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Completely irreducible meet decompositions in lattices, with applications to Grothendieck categories and torsion theories (II)

by
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*Dedicated to the memory of Laurenţiu Panaitopol (1940-2008)
on the occasion of his 70th anniversary*

Abstract

This is the second part of the paper with the same title published in *Bull. Math. Soc. Sci. Math. Roumanie* **52** (100), no.4, (2009), 393-419.

Key Words: Grothendieck category, irreducible subobject, completely irreducible subobject, coirreducible (uniform) object, irreducible decomposition, injective hull, Gabriel dimension, hereditary torsion theory, τ-irreducible submodule, τ-completely irreducible submodule, τ-coireducible module, module rich in τ-coirreducibles.

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2 Applications to Grothendieck categories and torsion theories

In this section we apply the lattice-theoretical results established in the previous sections to Grothendieck categories and module categories equipped with a hereditary torsion theory.

Throughout this section $\mathcal{G}$ will denote a fixed *Grothendieck category*, that is, an Abelian category with exact direct limits and with a generator. For any object $X \in \mathcal{G}$, $\mathcal{L}(X)$ will denote the lattice of all subobjects of X. It is well-known that $\mathcal{L}(X)$ is an upper continuous modular lattice (see e.g., Stenström [19, Chapter 4, Proposition 5.3, and Chapter 5, Section 1]). For all undefined notation and

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terminology on Abelian categories the reader is referred to Albu and Năstăsescu [3] and/or Stenström [19].

We say that an object $X \in \mathcal{G}$ is *subdirectly irreducible*, abbreviated SI, if the lattice $\mathcal{L}(X)$ is subdirectly irreducible. More generally, if $\mathbb{P}$ is any property on lattices, we say that an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of *coirreducible* (*uniform*) object, *completely coirreducible* object, *irreducible subobject* of an object, *completely irreducible* (CI) *subobject* of an object, object *rich in completely irreducibles* (RCI), object *rich in coirreducibles* (RC), etc. Similarly, a subobject Y of an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$.

If we specialize Corollary 0.6, Theorems 1.16, and Proposition 1.23 (of the first part of this paper) for $L = \mathcal{L}(X)$, we obtain at once.

Proposition 2.1. *If X is a semi-Artinian object of a Grothendieck category $\mathcal{G}$, then any irreducible subobject of X is* CI. □

Theorem 2.2. *The following assertions are equivalent for a nonzero object X of a Grothendieck category $\mathcal{G}$.*

(1) *X is* RC.

(2) *X is an essential extension of a direct sum of coirreducible subobjects of X.*

(3) *The injective hull $E(X)$ of X is an essential extension of a direct sum of indecomposable injective objects.*

(4) *0 has an irredundant irreducible decomposition in every nonzero subobjects of X.* □

Theorem 2.3. *The following statements are equivalent for a nonzero object X of a Grothendieck category $\mathcal{G}$.*

(1) *X is* RCC.

(2) *Every subobject of X contains a simple subobject.*

(3) *The socle* $\mathrm{Soc}(X)$ *of X is essential in X.*

(4) *For every nonzero subobject Y of X there exists a nonempty set I_Y such that 0 can be written as an irredundant intersection*

$$0 = \bigcap_{i \in I_Y} X_i$$

of maximal subobjects X_i of Y, $i \in I_Y$, in other words, the Jacobson radical $J(Y)$ of Y is zero and an irredundant intersection of maximal subobjects. □

As in Năstăsescu and Popescu [17], a Grothendieck category $\mathcal{G}$ is said to be an *L.C.-category* if each nonzero object X of $\mathcal{G}$ contains a coirreducible subobject, in other words, if the lattice $\mathcal{L}(X)$ is RC for each $0 \neq X \in \mathcal{G}$. The next result is a very particular case of Theorem 2.2.

Corollary 2.4. *The following statements are equivalent for a Grothendieck category* $\mathcal{G}$.

(1) $\mathcal{G}$ *is an* L.C.*-category.*

(2) *Every nonzero object* X *of* $\mathcal{G}$ *is an essential extension of a direct sum of coirreducible subobjects of* X.

(3) *For every nonzero object* X *of* $\mathcal{G}$*, the injective hull* $E(X)$ *of* X *is an essential extension of a direct sum of indecomposable injective objects.*

(4) *For every nonzero object* X *of* $\mathcal{G}$*,* 0 *has an irredundant irreducible decomposition in every nonzero subobjects of* X. □

Remarks 2.5. The equivalencies (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) in Corollary 2.4 are precisely the contents of Năstăsescu and Popescu [17, Proposition 1]. □

Proposition 2.6. *An object* X *of a Grothendieck category* $\mathcal{G}$ *is semi-Artinian if and only if every subobject* X *has an irredundant completely irreducible decomposition in* X. □

Proposition 2.7. *Let* $\mathcal{G}$ *be a Grothendieck category, and let* $X \in \mathcal{G}$*. If* X *has Gabriel dimension, then* X *is* RC.

Proof: Apply Examples 1.3 (3) to the lattice $L = \mathcal{L}(X)$. □

Recall that the concept of Gabriel dimension of an Abelian category $\mathcal{A}$, due to Gabriel [11], has been originally defined using a transfinite sequence of localizing subcategories of $\mathcal{A}$. For a Grothendieck category $\mathcal{G}$, the fact that $\mathcal{G}$ has Gabriel dimension can be equivalently expressed by saying that $\mathcal{G}$ possess a generator G having Gabriel dimension, that is, the lattice $\mathcal{L}(G)$ of all subobjects of G has Gabriel dimension.

Corollary 2.8. (Năstăsescu and Popescu [17, Remarques 1]). *Any Grothendieck category having Gabriel dimension is an* L.C.*-category.*

We end this paper by presenting some applications of our lattice theoretical results to module categories equipped with a hereditary torsion theory.

Throughout the remainder of the paper $\tau = (\mathcal{T}, \mathcal{F})$ will be a fixed hereditary torsion theory on Mod-R, and $\tau(M)$ will denote the τ-torsion submodule of a right R-module M. The set $F_\tau := \{\, I \leqslant R_R \mid R/I \in \mathcal{T}\,\}$ is called the *Gabriel topology* associated with τ.

For any M_R we denote $\mathrm{Sat}_\tau(M) = \{ N \mid N \leqslant M,\, M/N \in \mathcal{F} \}$, and for any $N \leqslant M$ we denote by $\overline{N} = \bigcap \{ C \mid N \leqslant C \leqslant M,\, M/C \in \mathcal{F} \}$ the *τ-closure* (or *τ-saturation*, or *τ-purification*) of N in M; N is called *τ-closed* (or *τ-saturated*, or *τ-pure*) if $N = \overline{N}$. Note that $\overline{N}/N = \tau(M/N)$ and

$$\mathrm{Sat}_\tau(M) = \{ N \mid N \leqslant M,\, N = \overline{N} \}.$$

It is known that $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice for any M_R (see Stenström [19, Chapter 9, Proposition 4.1]).

Recall that a module M_R is said to be *τ-simple* if the lattice $\mathrm{Sat}_\tau(M)$ has exactly two elements; i.e., $\mathrm{Sat}_\tau(M) = \{\tau(M), M\}$ and $M \notin \mathcal{T}$. A τ-simple τ-torsionfree module is called *τ-cocritical*. Note that the atoms of the lattice $\mathrm{Sat}_\tau(M)$ are exactly the τ-closed τ-simple submodules of M. A right ideal I of R is called *τ-critical* if the right R-module R/I is τ-cocritical. The *τ-socle* of M is defined by $\mathrm{Soc}_\tau(M) = \overline{\sum \{ C \mid C \leqslant M,\, C \text{ is } \tau\text{-cocritical}\}}$. Note that, by Albu [1, Proposition 1.15], $\mathrm{Soc}_\tau(M)$ is exactly the socle of the lattice $\mathrm{Sat}_\tau(M)$. A submodule N of M is said to be *τ-maximal* if the module M/N is τ-cocritical. The meet of all *τ-maximal* submodules is called the *τ-Jacobson radical* of M and denoted by $J_\tau(M)$; if M fails to have any τ-maximal submodules then we set $J_\tau(M) = M$.

For all undefined notation and terminology on torsion theories the reader is referred to Albu and Năstăsescu [3], Golan [12], and/or Stenström [19].

As in Albu, Iosif, and Teply [2], a module M_R is said to be *τ-subdirectly irreducible*, abbreviated τ-SI, if the lattice $\mathrm{Sat}_\tau(M)$ is subdirectly irreducible. More generally, if $\mathbb{P}$ is any property on lattices, we say that a module M_R is/has τ-$\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$. Since the lattices $\mathrm{Sat}_\tau(M)$ and $\mathrm{Sat}_\tau(M/\tau(M))$ are canonically isomorphic, we deduce that M_R is τ-$\mathbb{P}$ if and only if $M/\tau(M)$ is τ-$\mathbb{P}$. Thus, we obtain the concepts of a *τ-Artinian* module, *τ-Noetherian* module, *τ-semi-Artinian* module, *τ-coirreducible* (*uniform*) module, *τ-completely coirreducible* module, module *rich in τ-coirreducibles*, abbreviated τ-RC, module *rich in τ-completely coirreducibles*, abbreviated τ-RCC, module *rich in τ-completely irreducibles*, abbreviated τ-RCI, etc. We say that a submodule N of M_R is/has τ-$\mathbb{P}$ if its closure $\overline{N}$, which is an element of $\mathrm{Sat}_\tau(M)$, is/has $\mathbb{P}$. Thus, we obtain the concepts of a *τ-irreducible submodule* of a module, *τ-completely irreducible submodule* of a module, abbreviated τ-CI, etc. Since $\overline{N} = \overline{\overline{N}}$, it follows that N is/has τ-$\mathbb{P}$ if and only if $\overline{N}$ is/has τ-$\mathbb{P}$.

Before giving specializations of the latticial results from the previous section to the lattice $\mathrm{Sat}_\tau(M)$ we will present some intrinsic characterizations, that is, without explicitly referring to the lattice $\mathrm{Sat}_\tau(M)$, of τ-irreducible and τ-completely irreducible submodules of a module.

Proposition 2.9. *The following assertions are equivalent for a submodule N of a module M_R.*

(1) *N is τ-irreducible.*

(2) *$M/N \notin \mathcal{T}$ and for any submodules P and Q of M with $N \subseteq P \cap Q$ and $(P \cap Q)/N \in \mathcal{T}$ one has $P/N \in \mathcal{T}$ or $Q/N \in \mathcal{T}$.*

(3) *$M/N \notin \mathcal{T}$ and for any submodules P and Q of M with $\overline{N} = P \cap Q$ one has $P/N \in \mathcal{T}$ or $Q/N \in \mathcal{T}$.*

Proof: (1) $\Longrightarrow$ (2): First, note that since N is τ-irreducible, $\overline{N} \neq M$, i.e., $M/N \notin \mathcal{T}$. If $N \subseteq P \cap Q$ and $(P \cap Q)/N \in \mathcal{T}$, then $\overline{N} = \overline{P \cap Q} = \overline{P} \cap \overline{Q}$, hence $\overline{N} = \overline{P}$ or $\overline{N} = \overline{Q}$ because N is τ-irreducible, i.e., $\overline{N}$ is an irreducible element of the lattice $\mathrm{Sat}_\tau(M)$. Thus $P/N \subseteq \overline{P}/N = \overline{N}/N \in \mathcal{T}$ or $Q/N \subseteq \overline{Q}/N = \overline{N}/N \in \mathcal{T}$, and so, $P/N \in \mathcal{T}$ or $Q/N \in \mathcal{T}$, as desired.

(2) $\Longrightarrow$ (3): Let $P, Q \leqslant M$ with $\overline{N} = P \cap Q$. Then $\overline{N}/N = (P \cap Q)/N \in \mathcal{T}$, so $P/N \in \mathcal{T}$ or $Q/N \in \mathcal{T}$.

(3) $\Longrightarrow$ (1): If $\overline{N} = X \cap Y$ with $X, Y \in \mathrm{Sat}_\tau(M)$, then $X/N \in \mathcal{T}$ or $Y/N \in \mathcal{T}$ by hypothesis, and so $\overline{N} = \overline{X} = X$ or $\overline{N} = \overline{Y} = Y$. Now observe that $\overline{N} \neq M$ since $M/N \notin \mathcal{T}$. Consequently $\overline{N}$ is an irreducible element of the lattice $\mathrm{Sat}_\tau(M)$, in other words, N is τ-irreducible. □

Corollary 2.10. *The following assertions are equivalent for a module M_R.*

(1) *M is τ-coirreducible.*

(2) *$M \notin \mathcal{T}$ and for every $A, B \leqslant M$ with $A \cap B \in \mathcal{T}$ one has $A \in \mathcal{T}$ or $B \in \mathcal{T}$.*

In particular, if $M \in \mathcal{F}$, then M is τ-coirreducible $\Longleftrightarrow$ M is coirreducible.

Proof: M is τ-coirreducible if and only if 0 is a τ-irreducible submodule of M, so apply Proposition 2.9 for $N = 0$. □

Remarks 2.11. A module $M \in \mathcal{F}$ which is τ-completely coirreducible is not necessarily completely coirreducible. Indeed, consider the torsion theory $\tau_0 = (\mathcal{T}_0, \mathcal{F}_0)$ on the ring $R = \mathbb{Z}$ associated with the Gabriel topology F_0 on $\mathbb{Z}$ consisted of all nonzero ideals of $\mathbb{Z}$. Note that this is the "localization at 0" Gabriel topology F_0 defined by the prime ideal 0 of $\mathbb{Z}$, $\mathcal{T}_0$ is the class of all usual torsion Abelian groups, and $\mathcal{F}_0$ is the class of all usual torsionfree Abelian groups. Observe that the lattice $\mathrm{Sat}_{\tau_0}(\mathbb{Z}) = \{0, \mathbb{Z}\}$ has a unique atom $\mathbb{Z}$, so $\mathbb{Z}$ is τ_0-SI, i.e., τ_0-completely coirreducible, but it is not completely coirreducible because $\bigcap_{n \in \mathbb{N}^*} n\mathbb{Z} = 0$ and $n\mathbb{Z} \neq 0$ for all $n \in \mathbb{N}^*$. □

In order to extend the characterization of τ-irreducible submodules in Proposition 2.9 to τ-completely irreducible submodules, we introduce below the following definition.

Definition 2.12. *Let M_R be a module. We say that a hereditary torsion theory τ on* Mod-R *satisfies the condition $(\dagger_M)$ if the closure operator on the lattice of all submodules $\mathcal{L}(M)$ of M commutes with arbitrary intersections, i.e.,*

$(\dagger_M)$ $\qquad \overline{\bigcap_{i\in I} X_i} = \bigcap_{i\in I} \overline{X_i}$ *for any family* $(X_i)_{i\in I}$ *of submodules of* M. □

Note that in condition $(\dagger_M)$ only the inclusion "$\supseteq$" is necessary since "$\subseteq$" always holds.

For a module M_R we set

$$F(M) := \{\, N \leqslant M \mid M/N \in \mathcal{T}\,\}.$$

Observe that for $N \leqslant M$, one has $N \in F(M) \iff \overline{N} = M$. Clearly, $F(R_R)$ is exactly the Gabriel topology F_τ associated with τ.

Lemma 2.13. *If the condition* $(\dagger_M)$ *is satisfied for a module* M_R*, then* $\bigcap_{N\in F(M)} N \in F(M)$.

Proof: If we consider the family $(N)_{N\in F(M)}$ of all elements of $F(M)$, by condition $(\dagger_M)$ we have

$$M = \bigcap_{N\in F(M)} \overline{N} \subseteq \overline{\bigcap_{N\in F(M)} N},$$

so $\overline{\bigcap\limits_{N\in F(M)} N} = M$, i.e., $\bigcap_{N\in F(M)} N \in F(M)$, as desired. □

Remarks 2.14. We do not know whether $\bigcap_{N\in F(M)} N \in F(M)$ implies the condition $(\dagger_M)$, but we suspect *no*. □

Recall that the torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is called *Jansian* (see Golan [12]) if the Gabriel topology F_τ associated with τ has a basis consisting of an idempotent two-sided ideal, or equivalently, if $\bigcap\limits_{D\in F_\tau} D \in F_\tau$.

Proposition 2.15. (Golan [12, Proposition 6.6]). *A hereditary torsion theory* $\tau = (\mathcal{T}, \mathcal{F})$ *on* Mod-R *is Jansian if and only if* τ *satisfies the condition* $(\dagger_M)$ *for any module* M_R.

Proof: For the reader's convenience we include the proof. Assume that τ is Jansian. Let M_R be a module, let $(X_i)_{i\in I}$ be a family of submodules of M, and let $x \in \bigcap_{i\in I} \overline{X_i}$. For each $i \in I$ there exists $D_i \in F_\tau$ such that $xD_i \subseteq X_i$. If we set $D := \bigcap_{i\in I} D_i$, then $D \in F_\tau$ since τ is Jansian, so $xD \subseteq X_i$ for all $i \in I$. This shows that $xD \in \bigcap_{i\in I} X_i$, and so, $x \in \overline{\bigcap_{i\in I} X_i}$. Therefore, $\bigcap_{i\in I} \overline{X_i} \subseteq \overline{\bigcap_{i\in I} X_i}$, in other words, τ satisfies the condition $(\dagger_M)$.

Conversely, if τ satisfies the condition $(\dagger_M)$ for any module M_R, then, in particular $(\dagger_R)$ is satisfied, so $\bigcap_{D\in F(R_R)} D \in F(R_R)$ by Lemma 2.13, i.e., $\bigcap_{D\in F_\tau} D \in F_\tau$, which means exactly that τ is Jansian. □

Proposition 2.16. *Let N be a submodule of a module M_R, and consider the following assertions:*

(1) *N is τ-CI.*

(2) *$M/N \notin \mathcal{T}$ and for any family $(P_i)_{i\in I}$ of submodules of M such that $N \subseteq \bigcap_{i\in I} P_i$ and $\big(\bigcap_{i\in I} P_i\big)/N \in \mathcal{T}$, one has $P_i/N \in \mathcal{T}$ for some $i \in I$.*

(3) *$M/N \notin \mathcal{T}$ and for any family $(P_i)_{i\in I}$ of submodules of M such that $\overline{N} = \bigcap_{i\in I} P_i$, one has $P_i/N \in \mathcal{T}$ for some $i \in I$.*

Then (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1), and (1) $\Longrightarrow$ (2) if the torsion theory τ satisfies the condition $(\dagger_M)$.

Proof: (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1): Proceed as in the proof of Proposition 2.9.

(1) $\Longrightarrow$ (2): Assume that τ satisfies the condition $(\dagger_M)$, and let N be as in (2). Then

$$\overline{N} = \overline{\bigcap_{i\in I} P_i} = \bigcap_{i\in I} \overline{P_i},$$

so $\overline{N} = \overline{P_i}$ for some $i \in I$ because $\overline{N}$ is a CI element of the lattice $\mathrm{Sat}_\tau(M)$. Thus $P_i/N \subseteq \overline{P_i}/N = \overline{N}/N \in \mathcal{T}$, and then $P_i/N \in \mathcal{T}$, as desired. □

Definition 2.17. *A submodule N of a module M is called* strongly τ-completely irreducible, *abbreviated* strongly τ-CI, *if $M/N \notin \mathcal{T}$ and for any family $(P_i)_{i\in I}$ of submodules of M such that $N \subseteq \bigcap_{i\in I} P_i$ and $\big(\bigcap_{i\in I} P_i\big)/N \in \mathcal{T}$, one has $P_i/N \in \mathcal{T}$ for some $i \in I$.* □

Remarks 2.18. (1) Let $\tau_0 = (\mathcal{T}_0, \mathcal{F}_0)$ be the torsion theory on the ring $R = \mathbb{Z}$ associated with the Gabriel topology F_0 considered in Remark 2.11. Then it is easy to see that 0 is a τ_0-CI submodule of $M = \mathbb{Z}$ which is not strongly τ_0-CI.

(2) Any strongly τ-CI submodule N of M, with $N \in \mathrm{Sat}_\tau(M)$ is a CI submodule of M. Indeed, if $(X_i)_{i\in I}$ is a family of submodules of M with $N = \bigcap_{i\in I} X_i$, then $\big(\bigcap_{i\in I} X_i\big)/N = 0 \in \mathcal{T}$, so $X_i/N \in \mathcal{T}$ for some $i \in I$. On the other hand $X_i/N \leqslant M/N \in \mathcal{F}$, so $X_i/N = 0$, i.e., $N = X_i$, which shows that N is a CI submodule of M.

(3) By Proposition 2.6, any τ-CI-submodule of M_R is strongly τ-CI in the presence of condition $(\dagger_M)$. □

We are now going to specialize the latticial results obtained for an arbitrary upper continuous modular lattice to the particular case of the lattice $\mathrm{Sat}_\tau(M)$. We will present only two such specializations. To do that, we need some preparatory results.

Lemma 2.19. *The following assertions hold for a module $M_R \in \mathcal{F}$ and a submodule $N \leqslant M$.*

(1) *If $M/N \in \mathcal{T}$, then N is an essential submodule of M.*

(2) *N is an essential submodule of $\overline{N}$.*

(3) *If $N \in \mathrm{Sat}_\tau(M)$, then N is an essential submodule of M if and only if N is an essential element of the lattice $\mathrm{Sat}_\tau(M)$.*

Proof: (1) Let $0 \neq x \in M$. Since $M/N \in \mathcal{T}$, there exists $I \in F_\tau$ such that $xI \subseteq N$. But $xI \neq 0$ because $M \in \mathcal{F}$, so there exists $r \in R$ with $0 \neq xr \in N$, which shows that N is essential in M.

(2) Since $\overline{N}/N \in \mathcal{T}$, we can apply (1) by taking $\overline{N}$ as M.

(3) See the proof of Albu [1, Corollary 1.3]. □

As we already have indicated, a module M_R is said to be *rich in τ-coirreducibles*, abbreviated τ-RC (resp. *rich in τ-completely coirreducibles*, abbreviated τ-RCC) if the lattice $\mathrm{Sat}_\tau(M)$ is RC (resp. RCC). Also, a module M_R is said to be *τ-atomic* if the lattice $\mathrm{Sat}_\tau(M)$ is atomic. Note that, by Examples 1.3 (1), M_R is τ-RCC if and only if it is τ-atomic.

Proposition 2.20. *A module $M_R \in \mathcal{F}$ is τ-*RC (*resp.* τ-RCC) *if and only if $M \neq 0$ and for every $0 \neq X \leqslant M$ there exists $C \leqslant X$ which is τ-coirreducible* (*resp. τ-cocritical*).

Proof: One implication is clear. For the other one, assume that M is τ-RC (resp. τ-RCC), and let $0 \neq X \leqslant M$. Then $0 \neq \overline{X} \in \mathrm{Sat}_\tau(M)$, so, by definition, there exists $D \in \mathrm{Sat}_\tau(M)$ such that $D \leqslant \overline{X}$ and D is a coirreducible element (resp. atom) of the lattice $\mathrm{Sat}_\tau(M)$, that is, D is τ-coirreducible (resp. τ-cocritical). Now, observe that $D \cap X$ is also τ-coirreducible (resp. τ-cocritical) because X is an essential submodule of $\overline{X}$ by Lemma 2.19 (2). □

Corollary 2.21. *Let $M_R \in \mathcal{F}$. Then M is τ-*RC $\Longleftrightarrow$ *M is* RC.

Proof: Apply Proposition 2.20 and Corollary 2.10. □

Lemma 2.22. *Let $M_R \in \mathcal{F}$ be a module, and let $(N_i)_{i\in I}$ be a family of submodules of M. Then $(N_i)_{i\in I}$ is an independent family of submodules of M if and only if $(\overline{N_i})_{i\in I}$ is an independent family of elements of the lattice $\mathrm{Sat}_\tau(M)$.*

Proof: The implication $\Longleftarrow$ is clear. Conversely, let $(N_i)_{i\in I}$ be an independent family of submodules of M. In order to prove that $(\overline{N_i})_{i\in I}$ is an independent family of elements of the lattice $\mathrm{Sat}_\tau(M)$, it is sufficient to assume that I is the finite set $\{1,\ldots,n\}$ for some $n\in\mathbb{N}$, $n\geqslant 2$, because the independence is a property of finitary character in any upper continuous lattice, as $\mathrm{Sat}_\tau(M)$ is. Denote by $\bigvee$ and $\bigwedge$ the join and meet, respectively, in the lattice $\mathrm{Sat}_\tau(M)$. Then, for each $1\leqslant k<n$, we have:

$$\Big(\bigvee_{1\leqslant i\leqslant k}\overline{N_i}\Big)\bigwedge\overline{N_{k+1}}=\Big(\overline{\sum_{1\leqslant i\leqslant k}N_i}\Big)\bigcap\overline{N_{k+1}}=\overline{\Big(\sum_{1\leqslant i\leqslant k}N_i\Big)\bigcap N_{k+1}}=\overline{0}=0.$$

This proves that $(\overline{N_i})_{1\leqslant i\leqslant n}$ is an independent family of $\mathrm{Sat}_\tau(M)$, as desired. □

Remarks 2.23. The results of Lemma 2.19, Corollary 2.21, and Lemma 2.22 may fail in the absence of the condition "$M\in\mathcal{F}$". To see that, let R be any ring, let $\chi=(\text{Mod-}R,\{0\})$ be the improper torsion theory on Mod-R, let M be any nonzero module, and let N be any submodule of M which is not essential in M. Then Lemma 2.19 fails in this case. An example of a proper torsion theory enjoying the same property is provided by Albu [1, Examples 1.16].

For the failure of Corollary 2.21, consider the same torsion theory χ and a module M which is not RC. Since $\mathrm{Sat}_\chi(M)=\{M\}$, M is vacuously χ-RC, but it is not RC.

Finally, for the failure of Lemma 2.22, let M be a (direct sum) decomposable module: $M=N_1+N_2$, $N_1\neq 0$, $N_2\neq 0$, $N_1\cap N_2=0$. Then $(N_i)_{i=1,2}$ is an independent family of submodules of M, but $\overline{N_1}=\overline{N_2}=M$ in $\mathrm{Sat}_\chi(M)=\{M\}$, where χ is the torsion theory considered above. □

Lemma 2.24. *The following statements are equivalent for a module $M_R\in\mathcal{F}$.*

(1) *M is τ-coirreducible* (*resp. τ-completely coirreducible*).

(2) *$E_R(M)$ is an indecomposable module* (*resp. $E_R(M)\simeq E_R(C)$ for some τ-cocritical module C*).

(3) *$E_R(M)\simeq E_R(R/I)$ where I is an irreducible* (*resp. τ-critical*) *right ideal of R.*

Proof: For τ-coirreducibles use Corollary 2.10 and a well known characterization of coirreducible modules via injective hulls, and for τ-completely coirreducibles use Albu, Iosif, and Teply [2, Proposition 2.2]. □

If we specialize Theorem 1.16 characterizing RC and RCC lattices L for $L=\mathrm{Sat}_\tau(M)$, we obtain at once the following characterizations of τ-RC and τ-RCC modules M.

Theorem 2.25. *The following statements are equivalent for a module $M_R \notin \mathcal{T}$.*

(1) *M is τ-RC (resp. τ-RCC).*

(2) *There exists in the lattice $\mathrm{Sat}_\tau(M)$ an independent family $(N_i)_{i\in I}$ of τ-coirreducible (resp. τ-completely coirreducible) submodules N_i of M, $i \in I$, such that $\overline{\sum_{i\in I} N_i}$ is an essential element in the lattice $\mathrm{Sat}_\tau(M)$.*

(3) *For every $\tau(M) \neq N \in \mathrm{Sat}_\tau(M)$ there exists a nonempty set I_N such that $\tau(M)$ can be written as an irredundant intersection*

$$\tau(M) = \bigcap_{i\in I_N} N_i$$

of τ-irreducible (resp. τ-completely irreducible) submodules N_i in N, $N_i \in \mathrm{Sat}_\tau(M)$, $i \in I_N$.

Moreover, the equivalent conditions $(1) - (3)$ for a τ-RCC module can be reformulated as follows:

(1)' *Any submodule N of M, $\tau(M) \neq N \in \mathrm{Sat}_\tau(M)$ contains a τ-simple submodule in $\mathrm{Sat}_\tau(M)$.*

(2)' *The τ-socle $\mathrm{Soc}_\tau(M)$ of M is an essential element in the lattice $\mathrm{Sat}_\tau(M)$.*

(3)' *For every $\tau(M) \neq N \in \mathrm{Sat}_\tau(M)$ there exists a nonempty set I_N such that $\tau(M)$ can be written as an irredundant intersection*

$$\tau(M) = \bigcap_{i\in I_N} N_i$$

of τ-maximal submodules N_i of N, $i \in I_N$, in other words, the τ-Jacobson radical $J_\tau(N)$ of N is $\tau(M)$ and an irredundant intersection of τ-maximal submodules of N. □

In case the given module M_R is τ-torsionfree, then characterizations in Theorem 2.25 have the following more simple form, that involve essentiality and independence in the very familiar lattice $\mathcal{L}(M)$ of all submodules of M instead of the ones in the lattice $\mathrm{Sat}_\tau(M)$ of all τ-closed submodules of M. In this way, one can add, as in the original Fort [8, Théoréme 3], a new characterization in terms of injective hulls.

Theorem 2.26. *The following statements are equivalent for a nonzero module $M_R \in \mathcal{F}$.*

(1) *M is τ-RC (resp. τ-RCC).*

(2) *There exists a sum of an independent family of coirreducible (resp. τ-completely coirreducible) submodules of M that is essential in M.*

(3) *The injective hull* $E_R(M)$ *of* M *is an essential extension of a direct sum of (indecomposable) injective modules of type* $E_R(C)$ *where* C *are coirreducible (resp.* τ*-completely coirreducible) modules.*

(4) *For every* $0 \neq N \in \mathrm{Sat}_\tau(M)$ *there exists a nonempty set* I_N *such that* 0 *can be written as an irredundant intersection*

$$0 = \bigcap_{i \in I_N} N_i$$

of τ*-irreducible (resp.* τ*-completely irreducible) submodules* N_i *in* N*,* $N_i \in \mathrm{Sat}_\tau(M)$*,* $i \in I_N$*.*

Moreover, the equivalent conditions $(1) - (4)$ *for a* τ-RCC *module can be reformulated as follows:*

(1)' *Any nonzero submodule of* M *contains a* τ*-cocritical submodule.*

(2)' *The* τ*-socle* $\mathrm{Soc}_\tau(M)$ *of* M *is essential in* M*.*

(3)' *The injective hull* $E_R(M)$ *of* M *is an essential extension of a direct sum of indecomposable injective modules of type* $E_R(C)$ *where* C *are* τ*-cocritical modules.*

(4)' *For every* $0 \neq N \in \mathrm{Sat}_\tau(M)$ *there exists a nonempty set* I_N *such that* 0 *can be written as an irredundant intersection*

$$0 = \bigcap_{i \in I_N} N_i$$

of τ*-maximal submodules* N_i *of* N*,* $i \in I_N$*, in other words, the* τ*-Jacobson radical* $J_\tau(N)$ *of* N *is zero and an irredundant intersection of* τ*-maximal submodules of* N*.*

Proof: Apply Lemma 2.19 (3), Corollary 2.21, Lemma 2.22, Lemma 2.24, and Theorem 2.25. □

Since M is τ-RC (resp. τ-RCC) if and only if $M/\tau(M)$ is so, we can of course formulate Theorem 2.25 in terms of essentiality and independence in the lattice $\mathcal{L}(M/\tau(M))$ instead of the ones in the lattice $\mathrm{Sat}_\tau(M)$. For instance, the condition (2) can be expressed as: (2)" *There exists an independent family* $(X_i)_{i\in I}$ *of* τ*-coirreducible (resp.* τ*-completely coirreducible) submodules* X_i *of* $M/\tau(M)$*,* $i \in I$*, such that* $\bigoplus_{i\in I} X_i$ *is an essential submodule of* $M/\tau(M)$*.*

Proposition 2.27. *A module* M_R *is* τ*-semi-Artinian if and only if every* $N \in \mathrm{Sat}_\tau(M)$ *has an* ICID *in* $\mathrm{Sat}_\tau(M)$*.*

Proof: Apply Proposition 1.23 to the lattice $L = \mathrm{Sat}_\tau(M)$. □

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