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On the Classification of Randers Manifolds of Constant Curvature

by
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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

The purpose of the present paper is to state a global classification theorem for a class of proper Randers manifolds of positive constant flag curvature. The model for the classification is the unit sphere S^{2n+1} endowed with a Sasakian space form structure of constant φ-sectional curvature $c \in (-3, 1)$.

Key Words: constant flag curvature, Finsler isometry, Randers manifolds, Randers spheres, Sasakian space forms.
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Introduction

The classification problem for Randers manifolds of constant flag curvature was raised by Ingarden [11] half of century ago. The first significant contribution, that actually stimulated the work on this problem, was brought by Yasuda and Shimada [17]. Later on, Bao and Robles [3] proved that the *Yasuda-Shimada Theorem* is true only for a special class of Randers manifolds of constant curvature. Then Bao and Shen [5] constructed Randers metrics of positive constant flag curvature on the sphere S^3 and Shen [13] investigated projectively flat Randers metrics with constant flag curvature. By using the Sasakian space form structures on odd dimensional spheres, Bejancu and Farran [6] have constructed Randers metrics of positive constant curvature on the sphere S^{2n+1}, $n \geq 2$. This result was used by the authors to state the first classification theorem for a class of Randers manifolds of positive constant flag curvature (cf. Bejancu-Farran [7]). Later on, Bao, Robles and Shen [4] have obtained a local classification theorem of Randers manifolds of constant flag curvature.

The purpose of the present paper is to prove a global classification theorem for a class of proper Randers manifolds of positive constant flag curvature. We claim that this is the only global classification theorem in this theory, and hope that it will bring some insights for further research work on the field. The main result (cf. Theorem 4.3) is based on our previous papers [6] and [7], but here we present details on the geometric meaning of the whole study.

Now, we outline the content of the paper. In the first section we arrange some results from the theory of Randers manifolds of positive constant flag curvature. Then, in Section 2 we recall some concepts and results from the theory of Sasakian space forms. A surprising relationship between Randers manifolds of positive constant flag curvature and Sasakian space forms has been discovered by us in the papers [6] and [7]. Adding more geometrical meaning to the study, we present this interrelation in the following two sections. First, in Section 3 we construct a family of proper Randers metrics of positive constant flag curvature on the unit sphere S^{2n+1}. Moreover, we show that these Randers metrics are not projectively flat. Here we introduce the concept of *Randers* (c, K)*-sphere*, which is the sphere S^{2n+1} endowed with the Sasakian space form structure of constant φ-sectional curvature $c \in (-3, 1)$ and with a family of Randers metrics of positive constant flag curvature K. Then, in Section 4 we prove that the Randers (c, K)-spheres are models for a class of proper Randers manifolds of positive constant flag curvature. More precisely, we prove that any Randers manifold from this class is *Finsler isometric* to a Randers (c, K)-sphere (cf. Theorem 4.3).

1 Finsler Manifolds of Constant Flag Curvature

Let M be an m-dimensional C^∞ manifold. Throughout the paper we denote by $\mathcal{F}(M)$ the algebra of C^∞ functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$-module of C^∞ sections of a vector bundle E over M. Also, we make use of the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their range.

Now, suppose that there exists a function $F : TM \to [0, \infty)$ which vanishes only on the zero section of TM and it is C^∞ on the slit tangent bundle $TM^\circ = TM \setminus \{0\}$. Moreover, we suppose that F satisfies the following conditions:

(i) It is positively homogeneous of degree one with respect to the fibre coordinates, that is, we have

$$F(x, ky) = kF(x, y), \quad \text{for any } x \in M,\ y \in T_xM, \text{ and } k > 0.$$

(ii) The $m \times m$ matrix

$$[g_{ij}(x, y)] = \left[\frac{1}{2}\ \frac{\partial^2 F^2}{\partial y^i \partial y^j}\right], \quad i, j \in \{1, ..., m\}, \tag{1.1}$$

is positive definite at every point (x, y) of TM°.

Then, $\mathbb{F}^m = (M, F)$ is called a *Finsler manifold with Finsler metric* F. We denote by (x^i, y^i) the local coordinates on TM°, where (x^i) are the local coordinates on M and (y^i) are the coordinates on the fibre at (x^i). Then, the natural frame field on TM° is $\{\partial/\partial x^i, \partial/\partial y^i\}$, $i \in \{1, ..., m\}$. The unit Liouville vector field is a global section ℓ of the vertical vector bundle VTM° given by

$$\ell = \frac{y^i}{F}\ \frac{\partial}{\partial y^i} \cdot$$

Thus, we have

$$g_{ij}(x, y)\ell^i \ell^j = 1, \quad \text{where } \ell^i = \frac{y^i}{F} \cdot$$

The geometry of $\mathbb{F}^m$ is studied by using the canonical nonlinear connection GTM° on TM°. This is a complementary distribution to VTM° in TTM° whose local frame field is given by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^j_i\ \frac{\partial}{\partial y^j}\ ,$$

where we put

$$G^j_i = \frac{\partial G^j}{\partial y^i}\ ,\ G^j = \frac{1}{4}\ g^{jh}\left(\frac{\partial^2 F^2}{\partial y^h \partial x^k}\ y^k - \frac{\partial F^2}{\partial x^h}\right) \cdot \tag{1.2}$$

The following Finsler tensor fields are useful for the study of the curvature of $\mathbb{F}^m$:

$$\text{(a) } R^k{}_j = \ell^h \left\{\frac{\delta}{\delta x^j}\left(\frac{G^k_h}{F}\right) - \frac{\delta}{\delta x^h}\left(\frac{G^k_j}{F}\right)\right\}, \qquad \text{(b) } R_{ij} = g_{ik} R^k{}_j. \tag{1.3}$$

Next, we consider a flag $\ell \wedge V$ at $x \in M$ determined by ℓ and the tangent vector $V = V^i(\partial/\partial x^i)$. Then the *flag curvature* of $\mathbb{F}^m$ at the point x with respect to the flag $\ell \wedge V$ is the number

$$K(\ell, V) = \frac{R_{ij}V^iV^j}{g_{ij}V^iV^j - (g_{ij}\ell^iV^j)^2}\,.$$

If $K(\ell, V)$ has no dependence on $(x^i, y^i, V^i),\ i \in \{1, ..., m\}$, that is, $K(\ell, V)$ is a constant function, we say that $\mathbb{F}^m$ is *Finsler manifold of constant flag curvature.* It is known that $\mathbb{F}^m$ is of constant flag curvature K if and only if (cf. Bao-Chern-Shen [2], p. 313)

$$R_{ij} = Kh_{ij}, \tag{1.4}$$

where h_{ij} are the local components of the angular metric on $\mathbb{F}^m$ given by

$$h_{ij} = g_{ij} - \ell_i\ell_j, \quad \text{where } \ell_i = g_{ij}\ell^j. \tag{1.5}$$

Randers has introduced a special Finsler structure as follows.

Let $\mathbf{a} = (a_{ij}(x))$ be a Riemannian metric and $\mathbf{b} = (b_i(x))$ a 1-form on M. Then, we define on TM the function

$$F(x, y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i. \tag{1.6}$$

It is proved that F defines a Finsler structure on M if and only if

$$\|\mathbf{b}\|^2 = a^{ij}(x)b_i(x)b_j(x) < 1, \tag{1.7}$$

where $[a^{ij}(x)]$ is the inverse matrix of $[a_{ij}(x)]$. A Finsler metric given by (1.6) is called a *Randers metric* and $\mathbb{F}^m = (M, F, a_{ij}, b_i)$ is called a *Randers manifold.* If the 1-form $\mathbf{b}$ is nowhere zero on M, then we say that F is a *proper Randers metric* and $\mathbb{F}^m$ is a *proper Randers manifold.*

Now, by using $\mathbf{a}$ and $\mathbf{b}$, we define a vector field B and a 1-form θ by

$$B = b^i\,\frac{\partial}{\partial x^i}\,, \quad \text{where } b^i = a^{ij}b_j, \tag{1.8}$$

and

$$\theta = b^i(b_{i|j} - b_{j|i})dx^j, \tag{1.9}$$

where "$|$" denotes the covariant derivative with respect to the Levi-Civita connection ∇ on $(M, \mathbf{a})$. The curvature tensor field R of ∇ is given by

$$R(X, Y)Z = \nabla_X\nabla_YZ - \nabla_Y\nabla_XZ - \nabla_{[X,Y]}Z, \quad \forall X, Y, Z \in \Gamma(TM). \tag{1.10}$$

Locally, we put

$$R_{hijk} = \mathbf{a}\left(R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^i}\right). \tag{1.11}$$

Next, we recall that Bao and Robles [3] have obtained necessary and sufficient conditions for a Randers manifold to have constant flag curvature. Also they showed that the Yasuda-Shimada Theorem stated in [17] needs the additional condition $\theta = 0$ on M. Taking into account the papers of Bao and Robles [3], Matsumoto and Shimada [12], and Shimada [14] we recall the following from the "*Corrected Yasuda-Shimada Theorem*".

Theorem 1.1. *Let $\mathbb{F}^m = (M, F, a_{ij}, b_i)$ be a Randers manifold. Then $\mathbb{F}^m$ is of positive constant flag curvature and $\theta = 0$ on M if and only if the following conditions are satisfied:*

(i) *The length $\|\mathbf{b}\|$ of $\mathbf{b}$ is a constant on M and $\mathbf{b}$ is not parallel with respect to ∇.*

(ii) *The covariant derivative of* $\mathbf{b}$ *with respect to* ∇ *satisfies*

$$b_{i|j} + b_{j|i} = 0. \tag{1.12}$$

(iii) *The curvature tensor field of the Levi-Civita connection* ∇ *is given by*

$$\begin{aligned} R_{hijk} &= K(1 - \|\mathbf{b}\|^2)\{a_{hj}a_{ik} - a_{hk}a_{ij}\} \\ &+ K\{b_i b_k a_{hj} + b_h b_j a_{ik} - b_i b_j a_{hk} - b_h b_k a_{ij}\} \\ &+ b_{h|k}b_{i|j} - b_{h|j}b_{i|k} + 2b_{h|i}b_{k|j}. \end{aligned} \tag{1.13}$$

Remark 1.1. a. Clearly, condition (ii) is equivalent to the following:

(ii′) B *given by* (1.8) *is a Killing vector field on* $(M, \mathbf{a})$.

b. By using the conditions (i), (ii) and (iii) it is proved (cf. Bejancu-Farran [6]) that

$$b_{i|j|k} = K(b_j a_{ik} - b_i a_{jk}). \tag{1.14}$$

Now, we prove the following.

Theorem 1.2. *Let* $\mathbb{F}^m = (M, F, a_{ij}, b_i)$ *be a Randers manifold of positive constant flag curvature* K. *Then, for any constant* $K^* > 0$, *there exists on* M *a Randers metric* $F^* = (a^*_{ij}, b^*_i)$ *of flag curvature* K^*.

Proof: First, we define on M the Riemannian metric $\mathbf{a}^*$ and the 1-form $\mathbf{b}^*$ by

$$a^*_{ij} = \frac{K}{K^*}\, a_{ij} \quad \text{and} \quad b^*_i = \sqrt{\frac{K}{K^*}}\, b_i.$$

Then, it is easy to check the condition (1.7) for (a^*_{ij}, b^*_i). Thus the function

$$F^*(x, y) = \sqrt{a^*_{ij}(x)y^i y^j} + b^*_i(x)y^i = \sqrt{\frac{K}{K^*}}\, F(x, y), \tag{1.15}$$

defines a new Randers metric on M. By using (1.1) for both F and F^* and taking into account (1.15), we deduce that

$$\text{(a)} \ \ g_{ij}(x, y) = \frac{K^*}{K}\, g^*_{ij}(x, y) \quad \text{and} \quad \text{(b)} \ \ g^{ij}(x, y) = \frac{K}{K^*}\, g^{ij*}(x, y). \tag{1.16}$$

Next, by using (1.15), (1.16b) and (1.2), we deduce that F and F^* define the same canonical nonlinear connection, that is, we have $G^j_i = G^{j*}_i$. Thus, (1.3), (1.15) and (1.16a) imply

$$R_{ij} = R^*_{ij}. \tag{1.17}$$

On the other hand, by (1.5), (1.15) and (1.16a) we deduce that the angular metrics corresponding to F and F^* are related by

$$h_{ij} = \frac{K^*}{K}\, h^*_{ij}. \tag{1.18}$$

Finally, taking into account that the Randers metric F is of constant curvature K (see (1.4)), from (1.17) and (1.18) we deduce that

$$R^*_{ij} = K^* h^*_{ij},$$

that is, F^* is a Randers metric of constant curvature K^*. □

2 Sasakian Space Forms

Let $M(\varphi, \xi, \eta, \mathbf{a})$ be a $(2n+1)$–dimensional contact metric manifold, where φ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form, and $\mathbf{a}$ is a Riemannian metric satisfying (cf. Blair [8], p. 27)

$$\begin{aligned} &\text{(a) } \varphi^2 = -I + \eta \otimes \xi, && \text{(b) } \eta(\xi) = 1, \\ &\text{(c) } \mathbf{a}(\varphi X, \varphi Y) = \mathbf{a}(X,Y) - \eta(X)\eta(Y), && \text{(d) } d\eta(X,Y) = \mathbf{a}(X, \varphi Y), \end{aligned} \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$. From (2.1b) we see that both η and ξ are nowhere zero on M. The equations (2.1) imply

$$\begin{aligned} &\text{(a) } \varphi\xi = 0, \quad \text{(b) } \mathbf{a}(X, \varphi Y) + \mathbf{a}(Y, \varphi X) = 0, \\ &\text{(c) } \eta \circ \varphi = 0, \quad \text{(d) } d\eta(\xi, X) = 0, \quad \text{(e) } \eta(X) = \mathbf{a}(X, \xi). \end{aligned} \tag{2.2}$$

The contact metric manifold $M(\varphi, \xi, \eta, \mathbf{a})$ is called a *Sasakian manifold* if the following condition is satisfied:

$$(\nabla_X \varphi)Y = \mathbf{a}(X,Y)\xi - \eta(Y)X, \quad \forall X, Y \in \Gamma(TM). \tag{2.3}$$

In this case $(\varphi, \xi, \eta, \mathbf{a})$ is called a *Sasakian structure* on M. By direct calculations we deduce that on a Sasakian manifold we have:

$$\begin{aligned} &\text{(a) } \ \nabla_X \xi = -\varphi X, \quad \text{(b) } \ (\nabla_X \eta)Y = \mathbf{a}(X, \varphi Y), \\ &\text{(c) } \ (\nabla_X \eta)Y + (\nabla_Y \eta)X = 0, \\ &\text{(d) } \ (\nabla_Z \nabla_X \eta)Y = \mathbf{a}(Y,Z)\eta(X) - \mathbf{a}(X,Z)\eta(Y) \end{aligned} \tag{2.4}$$

for any $X, Y, Z \in \Gamma(TM)$.

Now, we recall a result on the existence of Sasakian structures for later use in our study.

Theorem 2.1. (Hatakeyama-Ogawa-Tanno [10]) *Let $(M, \mathbf{a})$ be a $(2n+1)$–dimensional Riemannian manifold admitting a unit Killing vector field ξ such that*

$$R(X,Y)\xi = \mathbf{a}(Y,\xi)X - \mathbf{a}(X,\xi)Y, \quad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

where R is the curvature tensor field of the Levi-Civita connection ∇ on $(M, \mathbf{a})$. Then M has a Sasakian structure $(\varphi, \xi, \eta, \mathbf{a})$, where φ and η are given by

$$\varphi X = -\nabla_X \xi \quad \textit{and} \quad \eta(X) = \mathbf{a}(X, \xi), \quad \forall X \in \Gamma(TM).$$

Next, we denote by $\mathcal{D}$ the contact distribution on M, that is, $\mathcal{D}$ is the orthogonal complementary distribution to the distribution spanned by ξ on M. Let $x \in M$ and Π be a plane section in T_xM. We say that Π is a *φ-section* if it is spanned by X and φX, where $X \in \mathcal{D}_x$. The sectional curvature of M at x determined by a φ-section Π is called *φ-sectional curvature.* A Sasakian manifold M of constant φ-sectional curvature c is called a *Sasakian space form* and it is denoted by $M[c]$. In this case, we say that $(\varphi, \xi, \eta, \mathbf{a}, c)$ is a *Sasakian space form structure* on M. By using the well known formula for the curvature tensor field R of $M[c]$ (cf. Blair [8], p. 97), and taking into account (2.4b) we deduce that R is given by

$$\begin{aligned} &\mathbf{a}(R(X,Y)Z,W) = \frac{c+3}{4} \ \{\mathbf{a}(Y,Z)\mathbf{a}(X,W) - \mathbf{a}(X,Z)\mathbf{a}(Y,W)\} \\ &+ \frac{1-c}{4}\big\{\eta(Y)\eta(Z)\mathbf{a}(X,W) + \eta(X)\eta(W)\mathbf{a}(Y,Z) \\ &- \eta(X)\eta(Z)\mathbf{a}(Y,W) - \eta(Y)\eta(W)\mathbf{a}(X,Z) + (\nabla_X\eta)(Z)(\nabla_Y\eta)(W) \\ &- (\nabla_Y\eta)(Z)(\nabla_X\eta)(W) + 2(\nabla_X\eta)(Y)(\nabla_Z\eta)(W)\big\}, \end{aligned} \tag{2.6}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

The standard model for Sasakian space forms of constant φ-sectional curvature $c > -3$ is the unit sphere S^{2n+1}. Let $(\varphi_0, \xi_0, \eta_0, \mathbf{a}_0)$ be the standard Sasakian structure on S^{2n+1} induced by the Kähler structure of $\mathbb{R}^{2n+2}$ (cf. Blair [8], p. 89). Then, for any $\varepsilon > 0$ we consider the following deformed structures

$$\varphi = \varphi_0, \ \xi = \frac{1}{\varepsilon}\xi_0, \ \eta = \varepsilon\eta_0, \ \mathbf{a} = \varepsilon\mathbf{a}_0 + \varepsilon(\varepsilon - 1)\eta_0 \otimes \eta_0. \tag{2.7}$$

Since the metrics restricted to the contact deformations are homothetic, the transformations (2.7) are called *$\mathcal{D}$-homothetic deformations.* Tanno [15] has proved that S^{2n+1} endowed with $(\varphi, \xi, \eta, \mathbf{a})$ given by (2.7) is a Sasakian space form $S^{2n+1}[c]$ of constant φ-sectional curvature

$$c = \frac{4}{\varepsilon} - 3. \tag{2.8}$$

Next, we suppose that $M[c]$ and $\widetilde{M}[\widetilde{c}]$ are two Sasakian space forms with Sasakian space form structures $(\varphi, \xi, \eta, \mathbf{a}, c)$ and $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{\mathbf{a}}, \widetilde{c})$, respectively. Then we say that $M[c]$ and $\widetilde{M}[\widetilde{c}]$ are *isomorphic* if $\widetilde{c} = c$ and there exists a C^∞ diffeomorphism $f : M[c] \to \widetilde{M}[\widetilde{c}]$ which maps the tensor fields from $(\varphi, \xi, \eta, \mathbf{a}, c)$ into the corresponding tensor fields from $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{\mathbf{a}}, \widetilde{c})$. In particular, two isomorphic Sasakian space forms are isometric Riemannian manifolds. This enables us to recall the following important result.

Theorem 2.2. (Tanno [16]) *Let $M[c]$ be a simply connected and complete Sasakian space form of constant φ-sectional curvature $c > -3$. Then $M[c]$ is isomorphic to $S^{2n+1}[c]$.*

Remark 2.1. *In particular, $M[c]$ and $S^{2n+1}[c]$ are isometric, but we should note that we consider on S^{2n+1} a Riemannian metric $\mathbf{a}$ given by the D-homothetic transformation* (2.7).

As we want to apply the theory of Sasakian space forms in Finsler geometry, the expressions of some of the above formulas in local coordinates are imperiously required. First, we set:

$$a_{ij} = \mathbf{a}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \qquad \eta_i = \eta\left(\frac{\partial}{\partial x^i}\right), \quad \varphi\left(\frac{\partial}{\partial x^i}\right) = \varphi_i^j \, \frac{\partial}{\partial x^j},$$
$$\eta_{i|j} = \left(\nabla_{\frac{\partial}{\partial x^j}}\eta\right)\left(\frac{\partial}{\partial x^i}\right), \quad \eta_{i|j|k} = \left(\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^j}}\eta\right)\left(\frac{\partial}{\partial x^i}\right).$$

Then, (2.4b), (2.4c) and (2.4d) become

$$\eta_{i|j} = g_{ik}\varphi_j^k, \tag{2.9}$$

$$\eta_{i|j} + \eta_{j|i} = 0, \tag{2.10}$$

and

$$\eta_{i|j|k} = a_{ik}\eta_j - a_{jk}\eta_i, \tag{2.11}$$

respectively. Also, as a consequence of (2.9), (2.2a) and (2.2c) we obtain

$$\eta_{i|j}\xi^j = 0 \quad \text{ and } \quad \eta_{i|j}\xi^i = 0. \tag{2.12}$$

Now, we take $X = \partial/\partial x^j$, $Y = \partial/\partial x^i$ and $\xi = \xi^k(\partial/\partial x^k)$ in (2.5) and by using (1.10) we infer that (2.5) is equivalent to

$$\xi^k{}_{|i|j} - \xi^k{}_{|j|i} = \eta_i\delta_j^k - \eta_j\delta_i^k. \tag{2.13}$$

Finally, we take $X = \partial/\partial x^k$, $Y = \partial/\partial x^j$, $Z = \partial/\partial x^h$ and $W = \partial/\partial x^i$ in (2.6) and by using (1.11) and (2.10) we deduce that (2.6) is equivalent to

$$\begin{aligned} R_{hijk} = \frac{c+3}{4}\,\{a_{hj}a_{ik} - a_{hk}a_{ij}\} + \frac{1-c}{4}\,\{&\eta_i\eta_k a_{hj} + \eta_h\eta_j a_{ik} \\ -\eta_i\eta_j a_{hk} - \eta_h\eta_k a_{ij} + \eta_{h|k}\eta_{i|j} - &\eta_{h|j}\eta_{i|k} + 2\eta_{h|i}\eta_{k|j}\}. \end{aligned} \tag{2.14}$$

3 Randers Metrics of Positive Constant Flag Curvature on S^{2n+1}

Let $S^{2n+1}[c]$ be the unit $(2n+1)$–dimensional sphere endowed with the Sasakian space form structure $(\varphi, \xi, \eta, \mathbf{a}, c)$ described in the previous section. We recall that $\mathbf{a}$ is not the standard metric on S^{2n+1} induced by the Euclidean metric of $\mathbb{R}^{2n+2}$. More precisely, $\mathbf{a}$ is a metric on S^{2n+1} defined by a $\mathcal{D}$-homothetic deformation (2.7). *Throughout this section, we suppose that* $c \in (-3, 1)$*, which by* (2.8) *is equivalent to* $\varepsilon \in (1, \infty)$. This enables us to define for each $\varepsilon > 1$ a new 1-form on S^{2n+1} as follows

$$\mathbf{b} = \alpha\eta, \quad \text{where } \alpha = \sqrt{1 - \frac{1}{\varepsilon}}\,. \tag{3.1}$$

Then we consider on the tangent bundle of S^{2n+1} the functions

$$F(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i, \tag{3.2}$$

where $a_{ij}(x)$ and $b_i(x)$ are the local components of the Riemannian metric $\mathbf{a}$ and of the 1-form $\mathbf{b}$, respectively. Taking into account that ξ is a unit vector field and by using (2.2e), we obtain $\|\eta\| = 1$. Thus, from (3.1) we deduce that

$$\|\mathbf{b}\| = \alpha < 1, \tag{3.3}$$

that is, F given by (3.2) defines a Randers metric on S^{2n+1}. Moreover, we prove the following theorems.

Theorem 3.1. *The sphere* S^{2n+1}*,* $n \geq 1$*, endowed with any of the Randers metrics given by* (3.2) *is a proper Randers manifold of constant flag curvature* $K = 1$*. Moreover,* $\mathbb{F}^{2n+1} = (S^{2n+1}, F)$ *is not a projective flat Finsler manifold.*

Proof: First, from (3.1) we obtain

$$b^i = \alpha\xi^i, \quad \text{where } b^i = a^{ij}b_j. \tag{3.4}$$

Then, by using (2.12), (3.1) and (3.4) we deduce that θ given by (1.9) vanishes identically on S^{2n+1}. Next, from (3.3) we see that the 1-form $\mathbf{b}$ is of constant length. Also, by using (2.4b) and (3.1), we obtain

$$(\nabla_{\varphi Y}\mathbf{b})Y = \mathbf{a}(\varphi Y, \varphi Y) > 0,$$

for any non zero vector field $Y \in \Gamma(\mathcal{D})$. Hence the condition (i) from Theorem 1.1 is satisfied. The condition (ii) of the same theorem is a direct consequence of (2.10) and (3.1). Now, by using (2.8), (3.1) and (3.3), we infer that

$$\begin{aligned}
&\frac{c+3}{4} = \frac{1}{\varepsilon} = 1 - \alpha^2 = 1 - \|\mathbf{b}\|^2, \\
&\frac{1-c}{4}\,\eta_i\eta_k = \left(1 - \frac{1}{\varepsilon}\right)\eta_i\eta_k = \alpha^2\eta_i\eta_k = b_i b_k, \\
&\frac{1-c}{4}\,\eta_{h|k}\eta_{i|j} = \alpha^2\eta_{h|k}\eta_{i|j} = b_{h|k}b_{i|j}.
\end{aligned} \tag{3.5}$$

Then, using (3.5) into (2.14), we deduce that (1.13) is true for $K = 1$. Hence the condition (iii) of Theorem 1.1 is satisfied. Thus, any F given by (3.2) on S^{2n+1} is a Randers metric of constant flag curvature $K = 1$. For the last part of the theorem we recall from Douglas [9] that a Finsler manifold is projectively flat if and only if its projective Weyl and Douglas tensors vanish. On the other hand, from Bacsó-Matsumoto [1] we know that the Douglas tensor of a Randers manifold vanishes if and only if the 1-form $\mathbf{b}$ is closed. In our case, by using (3.1) and (2.1d) we obtain

$$d\mathbf{b}(X, Y) = \alpha d\eta(X, Y) = \alpha\mathbf{a}(X, \varphi Y).$$

Then we take a non zero vector field $Y \in \Gamma(\mathcal{D})$ and deduce that

$$d\mathbf{b}(\varphi Y, Y) > 0.$$

Thus any Randers metric F given by (3.2) on S^{2n+1} is not projectively flat. This completes the proof of the theorem. $\square$

Theorem 3.2. *For any constant $K > 0$ there exists on S^{2n+1} a family of proper Randers metrics that are of flag curvature K and are not projectively flat.*

Proof: By Theorem 3.1, for any $\varepsilon > 1$ there exists a Randers metric F^* of constant flag curvature $K^* = 1$. Then we apply Theorem 1.2 and obtain a Randers metric $F = (1/\sqrt{K})F^*$ of constant flag curvature K. Next, from the proof of Theorem 1.2 we know that $\mathbf{b} = (1/\sqrt{K})\mathbf{b}^*$. As $\mathbf{b}^*$ is not closed, we conclude that $\mathbf{b}$ is also not closed. Thus F is not projectively flat. $\square$

Let us explain what Riemannian metric and 1-form we consider on S^{2n+1} to obtain the Randers metric F of constant flag curvature K. As in Section 2, we consider the standard Sasakian structure $(\varphi_0, \xi_0, \eta_0, \mathbf{a}_0)$ on S^{2n+1}. We note that S^{2n+1} is of constant sectional curvature 1 with respect to the Riemannian metric $\mathbf{a}_0$. Then we take $\varepsilon > 1$ and from (2.8) we deduce that

$$\varepsilon = \frac{4}{c+3}, \quad c \in (-3, 1). \tag{3.6}$$

Replace ε from (3.6) into (2.7) and we obtain on S^{2n+1} the Sasakian space form structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{\mathbf{a}}, c)$ given by

$$\begin{aligned} &\text{(a) } \overline{\varphi} = \varphi_0, \quad \text{(b) } \overline{\xi} = \frac{c+3}{4}\,\xi_0, \quad \text{(c) } \overline{\eta} = \frac{4}{c+3}\,\eta_0, \\ &\text{(d) } \overline{\mathbf{a}} = \frac{4}{c+3}\left\{\mathbf{a}_0 + \frac{1-c}{c+3}\,\eta_0 \otimes \eta_0\right\}. \end{aligned} \tag{3.7}$$

Thus, by Theorem 3.1, the function

$$\text{(a) } \overline{F}(x,y) = \sqrt{\overline{a}_{ij}y^iy^j} + \overline{b}_i(x)y^i; \quad \text{(b) } \overline{b}_i = \frac{\sqrt{1-c}}{2}\,\overline{\eta}_i, \tag{3.8}$$

defines on S^{2n+1} a Randers metric of constant flag curvature $\overline{K} = 1$. Finally, we consider the Riemannian metric $\widetilde{\mathbf{a}}$ and the 1-form $\widetilde{\mathbf{b}}$ given by

$$\text{(a) } \widetilde{\mathbf{a}} = \frac{1}{K}\,\overline{\mathbf{a}}, \quad \text{and} \quad \text{(b) } \widetilde{\mathbf{b}} = \frac{1}{\sqrt{K}}\,\overline{\mathbf{b}}. \tag{3.9}$$

Then, for any $c \in (-3, 1)$, the function

$$\widetilde{F}(x,y) = \sqrt{\widetilde{a}_{ij}(x)y^iy^j} + \widetilde{b}_i(x)y^i = \frac{1}{\sqrt{K}}\,\overline{F}(x,y) \tag{3.10}$$

defines on $S^{2n+1}[c]$ a Randers metric of constant flag curvature K. We call the Randers manifold $\widetilde{\mathbb{F}}^{2n+1} = (S^{2n+1}[c], \widetilde{F})$, where $\widetilde{F}$ is given by (3.10), the *Randers (c, K)-sphere*. As we shall see in the next section, this is going to be the standard model for Randers manifolds of positive constant flag curvature K and with $\theta = 0$.

4 The Classification Theorem

Let $\mathbb{F}^m = (M, F, a_{ij}, b_i)$, $m \geq 2$, be an m-dimensional proper Randers manifold whose 1-form θ given by (1.9) vanishes identically on M. Moreover, we suppose that $\mathbb{F}^m$ is of constant flag curvature $K = 1$.

In the first part we show an interesting interrelation between the geometry of the proper Randers manifold $\mathbb{F}^m$ and a natural Sasakian space form structure on M. Then we use this to obtain a global classification theorem for proper Randers manifolds of positive constant flag curvature with $\theta = 0$ on M.

First, we define the unit 1-form η on M by

$$\eta = \frac{1}{\|\mathbf{b}\|}\ \mathbf{b}. \tag{4.1}$$

Thus we have

$$a^{ij}\eta_i\eta_j = 1. \tag{4.2}$$

Then we note that Theorem 1.1 applies to the above $\mathbb{F}^m$. Thus, by using (4.1) in (1.12), (1.13) and (1.14), and taking into account that $\|\mathbf{b}\| =$ constant and $K = 1$, we obtain

$$\eta_{i|j} + \eta_{j|i} = 0, \tag{4.3}$$

$$\begin{aligned} R_{hijk} = (1 - \|\mathbf{b}\|^2)\{a_{hj}a_{ik} - a_{hk}a_{ij}\} + \|\mathbf{b}\|^2\{\eta_i\eta_k a_{hj} + \eta_h\eta_j a_{ik} \\ -\eta_i\eta_j a_{hk} - \eta_h\eta_k a_{ij} + \eta_{h|k}\eta_{i|j} - \eta_{h|j}\eta_{i|k} + 2\eta_{h|i}\eta_{k|j}\}, \end{aligned} \tag{4.4}$$

and

$$\eta_{i|j|k} = \eta_j a_{ik} - \eta_i a_{jk}, \tag{4.5}$$

respectively. Next, we define on M the unit vector field $\xi = \xi^i(\partial/\partial x^i)$, where we set

$$\xi^i = a^{ij}\eta_j. \tag{4.6}$$

Then, by direct calculations using (4.6), (4.3) and (4.5), we deduce that

$$a_{ik}\xi^k{}_{|j} + a_{jk}\xi^k{}_{|i} = 0, \tag{4.7}$$

and

$$a_{ih}\xi^h{}_{|j|k} = (a_{ik}a_{jh} - a_{jk}a_{ih})\xi^h. \tag{4.8}$$

Also, we define on M a tensor field φ of type $(1,1)$, whose local components are given by

$$\varphi^i_j = -\xi^i{}_{|j}. \tag{4.9}$$

Finally, we consider the number

$$c = 1 - 4\|\mathbf{b}\|^2, \tag{4.10}$$

and, taking into account that $0 < \|\mathbf{b}\| < 1$, we deduce that

$$-3 < c < 1. \tag{4.11}$$

Summing up, we can say that we constructed on M the structure $(\varphi, \xi, \eta, \mathbf{a}, c)$, where $\mathbf{a} = (a_{ij})$ is the Riemannian metric on M and φ, ξ, η, c are given by (4.9), (4.6), (4.1) and (4.10), respectively.

Now, we prove the following.

Theorem 4.1. *Let $\mathbb{F}^m = (M, F, a_{ij}, b_i)$, $m \geq 2$, be an m-dimensional proper Randers manifold of constant flag curvature $K = 1$ and with $\theta = 0$ on M. Then m must be an odd number $2n + 1$, and $(\varphi, \xi, \eta, \mathbf{a}, c)$ is a Sasakian space form structure on M.*

Proof: First, by using (4.9), (4.6) and (4.3), we obtain

$$\varphi^i_j \varphi^j_k = -a^{ih} a^{js} \eta_{j|h} \eta_{s|k}. \tag{4.12}$$

Then, from (4.2) we deduce that

$$a^{js} \eta_{j|h} \eta_s = 0. \tag{4.13}$$

Next, we take the covariant derivative in (4.13) and by using (4.5) and (4.2) we infer that

$$a^{js} \eta_{j|h} \eta_{s|k} = a_{hk} - \eta_h \eta_k.$$

Thus (4.12) becomes

$$\varphi^i_j \varphi^j_k = -\delta^i_k + \xi^i \eta_k. \tag{4.14}$$

Now, denote by $\mathcal{D}$ the complementary orthogonal distribution to span$\{\xi\}$ in TM. Then, by using (4.14) and (4.6), we obtain

$$\varphi^i_j \varphi^j_k X^k = -X^i,$$

for any $X = X^i(\partial/\partial x^i)$ that lies in $\Gamma(\mathcal{D})$. Hence, the restriction of φ to $\mathcal{D}$ is an almost complex structure. Thus the fibres of $\mathcal{D}$ must be of even dimension and therefore $m = 2n+1$, $n \geq 1$. Next, from (4.2), (4.6) and (4.7) we deduce that on the Riemannian manifold (M, a_{ij}) there exists a unit Killing vector field ξ. Moreover, from (4.8) we infer that

$$\xi^i{}_{|j|k} = \delta^i_k \eta_j - a_{jk} \xi^i,$$

which implies (2.13). Hence, by Theorem 2.1, $(\varphi, \xi, \eta, \mathbf{a})$ is a Sasakian structure on M. Finally, for c given by (4.10) we obtain

$$\frac{c+3}{4} = 1 - \|\mathbf{b}\|^2 \quad \text{and} \quad \frac{1-c}{4} = \|\mathbf{b}\|^2.$$

Thus, from (4.4) we deduce (2.14), that is, $(\varphi, \xi, \eta, \mathbf{a}, c)$ is a Sasakian space form structure on M. This completes the proof of the theorem. □

Next, we consider a proper Randers manifold $\mathbb{F}^m = (M, F, a_{ij}, b_i)$, $m \geq 2$, of constant flag curvature $K > 0$ and $\theta = 0$ on M. Then, we define on M the Riemannian metric $\mathbf{a}^*$ and the 1-form $\mathbf{b}^*$ by their local components

$$\text{(a) } a^*_{ij} = K a_{ij} \quad \text{and} \quad \text{(b) } b^*_i = \sqrt{K}\; b_i. \tag{4.15}$$

According to Theorem 1.2, the Randers metric

$$F^*(x, y) = \sqrt{K}\; F(x, y), \tag{4.16}$$

is of constant flag curvature $K^* = 1$. Thus, by Theorem 4.1, we infer that $m = 2n + 1$, and $(\varphi^*, \xi^*, \eta^*, \mathbf{a}^*, c^*)$ is a Sasakian space form structure, where $\mathbf{a}^*$ is the Riemannian metric given by (4.15a) and the others are defined as follows

$$\begin{aligned} &\text{(a) } \eta^*_i = \frac{1}{\|\mathbf{b}^*\|}\, b^*_i, \quad \text{(b) } \xi^{i*} = a^{ij*} \eta^*_j, \\ &\text{(c) } \varphi^{i*}_j = -\xi^{i*}{}_{|^*j}, \quad \text{(d) } c^* = 1 - 4\|\mathbf{b}^*\|^2. \end{aligned} \tag{4.17}$$

Here, the norm is taken with respect to the Riemannian metric $\mathbf{a}^*$ and the covariant derivative is taken with respect to the Levi-Civita connection defined by $\mathbf{a}^*$. By using (4.15), we rewrite the right sides of formulas in (4.17) in terms of the geometric objects defined by a_{ij} and b_i. First, by using (4.15b), we obtain

$$\|\mathbf{b}^*\|^2 = a^{ij*} b^*_i b^*_j = \frac{1}{K}\; a^{ij} \sqrt{K}\; b_i \sqrt{K}\; b_j = a^{ij} b_i b_j = \|\mathbf{b}\|^2.$$

Thus, we have

$$\begin{aligned}
&\text{(a)}\ \ \eta_i^* = \frac{\sqrt{K}}{\|\mathbf{b}\|}\ b_i, && \text{(b)}\ \ \xi^{i*} = \frac{1}{\sqrt{K}\|\mathbf{b}\|}\ a^{ik}b_k,\\
&\text{(c)}\ \ \varphi_j^{i*} = -\frac{1}{\sqrt{K}\|\mathbf{b}\|}\ a^{ik}b_{k|j}, && \text{(d)}\ \ c^* = 1 - 4\|\mathbf{b}\|^2.
\end{aligned} \tag{4.18}$$

Note that we used in (4.18c) the fact that the Levi-Civita connections of $\mathbf{a}$ and $\mathbf{a}^*$ coincide.

Summing up these results, we can state the following.

Theorem 4.2. *Let $\mathbb{F}^m = (M, F, a_{ij}, b_i)$, $m \geq 2$, be an m-dimensional proper Randers manifold of constant flag curvature $K > 0$ and with $\theta = 0$ on M. Then, m must be an odd number $2n+1$, and M carries a Sasakian space form structure $(\varphi^*, \xi^*, \eta^*, \mathbf{a}^*, c^*)$ given by* (4.18) *and* (4.15a).

Next, we consider two m-dimensional Finsler manifolds $\mathbb{F}^m = (M, F)$ and $\widetilde{\mathbb{F}}^m = (\widetilde{M}, \widetilde{F})$. Then we say that $\mathbb{F}^m$ and $\widetilde{\mathbb{F}}^m$ are *Finsler isometric* if there exists a C^∞ diffeomorphism $f : M \to \widetilde{M}$ such that

$$F = \widetilde{F} \circ df, \tag{4.19}$$

where $df : TM \to T\widetilde{M}$ is the differential of f. Now, we can state the following.

Theorem 4.3. (Global Classification Theorem) *Let $\mathbb{F}^m = (M, F, a_{ij}, b_i)$, $m \geq 2$, be an m-dimensional proper Randers manifold, where $(M, \mathbf{a} = (a_{ij}))$ is a simply connected and complete Riemannian manifold. Suppose that $\mathbb{F}^m$ is of positive constant flag curvature K and that $\theta = 0$ on M. Then, m must be an odd number $2n+1$, and $\mathbb{F}^{2n+1} = (M, F, a_{ij}, b_i)$ is Finsler isometric to the Randers (c, K)-sphere $\widetilde{F}^{2n+1} = (S^{2n+1}[c], \widetilde{F})$, where $c = 1 - 4\|\mathbf{b}\|^2$ and $\widetilde{F}$ is given by* (3.10).

Proof: By Theorem 4.2 we know that m must be an odd number $2n+1$, and M carries a Sasakian space form structure $(\varphi^*, \xi^*, \eta^*, \mathbf{a}^*, c)$, where $c = 1 - 4\|\mathbf{b}\|^2$ (cf. (4.18d)). Also, from the same theorem we deduce that the Randers metric F is expressed as follows (cf. (4.16))

$$F(x,y) = \frac{1}{\sqrt{K}}\ \left\{\sqrt{a_{ij}^* y^i y^j} + b_i^*(x) y^i\right\}, \tag{4.20}$$

where a_{ij}^* and b_i^* are the local components of the Riemannian metric $\mathbf{a}^*$ of the Sasakian space form $M[c]$ and of the 1-form $\mathbf{b}^*$ on M (see (4.15)). Next, by Theorem 2.2 we know that there exists a diffeomorphism $f : M[c] \to S^{2n+1}[c]$ which transforms the Sasakian space form structure $(\varphi^*, \xi^*, \eta^*, \mathbf{a}^*, c)$ of $M[c]$ into the Sasakian space form structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{\mathbf{a}}, c)$ of $S^{2n+1}[c]$ given by (3.7). Moreover, the Randers metric $\widetilde{F}$ of constant flag curvature K is given by (cf. (3.10))

$$\widetilde{F}(\widetilde{x}, \widetilde{y}) = \frac{1}{\sqrt{K}}\ \left\{\sqrt{\overline{a}_{ij}\widetilde{y}^i\,\widetilde{y}^j} + \overline{b}_i(\widetilde{x})\widetilde{y}^i\right\}, \tag{4.21}$$

where $(\widetilde{x}, \widetilde{y}) = df(x,y)$, for $(x,y) \in TM$. Now, according to the properties of f we deduce that

$$a_{ij}(x) y^i y^j = \overline{a}_{ij}(\widetilde{x})\widetilde{y}^i\,\widetilde{y}^j. \tag{4.22}$$

Also, by using (4.17a), (3.8) and taking into account that η^* is transformed in $\overline{\eta}$ by the diffeomorphism f, we obtain

$$b_i^*(x) y^i = \|\mathbf{b}\|\eta_i^*(x) y^i = \|\mathbf{b}\|\overline{\eta}_i(\widetilde{x})\widetilde{y}^i = \|\mathbf{b}\|\ \frac{2}{\sqrt{1-c}}\ \overline{b}_i(\widetilde{x})\widetilde{y}^i = \overline{b}_i(\widetilde{x})\widetilde{y}^i, \tag{4.23}$$

since

$$\|\mathbf{b}\| = \frac{\sqrt{1-c}}{2}\,.$$

Taking into account (4.20)–(4.23), we deduce that

$$F(x,y) = \widetilde{F}(\widetilde{x}, \widetilde{y}), \quad \forall (x,y) \in TM,$$

that is, F^{2n+1} is isometric to $\widetilde{F}^{2n+1}$. This completes the proof of the theorem. $\square$

Remark 4.1. If we drop the condition for M to be simply connected and complete Riemannian manifold, then Theorem 4.3 becomes a local classification theorem. This means that $\mathbb{F}^m$ is locally Finsler isometric to the Randers (c, K)-sphere $\widetilde{\mathbb{F}}^{2n+1}$.

Remark 4.2. The global classification of Randers manifolds of constant flag curvature for the cases that were not considered in the present paper is still open. We hope that a result similar to Theorem 4.3 can be proved in general.

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