Some equations over generalized quaternion and octonion division algebras

by

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Abstract

It is known that any polynomial of degree $n$ with coefficients in a field $K$ has at most $n$ roots in $K$. If the coefficients are in $\mathbb{H}$ (the quaternion algebra), the situation is different. For $\mathbb{H}$ over the real field, there is a kind of a fundamental theorem of algebra: If a polynomial has only one term of the greatest degree then it has at least one root in $\mathbb{H}$. A similar theorem is also true for the octonions.

In this paper we try to solve, in general or in particular cases, some quadratic and linear equations with two different terms of greatest degree and the coefficients in the generalized division quaternion and octonion algebras $\mathbb{H}(\alpha, \beta)$ and $\mathbb{O}(\alpha, \beta, \gamma)$ over an arbitrary field $K$, $\text{char}K \neq 2$.

Key Words: Quaternion algebra; Division algebra; Octonion algebra.

2000 Mathematics Subject Classification: Primary 17D05, Secondary 17D99.

1 Introduction

In the following $K$ is a field with $\text{char}K \neq 2$. We recall some known facts on generalized quaternion and octonion algebras.

Let $\mathbb{H}(\alpha, \beta)$ be a generalized quaternion $K$–algebra with basis $\{1, e_1, e_2, e_3\}$ and the multiplication given by

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and $\mathbb{O}(\alpha, \beta, \gamma)$ be a generalized octonion $K$–algebra, with basis $\{f_1, ..., f_7\}$ and the multiplication given by
Cristina Flaut and Mirela Ștefănescu

The algebra \( \mathbb{H} (\alpha, \beta) \) is associative but non-commutative and the algebra \( \mathbb{O} (\alpha, \beta, \gamma) \) is non-commutative, non-associative but it is alternative (i.e. \( x^2 y = x (xy) \)) and \( y x^2 = (yx) x, \forall x, y \in \mathbb{O} (\alpha, \beta, \gamma) \), flexible (i.e. \( x (yx) = (xy) x, \forall x, y \in \mathbb{O} (\alpha, \beta, \gamma) \)), power-associative (i.e. for each \( x \in \mathbb{O} (\alpha, \beta, \gamma) \) the subalgebra generated by \( x \) is an associative algebra).

For \( a \in \mathbb{H} (\alpha, \beta) \), \( a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \), the element \( \bar{a} = a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3 \) is called the conjugate of the element \( a \). If \( a \in \mathbb{O} (\alpha, \beta, \gamma) \), \( a = a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7 \) then \( \bar{a} = a_0 - a_1 f_1 - a_2 f_2 - a_3 f_3 - a_4 f_4 - a_5 f_5 - a_6 f_6 - a_7 f_7 \) is called the conjugate of the element \( a \). Let \( A = \mathbb{H} (\alpha, \beta) \) or \( A = \mathbb{O} (\alpha, \beta, \gamma) \) and \( a \in A \). We have that \( t(a) \cdot 1 = a + \pi \in K, n(a) = a + \pi \in K \) and these are called the trace, respectively, the norm of the element \( a \in A \). It follows that \( (a + \pi) a = a^2 + \pi a = a^2 + n(a) \cdot 1 \) and \( a^2 - t(a) a + n(a) = 0, \forall a \in A \), therefore the generalized quaternion and octonion algebras are quadratic.

The subset \( A_0 = \{ x \in A \mid t(x) = 0 \} \) of \( A \) is a subspace of the algebra \( A \). It is obvious that \( A = K \cdot 1 + A_0 \), therefore each element \( x \in A \) has the form \( x = \lambda \cdot 1 + x_0 \), with \( \lambda \in K \) and \( x_0 \in A_0 \).

If for \( x \in A \), the relation \( n(x) = 0 \) implies \( x = 0 \), the algebra \( A \) is called a division algebra.

For the algebra \( \mathbb{H} (\alpha, \beta) \), if \( x = a + b e_1 + c e_2 + d e_3 \in \mathbb{H} (\alpha, \beta) \) such that \( n(x) = 0 \), we obtain \( a^2 + \alpha b^2 + \beta c^2 + \alpha \beta d^2 = 0 \) or equivalently

\[
a^2 + \alpha b^2 = -\beta c^2 - \alpha \beta d^2 = -\beta (c^2 + \alpha d^2).
\]

Therefore

\[
\beta = -\frac{n(a + b e_1)}{n(c + d e_1)} = -n \left( \frac{a + b e_1}{c + d e_1} \right) = -n (\varepsilon + \delta e_1) = -\varepsilon^2 - \alpha \delta^2,
\]

where

\[
\varepsilon + \delta e_1 = \frac{a + b e_1}{c + d e_1},
\]

or

\[
n(z) = -\beta,
\]

with

\[
z = \varepsilon + \delta e_1 \in \mathbb{K}(\alpha).
\]
Some equations over generalized quaternion and octonion division algebras

Then $\mathbb{H}(\alpha, \beta)$ is a division algebra if and only if $\mathbb{K}(\alpha)$ is a separable and quadratic extension of the field $K$ and the equation $n(x) = -\beta$ does not have solutions in $\mathbb{K}(\alpha)$.

For the algebra $\mathbb{D}(\alpha, \beta, \gamma)$, if $a \in \mathbb{D}(\alpha, \beta, \gamma)$ such that $n(a) = 0$, we obtain

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 + \alpha a_1 a_2 + \beta a_2 a_3 + \gamma a_3 a_0 + \alpha \beta a_3 a_1 + \alpha \gamma a_1 a_3 + \beta \gamma a_2 a_0 + \alpha \beta \gamma a_0 a_1 = 0$$

and therefore $a_0^2 + a_1^2 + a_2^2 + a_3^2 + \alpha a_1 a_2 + \beta a_2 a_3 + \gamma a_3 a_0 = 0$.

It results

$$\gamma = -\frac{n(a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3)}{n(a_4 + a_5 f_1 + a_6 f_2 + a_7 f_3)} = -n(b_0 + b_1 f_1 + b_2 f_2 + b_3 f_3) = -b_0^2 - \alpha b_1^2 - \beta b_2^2 - \alpha \beta b_3^2,$$

where

$$b_0 + b_1 f_1 + b_2 f_2 + b_3 f_3 = \frac{a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3}{a_4 f_1 + a_5 f_2 + a_6 f_3 + a_7 f_4} \in \mathbb{H}(\alpha, \beta).$$

Then $\mathbb{D}(\alpha, \beta)$ is a division algebra if and only if $\mathbb{H}(\alpha, \beta)$ is a division algebra and the equation $n(x) = -\gamma$ does not have solutions in $\mathbb{H}(\alpha, \beta)$.

**Theorem (Artin).** The subalgebra generated by any two elements of the alternative algebra $A$ is an associative algebra.

In an alternative algebra $A$, the identity $(xy)(zx) = x(yz)x, \forall x, y, z \in A$, called the Moufang identity, holds.

**Proposition 1.1.** Let $A$ be a unitary power-associative division algebra. Then, each subalgebra $B$ of $A$ is a unitary algebra and $K \subseteq B$.

**Proof:** Let $B$ be a subalgebra of the algebra $A, b \in B, b \neq 0$. Let $B(b)$ the subalgebra of $B$ generated by $b$. From power-associativity, it results that the algebra $B(b)$ is an associative algebra. But $B(b)$ is a division algebra, therefore is a unitary algebra. We have that $B$ is a unitary algebra. $\square$

**Remark 1.2.** Since the generalized quaternion algebra $\mathbb{H}(\alpha, \beta)$ and generalized octonion algebra $\mathbb{D}(\alpha, \beta)$ are unitary and power-associative algebras, and, in this paper, they are division algebras, we have that $K \subseteq B$, for each subalgebra $B$ of $\mathbb{H}(\alpha, \beta)$ or $\mathbb{D}(\alpha, \beta)$. Then, if $a$ is a nonzero element of a subalgebra $B$ of $\mathbb{H}(\alpha, \beta)$ (or of $\mathbb{D}(\alpha, \beta)$), then its conjugate $\bar{a}$ belongs to $B$. Indeed, using the above notation, if $a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ (or $a = a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7$), then $w = a - a_0 \in B$(since $a_0 \in K$), so that $\bar{a} = a_0 - w \in B$. If $a \neq 0$, since $a^{-1} = \bar{a}(n(a))^{-1}$, it results that $a^{-1} \in B$.

In the following, we try to solve the following equations:

$$xa = bx + c, a, b, c \in A,$$

(1)
\[ x^2a = bx^2 + c, a, b, c \in A, \quad (2) \]
\[ xax = x^2b + c, \ a, b, c \in A, \quad (3) \]
\[ xax = xb + cx + d, \ a, b, c, d \in A, \quad (4) \]

where \( A \) is either the generalized quaternion division algebra or the generalized octonion division algebra.

2 Equations with coefficients in the generalized quaternion division algebra \( \mathbb{H}(\alpha, \beta) \)

The linear maps \( \lambda, \rho : \mathbb{H}(\alpha, \beta) \to \text{End}_K(\mathbb{H}(\alpha, \beta)) \), where \( \lambda(a) : \mathbb{H}(\alpha, \beta) \to \mathbb{H}(\alpha, \beta), \lambda(a)(x) = ax, a \in \mathbb{H}(\alpha, \beta) \), and \( \rho(a) : \mathbb{H}(\alpha, \beta) \to \mathbb{H}(\alpha, \beta), \rho(a)(x) = xa, a \in \mathbb{H}(\alpha, \beta) \), are called the left representation and the right representation of the algebra \( \mathbb{H}(\alpha, \beta) \).

The map \( \lambda : \mathbb{H}(\alpha, \beta) \to M_4(K), \lambda(a) = \begin{pmatrix}
a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\
a_1 & a_0 & -\beta a_3 & \beta a_2 \\
a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\
a_3 & -a_2 & a_1 & a_0
\end{pmatrix}, \)

where \( a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H}(\alpha, \beta) \), is an isomorphism from \( \mathbb{H}(\alpha, \beta) \) to the algebra of matrices of the form

\[
\begin{pmatrix}
a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\
a_1 & a_0 & -\beta a_3 & \beta a_2 \\
a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\
a_3 & -a_2 & a_1 & a_0
\end{pmatrix} \quad | \quad a_0, a_1, a_2, a_3 \in K
\]

Definition 2.1. \( \lambda(a) \) is called the left matrix representation of the element \( a \in \mathbb{H}(\alpha, \beta) \).

In the same manner, for the element \( a \in \mathbb{H}(\alpha, \beta), a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \), we define the right matrix representation

\[ \rho : \mathbb{H}(\alpha, \beta) \to M_4(K), \rho(a) = \begin{pmatrix}
a_0 & -\alpha a_1 & -\beta a_2 & -\alpha \beta a_3 \\
a_1 & a_0 & \beta a_3 & -\beta a_2 \\
a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\
a_3 & a_2 & -a_1 & a_0
\end{pmatrix}. \]

Proposition 2.2. ([8, 2]) Let \( x, y \in \mathbb{H}(\alpha, \beta) \) and \( r \in K \). Then the following statements are true:
\begin{itemize}
  \item[i)] \( x = y \iff \lambda(x) = \lambda(y) \);
  \item[ii)] \( x = y \iff \rho(x) = \rho(y) \);
\end{itemize}
Some equations over generalized quaternion and octonion division algebras

\[ iii) \lambda(x + y) = \lambda(x) + \lambda(y), \ \lambda(xy) = \lambda(x)\lambda(y), \ \lambda(rx) = r\lambda(x), \ \lambda(1) = I_4, \ r \in K; \]

\[ iv) \rho(x + y) = \rho(x) + \rho(y), \ \rho(xy) = \rho(x)\rho(y), \ \rho(rx) = r\rho(x), \ \rho(1) = I_4, \ r \in K; \]

\[ v) \lambda(x^{-1}) = (\lambda(x))^{-1}, \rho(x^{-1}) = (\rho(x))^{-1}, \text{ for } x \neq 0. \]

**Definition 2.3.** Let \( x \in \mathbb{H}(\alpha, \beta), \ x = a_0 + a_1e_1 + a_2e_2 + a_3e_3. \) The element \( \overline{\alpha} = (a_0, a_1, a_2, a_3)^	op \in M_{1 \times 4}(K) \) is called the vector representation of the element \( x. \)

**Proposition 2.4.** ([8]) With above notation, for each \( a, b, x \in \mathbb{H}(\alpha, \beta), \) the following statements are true:

1. \( \overline{ax} = \lambda(a) \overline{x}; \)
2. \( \overline{xb} = \rho(b) \overline{x}; \)
3. \( \overline{axb} = \lambda(a) \rho(b) \overline{x} = \rho(b)\lambda(a) \overline{x}; \)
4. \( \rho(b) \lambda(a) = \lambda(a) \rho(b). \)

**Proposition 2.5.** ([8, 2]) Let \( a, b \in \mathbb{H}(\alpha, \beta), a \neq 0, b \neq 0. \)

1. The linear equation \( ax = xb \) has non-zero solution in \( \mathbb{H}(\alpha, \beta) \) if and only if \( t(a) = t(b) \) and \( n(a - a_0) = n(b - b_0), \) where \( a = a_0 + a_1e_1 + a_2e_2 + a_3e_3, b = b_0 + b_1e_1 + b_2e_2 + b_3e_3. \)

2. If \( b \neq a, a, b \notin K, \) then the solutions of the equation \( ax = xb \) are in \( A(a, b) \) (the subalgebra generated by the elements \( a, b \in \mathbb{H}(\alpha, \beta) \) and have the form \( x = \lambda_1(a - a_0 + b - b_0) + \lambda_2(n(a - a_0) - (a - a_0)(b - b_0)), \) where \( \lambda_1, \lambda_2 \in K \) are arbitrary.

3. If \( b = a, \) then the general solution of the equation \( ax = xb \) is \( x = x_1e_1 + x_2e_2 + x_3e_3, \) where \( x_1, x_2, x_3 \in K \) and \( a_0x_1 + \beta a_2x_2 + \alpha a_3x_3 = 0. \)

**Remark 2.6.** ([9]) The general solution of the equation \( ax = xb \) could be written equivalently under the form \( x = aq - qb \) with an arbitrary \( q \in \mathbb{H}(\alpha, \beta). \)

### 2.1 The equation \( xa = bx + c \)

Now, we analyse the equation (1) where \( a \neq b, a = a_0 + a_1e_1 + a_2e_2 + a_3e_3, b = b_0 + b_1e_1 + b_2e_2 + b_3e_3. \) Using the left, the right and the vector representations, the equation (1) becomes \( (\lambda(a) - \rho(b))\overline{x} = \overline{c}. \)

**Case 1.** \( ya \neq by \) for all \( y \in \mathbb{H}(\alpha, \beta), \ y \neq 0. \) In this case, the equation \( ya = by \) has only the zero solution and, from Proposition 2.5, we have that \( \det (\lambda(a) - \rho(b)) \neq 0. \) Then the equation (1) has the unique solution \( \overline{x} = (\lambda(a) - \rho(b))^{-1}\overline{c}. \)

**Case 2.** If there are \( y \in \mathbb{H}(\alpha, \beta), y \neq 0, \) such that \( ya = by, \) then the equation \( ya = by \) has non-zero solutions. Therefore \( \det (\lambda(a) - \rho(b)) = 0 \) and this is equivalent with the conditions:

\[ t(a) = t(b) \text{ and } n(a - a_0) = n(b - b_0). \]
Lemma 2.7. The equation \((5)\) has solutions if and only if \(ad = d\). The general solution of the equation \((5)\) is
\[
z = 2t(a - a_0) - q, \tau \in K, \ q \in \mathbb{H}(\alpha, \beta).
\] (6)

Proof: Let us assume that the equation \((5)\) has a solution \(z_0\). If we multiply the equation \((5)\) by \(a\), to the right, we obtain \(az_0a = az_0a + d\). Since \(\bar{a} = t(a) - a\) and \(ad = n(a)\), we get \(n(a) z_0 = t(a) a z_0 = az_0a + d\).

If we multiply the equation \((5)\) by \(a\), to the left, we obtain \(az_0a = a^2 z_0 + ad\). But \(a^2 = t(a) a - n(a)\), therefore \(az_0a = t(a) az_0 - n(a) z_0 + ad\). Thus \(ad = d\).

For the converse, if \(ad = d\), using Remark 2.6, we obtain that there is an element \(q \in \mathbb{H}(\alpha, \beta)\) such that \(d = aq - qa\), so that the equation \((5)\) becomes
\[
za = az + aq - qa, \ \text{therefore} \ (z + q)a = a(z + q)
\]
Using Proposition 2.5, the equation \((z + q)a = a(z + q)\) has the general solutions of the form
\[
z + q = 2\gamma_1(a - a_0), \ \text{where} \ \gamma_1 \in K.
\]
Therefore
\[
z = 2\gamma_1(a - a_0) - q.
\]
\[\square\]

Proposition 2.8. With the above notation, the equation \(xa = bx + c\) has solutions if and only if \(bc = c\), and the general solution is
\[
x = 2\tau y_0(a - a_0) - y_0a, \tau \in K, \ y_0 \in \mathbb{H}(\alpha, \beta),
\] (7)
where \(b = y_0a y_0^{-1}, y_0 \in \mathbb{H}(\alpha, \beta), \ y_0 \neq 0\).

2.2 The equation \(x^2a = bx^2 + c\)

Proposition 2.9. ([2]) Let \(a \in \mathbb{H}(\alpha, \beta) \setminus K\) be such that there are \(r, s \in K\) with the properties \(n(a) = r^4\) and \(n(r^2 + a) = s^2\). Then the quadratic equation \(x^2a = bx^2 + c\) has two solutions of the form \(x = \pm \frac{r(r^2 + a)}{s} \).

If we denote \(x^2 = y\), then the equation \((2)\) has the form \(ya = by + c\), that is the equation \((1)\). If \(bc = c\), then the equation \((2)\) has solutions if we can find a solution of the equation \((1)\), denoted \(y_1\), such that there are \(r, s \in K\) with the properties \(n(y_1) = r^4\) and \(n(r^2 + y_1) = s^2\).

Proposition 2.10. If \(bc = c\) and there are \(r, s \in K\) and \(y_1 \in \mathbb{H}(\alpha, \beta), \ y_1\) is a solution of the equation \((1)\) with the properties \(n(y_1) = r^4\) and \(n(r^2 + y_1) = s^2\), then the equation \((2)\) has two solutions of the form \(x = \pm \frac{r(r^2 + y_1)}{s}\).
2.3 The equation $xax = x^2b + c$

We find sufficient conditions for the existence of solutions in the case $a \neq b$, $ab = ba, ac = ca, bc = cb$.

Case 1. If $a \in K$, then the equation becomes $x^2 = d$, where $d = (a - b)^{-1}c$ and we apply Proposition 2.9 if $d \in \mathbb{H}(\alpha, \beta) \setminus K$. If $d \in K$, we find quickly a solution.

Case 2. If $a \in \mathbb{H}(\alpha, \beta) \setminus K$ and $b \in K$, then the equation becomes $(x^2)^2 = g$, where $d = a - b$ and $g = c(a - b)$. If $g \in \mathbb{H}(\alpha, \beta) \setminus K$, we apply Proposition 2.9. If $g \in K$, we find quickly a solution.

Case 3. Let $a \in \mathbb{H}(\alpha, \beta) \setminus K$ and $b \in \mathbb{H}(\alpha, \beta) \setminus K$. Supposing that the equation (3) has a solution $x_0$, if we multiply the equation to the right by $a$, we obtain $x_0ax_0 = x_0^2ba + ca$. Denoting $y_0 = x_0a$, it results $y_0^2 = x_0^2ba + ca$. Since $y_0^2 = t(y_0)y_0 - n(y_0)$ and $x_0^2 = t(x_0)x_0 - n(x_0)$, we have $t(y_0)y_0 - n(y_0) = (t(x_0)x_0 - n(x_0))ba + ca$. Therefore $t(y_0)y_0 - n(y_0) = t(x_0)x_0ba - n(x_0)ba + ca$, hence $x_0(t(y_0) - t(x_0)b)a = n(y_0) - n(x_0)ba + ca$. The element $w = (t(y_0) - t(x_0)b)a \in \mathcal{A}(a, b, c)$ is nonzero because $t(y_0) \in K$ and $t(x_0) \in K$, while $b \notin K$. If $v = n(y_0) - n(x_0)ba + ca \in \mathcal{A}(a, b, c)$, we apply Remark 1.2 and it results that $x_0 = w^{-1}v \in \mathcal{A}(a, b, c)$, the algebra generated by the elements $a, b, c$. From the hypothesis, the algebra $\mathcal{A}(a, b, c)$ is commutative and then $x_0$ commutes with the elements $a, b, c$.

The equation (3) becomes $x_0^2(a - b) = c$. Denoting $d = c(a - b)^{-1}$, we have $x_0^2 = d$.

Proposition 2.11. If $a \neq b$, $ab = ba, ac = ca, bc = cb$ and $d = c(a - b)^{-1} \in \mathbb{H}(\alpha, \beta) \setminus K$ such that there are $r, s \in K$ with the properties $n(d) = r^4$ and $n(r^2 + d) = s^2$, then the equation (1.4) has two solutions $x = \pm \frac{r(r^2 + d)}{s}$.

2.4 The equation $xax = xb + cx + d$

Let us consider now the equation $xax = xb + cx + d$.

Proposition 2.12. ([2]) If there are $b, c \in \mathbb{H}(\alpha, \beta) \setminus K$ and $r \in K$ such that $bc = cb, b^2 - c \neq 0$, $n\left(\frac{b^2}{4} - c\right) = r^4$ and $n\left(r^2 + \frac{b^2}{4} - c\right) = s^2$, $s \neq 0$, then the equation $x^2 + bx + c = 0$ has a solution in $\mathbb{H}(\alpha, \beta)$.

Proof: We claim that any solution in $\mathbb{H}(\alpha, \beta)$ to equation $x^2 + bx + c = 0$ belongs to $\mathcal{A}(b, c)$. Then the equation $x^2 + bx + c = 0$ becomes $(x + \frac{b}{2})^2 - \frac{b^2}{4} + c = 0$ and we apply Proposition 2.9. To prove the claim, let $x_0 \in \mathbb{H}(\alpha, \beta)$ be a solution of the equation $x^2 + bx + c = 0$. Since $x_0^2 = t(x_0)x_0 - n(x_0)$ and $x_0^2 + bx_0 + c = 0$, it results $t(x_0)x_0 - n(x_0) + bx_0 + c = 0$. Therefore $(t(x_0) + b)x_0 = -c + n(x_0).$ Since $t(x_0) + b \neq 0$, $(t(x_0), n(x_0) \in K, 1 \in \mathcal{A}(b, c)$, we have $t(x_0) + b \in \mathcal{A}(b, c)$ and $-c + n(x_0) \in \mathcal{A}(b, c).$ Then $x_0 \in \mathcal{A}(b, c).$ Using $bc = cb$, we obtain that $\mathcal{A}(b, c)$ is commutative, so that $x_0$ commutes with each element of $\mathcal{A}(b, c).$
If we choose \( q, p \in \mathbb{H}(\alpha, \beta) \) such that \( qap = 1 \), then we make the following change of variable \( x = pyq \). Replacing it in the equation (4) we obtain \( (pyq)a(pyq) = (pyq)b + c(pyq) + d \). It results \( y^2 = yb_1 + c_1y + d_1 \), where \( b_1 = qbq^{-1}, c_1 = p^{-1}cp, d_1 = p^{-1}dq^{-1} \). If we denote \( y = z + b_1 \), we obtain \( z^2 = c_2z + d_2 \), where \( c_2 = c_1 - b_1 \) and \( d_2 = c_1b_1 + d_1 \). We apply now Proposition 2.12. for \( b = -c_2 \) and \( c = -d_2 \).

With the above notation, if there are \( r, s \in K \setminus \{0\} \) such that \( c_2d_2 = d_2c_2, \ n(\frac{c_2}{4} + d_2) = r^4, \ n(r^2 + \frac{c_2}{4} + d_2) = s^2 \), we get that the equation (4) has solutions of the form

\[
x_0 = -\frac{1}{2} \left( p^{-1}cp - qbq^{-1} \right) \pm \frac{r(r^2 + d_3)}{s},
\]

where

\[
d_3 = \frac{(p^{-1}cp - qbq^{-1})^2}{4} + p^{-1}cpqbq^{-1} - p^{-1}dq^{-1}.
\]

### 3 Equations with coefficients in the generalized division octonion algebra \( \mathbb{O}(\alpha, \beta, \gamma) \)

We now use the matrix representations of the generalized quaternions to introduce the matrix representations for the generalized octonions.

Using the multiplication tables for the generalized quaternion \( K \)-algebra and generalized octonion \( K \)-algebra, an element \( a \in \mathbb{O}(\alpha, \beta, \gamma) \) could be written under the form \( a = a' + a''f_4 \in \mathbb{O}(\alpha, \beta, \gamma), \) where \( a' = a_0 + a_1f_1 + a_2f_2 + a_3f_3, a'' = a_4 + a_5f_1 + a_6f_2 + a_7f_3 \in \mathbb{H}(\alpha, \beta). \) Then the matrix

\[
\Lambda (a) = \begin{pmatrix}
\lambda(a') & -\gamma \rho(a'') \\
\lambda(a'') & \rho(a')
\end{pmatrix}
\]

is called the left matrix representation of the element \( a \in \mathbb{O}(\alpha, \beta, \gamma), \) where

\[
M_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\in \mathcal{M}_4(K)
\]

and \( \lambda \) and \( \rho \) are the matrix representations for the quaternions.

Analogously, we define the right matrix representation

\[
\Delta (a) = \begin{pmatrix}
\rho(a') & -\gamma \lambda(a'') \\
\lambda(a') & \rho(a'')
\end{pmatrix} = A_1 \Lambda^t(a) A_2,
\]

where \( A_1, A_2 \in \mathcal{M}_8(K) \) are

\[
A_1 = \begin{pmatrix}
-\gamma D_1 & 0 \\
0 & C_1
\end{pmatrix},
\]

and

\[
A_2 = \begin{pmatrix}
\alpha & 0 \\
0 & \alpha
\end{pmatrix}.
\]

The matrices \( A_1, A_2, \Lambda, \rho \) and \( \lambda \) satisfy the relations

\[
\lambda(a') = \langle a', \rho \rangle, \quad \rho(a') = \langle a', \lambda \rangle, \quad \lambda(a'') = \langle a'', \rho \rangle, \quad \rho(a'') = \langle a'', \lambda \rangle.
\]
Some equations over generalized quaternion and octonion division algebras

\[ A_2 = \begin{pmatrix} -\gamma^{-1}D_2 & 0 \\ 0 & C_2 \end{pmatrix}, \text{ with } D_1, D_2, C_1, C_2 \in \mathcal{M}_4(K), \]

\[ C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 & \alpha^{-1}\beta^{-1} \end{pmatrix}, \]

\[ C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{pmatrix}. \]

\[ D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha^{-1} & 0 & 0 \\ 0 & 0 & -\beta^{-1} & 0 \\ 0 & 0 & 0 & -\alpha^{-1}\beta^{-1} \end{pmatrix}, \]

\[ D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\alpha\beta \end{pmatrix}. \]

and \( A_1A_2 = A_2A_1 = I_8 \)

**Proposition 3.1.** ([8]) Let \( x = x_0 + x_1f_1 + x_2f_2 + x_3f_3 + x_4f_4 + x_5f_5 + x_6f_6 + x_7f_7, a = a_0 + a_1f_1 + a_2f_2 + a_3f_3 + a_4f_4 + a_5f_5 + a_6f_6 + a_7f_7 \in O(\alpha, \beta, \gamma) \).

Let \( \bar{x} = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)^t \) be the vector representation of the element \( x \). Then we have \( \bar{a}\bar{x} = \Lambda(a)\bar{x} \) and \( \bar{x}a = \Delta(a)\bar{x}, \forall a, x \in O(\alpha, \beta, \gamma) \).

**Proposition 3.2.** ([8]) Let \( x, y \in O(\alpha, \beta, \gamma) \) and \( a, b \in K \). Then we have: i) \( x = y \iff \Lambda(x) = \Lambda(y) \); ii) \( x = y \iff \Delta(x) = \Delta(y) \); iii) \( \Lambda(x+y) = \Lambda(x) + \Lambda(y) \); iv) \( \Lambda(mx) = m\Lambda(x) \); v) \( \Delta(x+y) = \Delta(x) + \Delta(y) \); vi) \( \Delta(mx) = m\Delta(x) \); vii) \( \Lambda(x^{-1}) = \Lambda^{-1}(x) \); viii) \( \Delta(x^{-1}) = \Delta^{-1}(x) \).

Since the algebra \( O(\alpha, \beta, \gamma) \) is non-associative, the equalities \( \Lambda(xy) = \Lambda(x)\Lambda(y), \Delta(xy) = \Delta(x)\Delta(y) \) do not hold always.

**Proposition 3.3.** ([8, 2l]) Let \( a, b \in O(\alpha, \beta, \gamma), a \neq 0, b \neq 0 \). The linear equation \( ax = xb \) has a non-zero solution in \( O(\alpha, \beta, \gamma) \) if and only if \( t(a) = t(b) \) and \( n(a-b) = n(b-b) \), where \( a = a_0 + a_1f_1 + a_2f_2 + a_3f_3 + a_4f_4 + a_5f_5 + a_6f_6 + a_7f_7, b = b_0 + b_1f_1 + b_2f_2 + b_3f_3 + b_4f_4 + b_5f_5 + b_6f_6 + b_7f_7 \).

i) If \( b \neq a \), then the solutions of the equation \( ax = xb \) are in \( A(a, b) \) (the subalgebra generated by the elements \( a, b \in O(\alpha, \beta, \gamma) \)) and have the form \( x = \lambda_1(a-a_0) + \lambda_2(n(a-a_0) - (a-a_0)(b-b_0)) \), where \( \lambda_1, \lambda_2 \in K \) are arbitrary.

ii) If \( b = a \), then the general solution of the equation \( ax = xb \) is \( x = x_1e_1 + x_2e_2 + x_3e_3 \), where \( x_1, x_2, x_3 \in K \) and \( a\alpha_1x_1 + \beta a_2x_2 + \alpha\beta a_3x_3 + \gamma a_4x_4 + \alpha\gamma a_5x_5 + \beta\gamma a_6x_6 + \alpha\beta\gamma a_7x_7 = 0 \).

**Remark 3.4.** ([9]) The general solution of the equation \( ax = xb \) could be written equivalently under the form \( x = aq - qb \) with \( q \in O(\alpha, \beta, \gamma) \), \( q \) arbitrary. Indeed, for \( q = \frac{a+v}{2n(v)} \), where \( a = a_0 + \nu \), \( a_0 \in K \) and \( z \) is a solution of the equation \( az = xz \). If \( b \neq a \), then \( q \in A(a, b) \). If \( b = a \), then \( q \in O(\alpha, \beta, \gamma) \) is arbitrary.
3.1 The equation $xa = bx + c$

Next, we try to solve the equation (1) when $a \neq b$. Using the left, the right and the vectorial representations, the equation (1) becomes

$$(\Lambda(a) - \Delta(b))\hat{\alpha} = \hat{\tau}.$$  

**Case 1.** $ya \neq by$ for all $y \in \mathbb{O}(\alpha, \beta, \gamma)$, $y \neq 0$. In this case, the equation $ya = by$ has only the zero solution and, from Proposition 3.3, we have that $\det(\Lambda(a) - \Delta(b)) \neq 0$. Then the equation (1) has the unique solution

$$\hat{x} = (\Lambda(a) - \Delta(b))^{-1}\hat{\tau}.$$

**Case 2.** If there are $y \in \mathbb{O}(\alpha, \beta, \gamma)$, $y \neq 0$, such that $ya = by$, then the equation $ya = by$ has non-zero solutions. Therefore $\det(\Lambda(a) - \Delta(b)) = 0$ and this is equivalent with the conditions: $t(a) = t(b)$ and $n(a - a_0) = n(b - b_0)$. It results that there is a non-zero element $y_0 \in \mathbb{O}(\alpha, \beta, \gamma)$ such that $b = y_0a^{-1}_0$. If $bc = ca$, it results that there is $q \in \mathbb{A}(a, b)$ such that $c = bq - qa$. Therefore, the equation (1) becomes $(x + q)a = b(x + q)$ and we have solutions. These solutions are in $\mathbb{A}(a, b)$. Supposing that the equation (1) has solutions $x \in \mathbb{A}(a, b)$, then the equation (1) becomes $xa = (y_0a^{-1}_0)x + c$. We remark that $c \in \mathbb{A}(a, b)$. We obtain $y_0^{-1}xa = ay_0^{-1}_0x + y_0^{-1}c$, and the equation

$$za = az + d,$$

where $z = y_0^{-1}x, d = y_0^{-1}c$.

**Proposition 3.5.** The equation (8) has solutions if and only if $ad = d\hat{a}$. In this case, the general solution of the equation (8) is

$$z = 2\tau(a - a_0) - q, \tau \in K, q \in \mathbb{O}(\alpha, \beta, \gamma).$$

**Proof:** Supposing that the equation (8) has a solution $z_0 \in \mathbb{O}(\alpha, \beta, \gamma)$, if we multiply the equation (8) with $\hat{a}$ to the right, we obtain $(z_0a)\hat{a} = (a_0a)\hat{a} + d\hat{a}$. Since $\hat{a} = t(a) - a, a\hat{a} = n(a)$, by flexibility, we get $n(a)z_0 = t(a)az_0 - a_0z_0 + d\hat{a}$.

If we multiply the equation (8) with $a$ to the left, we obtain, using alternativity, $a_0z_0 = a^2z_0 + ad$. Since $a^2 = t(a)a - n(a) \cdot 1$, we have $a_0z_0 = t(a)az_0 - n(a)z_0 + ad$. Then $ad = d\hat{a}$.

Conversely, if $ad = d\hat{a}$, using Remark 3.4., we obtain that there is $q \in \mathbb{O}(\alpha, \beta, \gamma)$ such that $d = aq - qa$. Now, the equation (8) becomes $za = az + aq - qa$, therefore $(z + q)a = a(z + q).$ Using Proposition 3.3., the equation $(z + q)a = a(z + q)$ has the general solution of the form $z + q = 2\gamma_1(a - a_0)$, where $\gamma_1 \in K$. Therefore $z = 2\gamma_1(a - a_0) - g$. \hfill \Box

**Corollary 3.6.** With the above notation, the equation $xa = bx + c$ has solutions in $\mathbb{A}(a, b)$ if and only if $bc = ca$, and the general solution is

$$x = 2\gamma_1y_0(a - a_0) - y_0q,$$

where $\gamma_1 \in K, q \in \mathbb{A}(a, b), b = y_0a^{-1}_0, y_0 \in \mathbb{O}(\alpha, \beta, \gamma), y_0 \neq 0.$
3.2 The equation $x^2a = bx^2 + c$

**Proposition 3.7.** ([2]) Let $a \in \mathbb{O}(\alpha, \beta, \gamma) \setminus K$ be such that there are $r, s \in K$ with the properties $n(a) = r^4$ and $n(r^2 + \bar{a}) = s^2$, then the quadratic equation $x^2 = a$ has two solutions of the form $x = \pm \frac{r(r^2+\bar{a})}{s}$.

If we denote $x^2 = y$, then the equation (2) has the form $ya = by + c$, which is equation (1). If $bc = c\bar{a}$, then the equation (2) has solutions if we can find a solution of the equation (1), denoted $y_1$, such that there are $r, s \in K$ with the properties $n(y_1) = r^4$ and $n(r^2 + \bar{y}_1) = s^2$.

**Proposition 3.8.** If $bc = c\bar{a}$ and there are $r, s \in K$ and $y_1 \in \mathbb{O}(\alpha, \beta, \gamma)$, where $y_1$ is a solution of the equation (1) with the properties $n(y_1) = r^4$ and $n(r^2 + \bar{y}_1) = s^2$, then the equation (2) has two solutions of the form $x = \pm \frac{r(r^2+y_1)}{s}$.

3.3 The equation $xax = x^2b + c$

We find sufficient conditions for the existence of solutions in the case when $a \neq b$, $ab = ba$, $c \in \mathbb{A}(a, b)$.

Supposing that the equation (3) has a solution $x_0$, if we multiply the equation to the right by $a$, we obtain $(x_0ax_0)a = (x_0^2b)a + ca$. Denoting $y_0 = x_0a$ and using the Moufang identity, it results $y_0^2 = (x_0^2b)a + ca$. Since $y_0^2 = t(y_0)y_0 - n(y_0)$ and $x_0^2 = t(x_0)x_0 - n(x_0)$ we have

$$t(y_0)y_0 - n(y_0) = (t(x_0)x_0 - n(x_0))b + ca.$$

Therefore

$$t(y_0)x_0a - n(y_0) = (t(x_0)x_0b)a - n(x_0)ba + ca,$$

hence $(x_0(t(y_0) - t(x_0)b))a = n(y_0) - n(x_0)ba + ca$. It results that $x_0 \in \mathbb{A}(a, b, c)$, the algebra generated by the elements $a, b$. From the hypothesis, the algebra $\mathbb{A}(a, b)$ is commutative and then $x_0$ commutes and associates with the elements $a, b, c$.

The equation (3) becomes $x_0^2(a - b) = c$. Denoting $d = c(a - b)^{-1}$, we have $x_0^2 = d$.

**Proposition 3.9.** If $a \neq b$, $ab = ba$, $ac = ca$, $bc = cb$ and $d = c(a - b)^{-1} \in \mathbb{O}(\alpha, \beta, \gamma) \setminus K$ such that there are $r, s \in K$ with the properties $n(d) = r^4$ and $n(r^2 + d) = s^2$, then the equation (3) has two solutions $x = \pm \frac{r(r^2+d)}{s}$.
The equation $xax = xb + cx + d, b, c \in \mathcal{A}(a)$

**Proposition 3.10.** If there are $b, c \in \mathcal{O}(\alpha, \beta, \gamma) \setminus K, c \in \mathcal{A}(b)$ and $r \in K$ such that $\frac{c^2}{4} - c \neq 0, n\left(\frac{c^2}{4} - c\right) = r^2$ and $n\left(r^2 + \frac{c^2}{4} - \varepsilon\right) = s^2, s \neq 0$, then the equation $x^2 + bx + c = 0$ has solutions in $\mathcal{O}(\alpha, \beta, \gamma)$.

**Proof:** We claim that any solution in $\mathcal{O}(\alpha, \beta, \gamma)$ to the equation $x^2 + bx + c = 0$ commutes to each element of $\mathcal{A}(b, c)$. Then the equation $x^2 + bx + c = 0$ becomes $(x + \frac{b}{2})^2 - \frac{b^2}{4} + c = 0$ and we use Proposition 3.7. To prove the claim, let $x_0 \in \mathcal{O}(\alpha, \beta, \gamma)$ be a solution of the equation $x^2 + bx + c = 0$. Since $x_0^2 = t(x_0)x_0 - n(x_0)$ and $x_0^2 + bx_0 + c = 0$, we have $t(x_0)x_0 - n(x_0) + bx_0 + c = 0$, and therefore $(t(x_0) + b)x_0 = -c + n(x_0)$. From $t(x_0) + b \neq 0, t(x_0), n(x_0) \in K, 1 \in \mathcal{A}(b, c)$, it results $t(x_0) + b \in \mathcal{A}(b, c)$ and $-c + n(x_0) \in \mathcal{A}(b, c)$. Now, we apply Remark 1.2. It results that $x_0 \in \mathcal{A}(b, c)$. Since $c \in \mathcal{A}(b)$, we have that $\mathcal{A}(b, c) = \mathcal{A}(b)$ is a commutative algebra and $x_0$ commutes with each element of $\mathcal{A}(b, c)$. \hfill \Box

If we choose $q,p \in \mathcal{A}(a) \subset \mathcal{O}(\alpha, \beta, \gamma)$ such that $qap = 1$, then we make the following change of variables $x = pqq$ in the equation (4) and we obtain $(pq)\alpha(pqq) = (pq)\beta + c(pqq) + d$. It results $py^2q = pyqb + cpqq + d$. Since the subalgebra $\mathcal{A}(a, y)$ is an associative algebra, by Artin’s theorem, we have $y^2 = yb_1 + c_1y + d_1$, where $b_1 = qab^{-1}, c_1 = p^{-1}cp, d_1 = p^{-1}dq^{-1}$. If we denote $y = z + b_1, z = c_2z + d_2, c_2 = c_1 - b_1$ and $d_2 = c_1b_1 + d_1$. We apply now Proposition 3.10. for $b = -c_2$ and $c = -d_2$.

With above notation, if there are $r, s \in K \setminus \{0\}$ such that $c_2d_2 = d_2c_2, n\left(\frac{c^2}{4} + d_2\right) = r^2, n\left(r^2 + \frac{c^2}{4} + d_2\right) = s^2$, we get that the equation (4) has solutions of the form

$$x_0 = -\frac{1}{2}\left(p^{-1}cp - qbq^{-1}\right) \pm \frac{r(r^2 + d_3)}{s},$$

where

$$d_3 = \frac{(p^{-1}cp - qbq^{-1})^2}{4} + p^{-1}cpqbq^{-1} - p^{-1}dq^{-1}.$$

**References**


[3] R. E. Johnson, On the equation $\chi\alpha = \gamma\chi + \beta$ over algebraic division ring, J. of Algebra, 67(1941), 479-490.
Some equations over generalized quaternion and octonion division algebras


Received: 30.10.2008.