+[X]$.

The number

$$S_2(P) \;=\; \max\{(kb_1)^{(1/m_1)},\ldots,(kb_k)^{(1/m_k)}\}$$

is an aboslute positiveness bound for P *.*

Proof: We put $Q(X) = \frac{1}{d}P'(X)$ and we have

$$Q(X) \;=\; X^{d-1} - \frac{d-m_1}{d}b_1X^{d-m_1-1} - \cdots - \frac{d-m_k}{d}b_kX^{d-m_k-1} + g'(X)\,.$$

We observe that $g'\in\mathbb{R}_+[X]$ and

$$\frac{d-m_j}{d}b_j \;<\; b_j\,.$$

We conclude, as in Proposition 2, that $S_2(P)>S_2(P')$. ∎

Remark: The bound in Theorem 4 can be extended to polynomials having at least one sign variation. Most of such results are summarized in A. Akritas [2] and D. Ştefănescu [13]. These extensions are based on the formula in Theorem 4 and the proof of Theorem 6. It can be proved that they also are absolute.

Applications and Computational Aspects

Because classical orthogonal polynomials have real coefficients and all their zeros are real, any upper bound for the positive roots is a bound for abosolute positiveness.

,We consider the polynomials

$$P_n(X) \;=\; \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\,\frac{(2n-2k)!}{k!(n-k)!(n-2k)!}\,X^{n-2k}, \quad \text{Legendre}$$

$$L_n(X) \;=\; \sum_{k=0}^{n}\binom{n}{n-k}\frac{(-1)^k}{k!}\,X^k, \quad \text{Laguerre}$$

and the bounds $Nw(P) = \sqrt{a_1^2 - 2a_2a_0}$ (Newton, see [9]) and $S_1(P)$ (Theorem 4). They give the followinf bounds for absolute positiveness:

Proposition 7. *Let P_n, respectively L_n be the orthogonal polynomials of degree n of Legendre, respectively Laguerre.*

i. The numbers

$$S_1(P_n) = \sqrt{\frac{n(n-1)}{2(2n-1)}} \quad \text{and} \quad Nw(P_n) = \sqrt{\frac{2(2n-2)!}{(n-1)!(n-2)!}}$$

are bounds for the absolute positiveness P_n .

ii. The numbers

$$S_1(L_n) = n^2 \quad \text{and} \quad Nw(L_n) = \sqrt{n^4 - n^2(n-1)^2}$$

are bounds for the absolute positiveness L_n .

Proof: We use Theorem 4 and the bound of Newton. For Legendre polynomials we have the bound

$$\max\left\{\frac{(n-2k+1)(n-2k+2)}{k(2n-2k+1)}\,; 1 \le k \le \lfloor n/2 \rfloor\right\}$$

which gives $S_1(P_n) = \sqrt{\frac{n(n-1)}{2(2n-1)}}$. ∎

We note that $S_1(L_n) < Nw(L_n)$. Much better bounds can be obtained using specific properties of orthogonal polynomials, see W. H. Foster–I. Krasikov [4] and D. Ştefănescu [12].

Refinements of the Bounds for Absolute Positiveness

Upper bounds for positive roots can be used for computing refined absolute positiveness bounds. A fruitful method is to express the bounds as quotients between absolute values of negative and positive coefficients, as in Theorem 4.

A. Akritas, A. Strzeboński and P. Vigklas [1], [2], for example, consider a polynomial

$$P(X) = \alpha_n X^n + \alpha_{n-1}X^{n-1} + \cdots + \alpha_0 \in \mathbb{R}[X]\,, \quad \alpha_n > 0$$

and represents it as

$$P(X) = q_1(X) - q_2(X) + q_3(X) - q_4(X) + \cdots + q_{2m-1}(X) - q_{2m}(X) + g(X)\,, \quad (1)$$

where all polynomials q_i and g have positive coefficients.

This extends the proof of Theorem 6 in [11]. There we represented the polynomial

$$X^d - b_1 X^{d-m_1} - \cdots - b_k X^{d-m_k}$$

as

$$\frac{1}{k}\left(X^d - kb_1 X^{d-m_1}\right) + \cdots + \frac{1}{k}\left(X^d - kb_k X^{d-m_k}\right) , \qquad (2)$$

which allows using the same device as in Theorem 4.

In fact, A. Akritas a.o [1] use the same bound as in Theorem 6 for a polynomial $q_{2i-1} - q_{2i}$, as we did for $X^d - b_i X^{d-m_i}$.

In his Theorem 5 from [2], A. Akritas considers the concepts of *matching* (or *pairing*, introduced in [11]) of a positive coefficient with a convenient negative one, and that of *breaking up* a positive coefficient into several parts to be paired with negative coefficients of lower order terms.

In fact, *breaking up* proved to be useful for pairing positive and negative coeff-cients. It was used also by D. Ştefănescu [11] for proving that the number $S_2(P)$ is an upper bound for positive roots. In (2) (cf. [11]), the leading coefficient 1 is broken into k equal parts. A. Akritas [2] called this device "Cauchy's leading–coefficient implementation" of his Theorem 5.

Remark: In the Theorem of Akritas [2] it is assumed that $P(X)$ has the representation (1), where all polynomials q_i and g have positive coefficients. This can be realised for any polynomial with real coefficients having at least one sign variation, as proved by D. Ştefănescu [13]:

Lemma 8. *Any polynomial $P \in \mathbb{R}[X]$ having at least one sign variation can be represented as*

$$P(X) \,=\, c_1 X^{d_1} - b_1 X^{m_1} + c_2 X^{d_2} - b_2 X^{m_2} + \cdots + c_k X^{d_k} - b_k X^{m_k} + g(X) \,, \quad (3)$$

with $g \in \mathbb{R}_+$, $d_1 > d_2, \ldots, d_k$, $c_i > 0$, $b_i > 0$, $d_i > m_i$ for all i.

Note that the representation (3) is not unique. A. Akritas [2] used in [2] several representations of P. His main implementations, *"first–λ"* and *"local–max"* are based on the breaking of a coefficient into equal parts, respectively by powers of 2 .

Extending the proof of Theorem 6 from [11], we have obtained in [13] a more general ound that involves two families of parameters (γ_{ij}) and $(beta_i)$. A related procedure was used by P. Batra–V. Sharma [3] for multivariate polynomials, starting with the matrix of pounds (δ_{ij}) while in [13] we used a matrix (γ_{ij}) and a vector (β_j) .

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