

Geometric and algebraic aspects of 1-formality

by

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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

Formality is a topological property, defined in terms of Sullivan’s model for a space. In the simply-connected setting, a space is formal if its rational homotopy type is determined by the rational cohomology ring. In the general setting, the weaker 1-formality property allows one to reconstruct the rational pro-unipotent completion of the fundamental group, solely from the cup products of degree 1 cohomology classes.

In this note, we survey various facets of formality, with emphasis on the geometric and algebraic implications of 1-formality, and its relations to the cohomology jump loci and the Bieri–Neumann–Strebel invariant. We also produce examples of 4-manifolds W such that, for every compact Kähler manifold M , the product $M \times W$ has the rational homotopy type of a Kähler manifold, yet $M \times W$ admits no Kähler metric.

Key Words: Formality, fundamental group, cohomology jumping loci, holonomy Lie algebra, Bieri–Neumann–Strebel invariant, Malcev completion, lower central series, Kähler manifold, quasi-Kähler manifold, Milnor fiber, hyperplane arrangement, Artin group, Bestvina–Brady group, pencil, fibration, monodromy

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1 Introduction

1.1 Formality of spaces

The question whether one can reconstruct the homotopy type of a space from homological data goes back to the beginnings of Algebraic Topology. It was recognized by H. Poincaré himself that homology is not enough: for a path-connected space X , the first homology group, $H_1(X, \mathbb{Z})$, only records the abelianization of the fundamental group, $\pi_1(X)$. Even in the simply-connected setting, homology by itself fails to detect the Hopf map, $S^3 \rightarrow S^2$. On the other hand, if one looks at the de Rham algebra of differential forms on S^n , one can reconstitute in a purely algebraic fashion all the higher homotopy groups of S^n , modulo torsion.

These considerations lead to the notion of *formality* of a space, as formulated by D. Sullivan in his foundational work on rational homotopy theory [45]. The general definition (which we review in §2) involves a certain commutative differential graded algebra, $A_{\text{PL}}(X, \mathbb{R})$, attached to a path-connected space X . Formality amounts to this cdga being related by a chain of quasi-isomorphisms to the cohomology algebra, $H^*(X, \mathbb{R})$, viewed as a cdga with the zero differential. In the case when X is a smooth manifold, Sullivan’s A_{PL} algebra may be replaced by de Rham’s algebra, leading to the following basic principle in rational homotopy theory: “The manner in which a closed form which is zero in cohomology actually becomes exact contains geometric information”, cf. [10, p. 253].

If X is a simply-connected formal space with finite Betti numbers, then one can build the whole rational Postnikov tower of X (in particular, $\pi_*(X) \otimes \mathbb{Q}$), in a purely “formal” way, just from the rational cohomology ring. On the other hand, one cannot hope to do this for an arbitrary formal space of finite type, unless $\pi_1(X)$ is nilpotent, and acts unipotently on the higher homotopy groups. For example, consider the real projective plane, $\mathbb{R}P^2$. Clearly, $\tilde{H}^*(\mathbb{R}P^2, \mathbb{Q}) = 0$; in fact, $\mathbb{R}P^2$ is a formal space, with the \mathbb{Q} -homotopy type of a point. Yet $\pi_2(\mathbb{R}P^2) = \mathbb{Z}$, and so the projective plane does not have the same rational homotopy groups as a point.

1.2 1-Formality of groups

A fruitful way to look at formality in the non-simply-connected setting is through the prism of various Lie algebras attached to the fundamental group. If the space X is formal, then the group $G = \pi_1(X)$ is *1-formal*. This means that the Malcev completion of G , as defined by D. Quillen in [42], is isomorphic, as a filtered Lie algebra, to the completion with respect to degree of a quadratic Lie algebra. In other words, the Malcev completion of G —and thus, the rational pro-unipotent completion of G —can be reconstituted from the cup-product map, $\cup_G: H^1(G, \mathbb{Q}) \wedge H^1(G, \mathbb{Q}) \rightarrow H^2(G, \mathbb{Q})$, more precisely, from the corestriction to its image, μ_G . See §3 for more details.

The main goal of this (non-exhaustive) survey is to present answers to the following natural question: Given a 1-formal group G , what kind of algebraic/topological/geometric information about the group can be extracted from the map μ_G ?

In the presence of 1-formality, various other algebraic objects attached to G can be recovered from the cup-products in degree 1. Among these objects, we treat in §3 the graded Lie algebras (modulo torsion) associated to the lower central series of the solvable quotients of G .

1.3 Examples

We will come back to the above question in §1.4. But first, let us address another natural question: What sort of conditions insure that a space X is formal—or, that a group G is 1-formal? In essentially all known examples, formality follows from one of the following reasons: the cohomology ring of X has some special properties; the homotopy groups of X vanish up to a certain degree; X supports some special geometric structures; or G admits some distinguished type of presentation. Furthermore, there are a number of formality-preserving constructions which can be very useful in practice.

We provide a variety of examples, indicating how formality can be deduced from homological, geometric, or group-theoretic arguments. We start in §4 with the connection between the algebraic properties of the cohomology ring $A = H^*(X, \mathbb{C})$ and the formality properties of the space X . In §5, we discuss the formality properties of highly connected manifolds, as well as Kähler and quasi-Kähler manifolds, smooth affine varieties, and Milnor fibers. Finally, we delineate in §6 several classes of 1-formal groups, including Artin groups, Bestvina–Brady groups, and pure welded braid groups.

1.4 Cohomology jump loci

The 1-formality property of a group G puts strong restrictions on the structure of its cohomology jumping loci. We survey this subject in §7, where we discuss the characteristic varieties $\mathcal{V}_d(G)$, the resonance varieties $\mathcal{R}_d(G)$, and the relationship between the two. For a 1-formal group G , it turns out that the analytic germs at the origin of the characteristic varieties can be reconstructed from the map μ_G . Furthermore, the analysis of the qualitative properties of the resonance varieties of G reveals subtle constraints on the cup-product map, imposed by 1-formality. As we explain in Example 7.2, this is a striking phenomenon, peculiar to *non-simply-connected* rational homotopy theory.

We also discuss in §7 the Bieri–Neumann–Strebel invariant, $\Sigma^1(G)$ —a rather enigmatic object, which controls the finiteness properties of normal subgroups of G with abelian quotient. Again, the 1-formality assumption plays a significant role, and yields an upper bound for the BNS invariant $\Sigma^1(G)$, depending solely on μ_G .

Under suitable geometric assumptions, the restrictions imposed by 1-formality on the cohomology jumping loci are strong enough to lead to complete classification results. In §8, we describe how cup-products in degree 1 can detect the realizability of a 1-formal group G , as the fundamental group of a (quasi-) Kähler manifold. This method applies to a wide range of groups, including right-angled Artin groups, Bestvina–Brady groups, and 3-manifold groups.

1.5 Algebraic monodromy

Given a locally trivial, smooth fibration, $F \hookrightarrow M \rightarrow B$, there is an associated algebraic monodromy action of $\pi_1(B)$ on $H_*(F)$. Likewise, every group extension, $N \hookrightarrow G \twoheadrightarrow Q$, gives rise to a monodromy action of Q on $H_*(N)$, induced by conjugation in G . In §9, we examine the interplay between monodromy and 1-formality. For Artin kernels, triviality of the monodromy action insures 1-formality, whereas for fibrations over the circle, formality of the total space implies absence of monodromy Jordan blocks of size greater than 1 for the eigenvalue 1.

Needless to say, the formal theory has its limitations. We illustrate this point in the last section (where all the new material is concentrated), with a family of examples originating from symplectic

geometry. The upshot is the following theorem, which we prove in §10, by analyzing the algebraic monodromy of 3-dimensional mapping tori.

Theorem 1.1. *There exist infinitely many closed, orientable, formal 4-manifolds W such that, for every compact Kähler manifold M , the following hold.*

1. *The manifold $M \times W$ has the same rational homotopy type as the Kähler manifold $M \times T^2 \times S^2$.*
2. *The manifold $M \times W$ admits no Kähler metric.*

1.6 Conventions

Except otherwise stated, by a space we always mean a topological space having the homotopy type of a connected polyhedron with finite 1-skeleton. Similarly, all groups we consider here are assumed to be finitely generated. The typical examples we have in mind are compact, connected, smooth manifolds and their fundamental groups. We say that a manifold is *closed* if it is smooth, compact, connected, and boundaryless. Coefficients are usually taken in a field \mathbb{k} of characteristic zero; when coefficients are not mentioned, the default is $\mathbb{k} = \mathbb{C}$.

2 From spaces to differential graded algebras

We start with Sullivan’s construction of an algebraic model encoding the rational homotopy type of a space.

2.1 Differential graded algebras

Fix a ground field \mathbb{k} , of characteristic 0. A *commutative differential graded algebra* (for short, a cdga), is a graded \mathbb{k} -algebra A , endowed with a differential $d_A: A \rightarrow A$ of degree 1. We assume here commutativity in the graded sense, that is, $ab = (-1)^{|a||b|}ba$, for every homogeneous elements $a, b \in A$, where $|a|$ denotes the degree of a .

A cdga morphism is a *quasi-isomorphism* if it induces an isomorphism in cohomology. Two commutative differential graded algebras, A and B , are said to be *weakly equivalent* if there is a zig-zag of quasi-isomorphisms (going both ways), connecting A to B .

Definition 2.1. A cdga is *formal* if it is weakly equivalent to its cohomology algebra, endowed with the zero differential.

We will also consider the following natural notion of “partial” formality, up to some degree $q \geq 1$.

Definition 2.2. A cdga (A, d_A) is *q-formal* if there is a zig-zag of morphisms connecting (A, d_A) to $(H^*(A, d_A), d = 0)$, with each one of these maps inducing an isomorphism in cohomology up to degree q , and a monomorphism in degree $q + 1$.

2.2 Models of spaces

Let X be a space. (Recall we are tacitly assuming X is homotopy equivalent to a connected polyhedron with finite 1-skeleton.) In [45], Sullivan constructs an algebra $A_{\text{PL}}(X, \mathbb{k})$ of polynomial differential forms on X with coefficients in \mathbb{k} , and provides it with a natural cdga structure. A *model* for X , over the field \mathbb{k} , is a cdga weakly equivalent to $A_{\text{PL}}^*(X, \mathbb{k})$. Two spaces, X and Y , have the same \mathbb{k} -homotopy type if $A_{\text{PL}}^*(X, \mathbb{k})$ and $A_{\text{PL}}^*(Y, \mathbb{k})$ are weakly equivalent.

The space X is said to be formal (over \mathbb{k}) if Sullivan’s algebra $A_{\text{PL}}(X, \mathbb{k})$ is formal. Likewise, X is *q-formal* if this cdga is *q-formal*. When X is a smooth manifold, and $\mathbb{k} = \mathbb{R}$ or \mathbb{C} , we may replace

in the definition the algebra of polynomial forms by the corresponding de Rham algebra of differential forms, $\Omega_{\text{dR}}(X, \mathbb{k})$; see for instance [20].

Clearly, formality implies partial formality, and q -formality implies r -formality, for all $r \leq q$. We refer to Măcincic [28] for a study of q -formality in the range $q \geq 2$. Here, we are primarily interested in 1-formality.

Remark 2.3. The 1-formality property of a space X depends only on its fundamental group, $G = \pi_1(X)$. Indeed, let $f: X \rightarrow K(G, 1)$ be a classifying map. Then, the induced homomorphism, $f^*: H^i(G, \mathbb{k}) \rightarrow H^i(X, \mathbb{k})$, is an isomorphism for $i = 1$ and a monomorphism for $i = 2$. The claim follows.

2.3 Massey products

The first approach to 1-formality is in terms of certain higher order structures from the algebraic homotopy theory of differential graded algebras, called *Massey products*. This point of view has been extensively used to compare symplectic and Kähler structures on manifolds, see for instance [21, 22, 46].

Since we will not pursue this direction here, we will avoid precise definitions, and simply say that “a group G is 1-formal if and only if all Massey products of elements from $H^1(G, \mathbb{k})$ vanish uniformly, for length at least 3”; compare with [10, p. 262]. (The length 2 Massey products are simply the cup-products.)

For a differential graded algebra (A, d) , the Massey product of classes $\alpha_1, \alpha_2, \alpha_3 \in H^1(A)$ is defined, provided $\alpha_1\alpha_2 = \alpha_2\alpha_3 = 0$. Pick representative cocycles a_i for α_i , and elements $y, z \in A$ such that $dy = a_1a_2$ and $dz = a_2a_3$. It is readily seen that $ya_3 + a_1z$ is a cocycle. The set of cohomology classes of all such cocycles is the Massey triple product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$. The image of this set in the quotient ring $H^*(A)/\langle \alpha_1, \alpha_3 \rangle$ is a well-defined element in degree 2; we say $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is *non-vanishing* if this element is not 0.

Example 2.4. Let $M = G_{\mathbb{R}}/G_{\mathbb{Z}}$ be the 3-dimensional Heisenberg nilmanifold, where $G_{\mathbb{R}}$ is the group of real, unipotent 3×3 matrices, and $G_{\mathbb{Z}} = \pi_1(M)$ is the subgroup of integral matrices in $G_{\mathbb{R}}$. One may use invariant forms to obtain the following simple model (A, d) for the Heisenberg manifold: $A = \wedge(a, b, z)$ is the exterior algebra on the indicated generators in degree 1, and the differential is given by $da = db = 0, dz = ab$. Clearly, $\cup_M = 0$, and $\langle [a], [a], [b] \rangle = [az]$, with trivial indeterminacy. Since $[az] \neq 0$, the manifold M is not 1-formal.

3 From groups to Lie algebras

There is a dual approach to 1-formality, based on Lie algebras. In this section, we review the construction of several Lie algebras associated to a group, and how these Lie algebras are related to each other, in the case when the group is 1-formal.

3.1 Holonomy Lie algebras

Given a space X , let $\cup_X: H^1(X, \mathbb{k}) \wedge H^1(X, \mathbb{k}) \rightarrow H^2(X, \mathbb{k})$ be the cup-product map in degree one, and let $\partial_X: H_2(X, \mathbb{k}) \rightarrow H_1(X, \mathbb{k}) \wedge H_1(X, \mathbb{k})$ be the comultiplication map.

Definition 3.1 (K.-T. Chen [7]). The *holonomy Lie algebra* of X over \mathbb{k} is the quotient

$$\mathfrak{h}(X, \mathbb{k}) = \text{Lie}(H_1(X, \mathbb{k})) / \text{ideal}(\text{im}(\partial_X)),$$

where $\text{Lie}(H_1(X, \mathbb{k}))$ is the free Lie algebra over \mathbb{k} , generated by $H_1(X, \mathbb{k})$ and graded by bracket length.

Note that the defining ideal of $\mathfrak{h}(X, \mathbb{k})$ is a homogeneous (in fact, quadratic) ideal. Hence, the holonomy Lie algebra inherits a natural grading from the free Lie algebra. Further, note that $\mathfrak{h}(X, \mathbb{k})$ depends only on μ_X , the corestriction of \cup_X to its image.

Now assume X is a path-connected space, and let $G = \pi_1(X)$. Define the holonomy Lie algebra of the group G as that of a classifying space $K(G, 1)$:

$$\mathfrak{h}(G, \mathbb{k}) := \mathfrak{h}(K(G, 1), \mathbb{k}).$$

It is readily seen that $\mu_X = \mu_G$. It follows that $\mathfrak{h}(X, \mathbb{k}) = \mathfrak{h}(G, \mathbb{k})$.

3.2 Malcev Lie algebras

Next, we recall some notions from [42, Appendix A]. A *Malcev Lie algebra* is a Lie algebra over \mathbb{k} , endowed with a decreasing, complete \mathbb{k} -vector space filtration, satisfying certain axioms (see §3.3 below for more details). For example, the completion of $\mathfrak{h}(X, \mathbb{k})$ with respect to the degree filtration is a Malcev Lie algebra, denoted $\widehat{\mathfrak{h}}(X, \mathbb{k})$.

In [42], Quillen associates to every group G , in a functorial way, a Malcev Lie algebra, denoted $\mathfrak{m}(G, \mathbb{k})$. This object, called the *Malcev completion* of G , captures the properties of the torsion-free nilpotent quotients of G .

Here is a concrete way to describe it. The group algebra $\mathbb{k}G$ has a natural Hopf algebra structure, with comultiplication given by $\Delta(g) = g \otimes g$, and counit the augmentation map. Let I be the augmentation ideal. One verifies that the Hopf algebra structure on $\mathbb{k}G$ extends to the I -adic completion, $\widehat{\mathbb{k}G} = \varprojlim_r \mathbb{k}G/I^r$. Finally, $\mathfrak{m}(G, \mathbb{k})$ coincides with the Lie algebra of primitive elements in $\widehat{\mathbb{k}G}$, endowed with the inverse limit filtration.

Theorem 3.2 (Sullivan [45]). *A group G is 1-formal, over \mathbb{k} , if and only if $\mathfrak{m}(G, \mathbb{k}) \cong \widehat{\mathfrak{h}}(G, \mathbb{k})$, as filtered Lie algebras.*

Consequently, if G is a 1-formal group, then the co-restriction to the image of the cup-product map, μ_G , determines the Malcev completion $\mathfrak{m}(G)$.

The next result is folklore. A proof is given in [15, Lemma 2.9] for finitely presented groups, but the argument given there works as well for finitely generated groups.

Lemma 3.3. *A group G is 1-formal, over \mathbb{k} , if and only if the Malcev Lie algebra $\mathfrak{m}(G, \mathbb{k})$ is isomorphic, as a filtered Lie algebra, to the completion with respect to degree of a quadratic Lie algebra.*

Example 3.4. Let F_n be the free group of rank $n \geq 0$. Clearly, $H_1(F_n, \mathbb{k}) = \mathbb{k}^n$ and $H_2(F_n, \mathbb{k}) = 0$; in particular, $\cup_{F_n} = 0$. Thus, $\mathfrak{h}(F_n, \mathbb{k})$ is isomorphic to $\mathbb{L}_n = \text{Lie}(\mathbb{k}^n)$, the free Lie algebra of rank n , over \mathbb{k} . It is readily checked that $\mathfrak{m}(F_n, \mathbb{k}) = \widehat{\mathbb{L}}_n$. It follows from Theorem 3.2 (or Lemma 3.3) that F_n is 1-formal.

Example 3.5. Let Σ_g be the Riemann surface of genus $g \geq 1$. The group $G = \pi_1(\Sigma_g)$ is generated by $x_1, y_1, \dots, x_g, y_g$, subject to the single relation $[x_1, y_1] \cdots [x_g, y_g] = 1$, where $[x, y] = xyx^{-1}y^{-1}$ is the group commutator. It is readily checked that $\mathfrak{h}(G, \mathbb{k})$ is the quotient of the free Lie algebra on $x_1, y_1, \dots, x_g, y_g$ by the ideal generated by $[x_1, y_1] + \cdots + [x_g, y_g]$. A further computation shows that $\mathfrak{m}(G, \mathbb{k}) \cong \widehat{\mathfrak{h}}(G, \mathbb{k})$, and thus, G is 1-formal.

3.3 Associated graded Lie algebra

Given an arbitrary group G , the lower central series (for short, LCS) of G is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$. The *associated graded Lie algebra*, $\text{gr}(G)$, is the direct sum of the successive LCS quotients,

$$\text{gr}(G) = \bigoplus_{k \geq 1} \gamma_k G / \gamma_{k+1} G,$$

with Lie bracket induced from the group commutator.

The Malcev filtration $\{F_s\}_{s \geq 1}$ on $\mathfrak{m}(G, \mathbb{k})$ is required to satisfy $[F_s, F_t] \subseteq F_{s+t}$, for all s, t . Consequently, the associated graded vector space, $\text{gr}(\mathfrak{m}(G, \mathbb{k})) = \bigoplus_{s \geq 1} F_s/F_{s+1}$, inherits a natural Lie algebra structure, compatible with the grading. The basic property of the Malcev completion is that

$$\text{gr}(G) \otimes \mathbb{k} \cong \text{gr}(\mathfrak{m}(G, \mathbb{k})),$$

as Lie algebras with grading. We infer from Theorem 3.2 that μ_G also determines $\text{gr}(G)$ modulo torsion, in the 1-formal case.

Corollary 3.6. *If the group G is 1-formal, then $\text{gr}(G) \otimes \mathbb{k} \cong \mathfrak{h}(G, \mathbb{k})$, as Lie algebras with grading.*

The *derived series* of a group G is defined inductively by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. The derived series of a Lie algebra is constructed similarly. The next result extends Corollary 3.6 to solvable quotients.

Theorem 3.7 ([34]). *If the group G is 1-formal, then for each $i \geq 1$,*

$$\text{gr}(G/G^{(i)}) \otimes \mathbb{k} \cong \mathfrak{h}(G, \mathbb{k})/\mathfrak{h}^{(i)}(G, \mathbb{k}),$$

as Lie algebras with grading.

The proof, given in [34, Theorem 4.2] in the case when G is finitely presented, works as well when G is finitely generated.

4 Cohomology ring and formality

We now turn to the task of delineating a set of conditions, sufficient to guarantee the formality of a space, or the 1-formality of a group. We start in this section with some cohomological considerations.

4.1 Small first Betti number

As usual, let G be a finitely generated group, and let \mathbb{k} be a field of characteristic 0. Denote by $b_i(G) = \dim_{\mathbb{k}} H_i(G, \mathbb{k})$ the i -th Betti number of G . Let us start with a well-known (and easy to prove) fact.

Proposition 4.1. *If $b_1(G) \leq 1$, then G is 1-formal.*

Proof: When $b_1(G) = 0$, the homomorphism $f: G \rightarrow \{1\}$ induces an isomorphism on H^1 and a monomorphism on H^2 . Likewise, when $b_1(G) = 1$, we may pick a homomorphism $f: G \rightarrow \mathbb{Z}$ inducing an isomorphism on H^1 ; it follows then that f is injective on H^2 . Our claim follows from the 1-formality of free groups, discussed in Example 3.4. □

This result is optimal. Indeed, the Heisenberg group $G = G_{\mathbb{Z}}$ from Example 2.4 has $b_1(G) = 2$, and is not 1-formal.

In the case of 3-manifolds, the above proposition was sharpened by Fernández and Muñoz, as follows.

Theorem 4.2 ([24]). *Let M be a closed, orientable 3-manifold with $b_1(M) \leq 1$. Then M is formal, and has the same \mathbb{k} -homotopy type as S^3 or $S^1 \times S^2$.*

4.2 Regular sequences

Let A be a graded, graded-commutative algebra (for short, cga) over \mathbb{k} . A sequence r_1, \dots, r_n of elements of A is said to be a *regular sequence* if r_i is not a zero-divisor in $A/(r_1, \dots, r_{i-1})$, for each $i \leq n$.

Theorem 4.3 (Sullivan [45]). *If $H^*(M, \mathbb{k})$ is the quotient of a free cga by an ideal generated by a regular sequence, then M is a formal space. In particular, freeness of $H^*(M, \mathbb{k})$ implies formality of M .*

This result provides a large supply of formal spaces, such as

- rational cohomology spheres;
- rational cohomology tori;
- compact connected Lie groups G , as well as their classifying spaces, BG ;
- homogeneous spaces of the form G/K , with $\text{rank } G = \text{rank } K$;
- Eilenberg-MacLane spaces $K(\pi, n)$ with $n \geq 2$.

In particular, if X is the complement of a knotted sphere in S^n , $n \geq 3$, then X is a formal space.

On the other hand, not all homogeneous spaces are formal: for instance, $\text{Sp}(5)/\text{SU}(5)$ is not, see e.g. [21, p. 143]. And Eilenberg-MacLane spaces $K(\pi, 1)$ need not be formal: for example, if π is a torsion-free, finitely generated nilpotent group, then $K(\pi, 1)$ is formal if and only if π is abelian, see e.g. [21, p. 120].

4.3 From partial to full formality

In general, q -formality is strictly weaker than formality, for $q < \infty$. Nevertheless, under favorable circumstances the two notions are equivalent. One result of this type is related to the well-known Koszul property from homological algebra.

Let A be a graded \mathbb{k} -algebra with $A^0 = \mathbb{k}$. Then,

$$\text{Tor}^A(\mathbb{k}, \mathbb{k}) = \bigoplus_{s, t \geq 0} \text{Tor}_s^A(\mathbb{k}, \mathbb{k})_t$$

is a bigraded vector space: the index s denotes the usual homological degree, while t stands for the internal degree, coming from the grading of A^* . The algebra A is said to be a *Koszul algebra* if $\text{Tor}_s^A(\mathbb{k}, \mathbb{k})_t = 0$, for $s \neq t$.

Theorem 4.4 ([41, 35]). *Let X be a connected CW-complex with finite skeleta. Suppose $H^*(X, \mathbb{k})$ is a Koszul algebra. Then X is 1-formal if and only if X is formal.*

The following result of Măcincic ties partial formality to full formality, under a completely different homological assumption.

Theorem 4.5 ([28]). *Let X be a space with the property that $H^i(X, \mathbb{k}) = 0$, for $i > q + 1$. If X is q -formal, then X is formal.*

Here is an immediate consequence.

Corollary 4.6. *Every q -formal CW-complex of dimension at most $q + 1$ is formal.*

In particular, if G is a finitely-generated, 1-formal group, and K is a 2-complex with $\pi_1(K) = G$, then K is formal. See [25, Lemma 2.10] for a result similar to Corollary 4.6, involving a different (more restrictive) notion of q -formality, introduced by Fernández and Muñoz in [25].

5 Manifolds and geometric structures

In this section, we look at some of the ways in which formality of a space is implied by relevant topological properties or geometric structures.

5.1 Cell complexes and manifolds

We start with connectivity properties: roughly speaking, the more highly connected a finite-dimensional CW-complex is, the more likely it is to be formal. This was made precise by Stasheff [44], as follows. Let X be a k -connected CW-complex of dimension n ; if $n \leq 3k + 1$, then X is formal. This is the best possible bound: attaching a cell e^{3k+2} to the wedge $S^{k+1} \vee S^{k+1}$ via the iterated Whitehead product $[\iota_1, [\iota_1, \iota_2]]$ yields a non-formal CW-complex.

Formality is preserved under several standard operations on (based) CW-complexes with finite Betti numbers. For instance, if X and Y are formal, then so is the product $X \times Y$ and the wedge $X \vee Y$; moreover, a retract of a formal space is formal. We refer to [20, 21] for details.

For closed manifolds, the above dimension bound can be relaxed, by using Poincaré duality. As shown by Miller [30], if M is a closed, k -connected manifold ($k \geq 0$) of dimension $n \leq 4k + 2$, then M is formal. In particular, all simply-connected closed manifolds of dimension at most 6 are formal. Again, this is best possible: as shown by Fernández and Muñoz in [23], there exist closed, simply-connected, non-formal manifolds of dimension 7. On the other hand, if M is a closed, orientable, k -connected n -manifold with $b_{k+1}(M) = 1$, then the bound insuring formality can be improved to $n \leq 4k + 4$, see Cavalcanti [6].

Formality behaves well with respect to certain operations on manifolds. For instance, Stasheff [44] proved the following: If M is a closed, simply-connected manifold such that $M \setminus \{*\}$ is formal, then M is formal. Moreover, if M and N are closed, orientable, formal manifolds, so is their connected sum, $M \# N$; see [20].

5.2 Kähler manifolds and smooth algebraic varieties

Certain geometric structures influence favorably the formality of manifolds. For instance, on a compact Riemannian symmetric space M , the product of harmonic forms is again harmonic. Hodge theory, then, implies the formality of M [45].

Hodge theory also has strong implications on the topology of compact Kähler manifolds. If M is such a manifold, let d be the exterior derivative, J the complex structure, and $d^c = J^{-1}dJ$. In [10], Deligne, Griffiths, Morgan, and Sullivan showed that the following “ dd^c Lemma” holds for M : if α is a form which is closed for both d and d^c , and exact for either d or d^c , then α is exact for dd^c . Formality ensues:

Theorem 5.1 ([10]). *All compact Kähler manifolds are formal.*

A manifold M is said to be a *quasi-Kähler manifold* if $M = \overline{M} \setminus D$, where \overline{M} is a compact Kähler manifold, and D is a normal crossing divisor. For example, smooth, irreducible, quasi-projective complex varieties are quasi-Kähler. In [31, Corollary 10.3], Morgan establishes the following result.

Theorem 5.2 ([31]). *Let M be a smooth, quasi-projective variety. If the Deligne weight filtration space $W_1H^1(M, \mathbb{C})$ vanishes, then M is 1-formal.*

This happens, for instance, when M is the complement of a hypersurface in $\mathbb{C}\mathbb{P}^n$.

Example 5.3. Let \mathcal{C} be an algebraic curve in $\mathbb{C}\mathbb{P}^2$. By the above, the complement $M = \mathbb{C}\mathbb{P}^2 \setminus \mathcal{C}$ is 1-formal. On the other hand, M has the homotopy type of a finite CW-complex of dimension 2. Thus,

by Corollary 4.6 to Măcinic’s theorem, M is formal. For a different proof of this result, see [8, Theorem 6.4].

Also note that every smooth, irreducible complex curve C is formal. In the compact case, formality follows from the Kähler property, while in the non-compact case, formality follows from the fact that C is homotopy equivalent to a (finite) wedge of circles.

5.3 Affine varieties and Milnor fibrations

In contrast, smooth, irreducible *affine* varieties need not be 1-formal. A general construction illustrating this phenomenon is given in [16, Proposition 7.2]. Here is a concrete example, taken from [16].

Example 5.4. Consider the polynomials $g = x^3 + y^3 + z^3$ and $f = x + y^2 + z^3$. Then $M = V(g) \setminus V(f)$ is a smooth affine subvariety of \mathbb{C}^4 , yet M is not 1-formal.

An important construction in singularity theory is that of the *Milnor fibration*. In its simplest incarnation, this goes as follows. Let $f \in \mathbb{C}[z_0, \dots, z_n]$ be a homogeneous polynomial. The restriction $f: \mathbb{C}^{n+1} \setminus V(f) \rightarrow \mathbb{C}^*$ turns out to be a smooth bundle projection. Clearly, the fiber of this bundle, $F(f) := f^{-1}(1)$, is a smooth affine variety, having the homotopy type of an n -dimensional finite CW-complex. When the singularity $(V(f), 0)$ is reduced, the Milnor fiber $F(f)$ is connected. The above considerations naturally lead to the following question.

Question 5.5. Is the Milnor fiber of a reduced polynomial, $F(f)$, always 1-formal?

When $f = f_1 \cdots f_d$ completely splits as a product of distinct linear factors, the reduced variety $V(f)$ is a finite union of hyperplanes in \mathbb{C}^{n+1} . In this case, the complement $M = \mathbb{C}^{n+1} \setminus V(f)$ is a formal space. This follows from work by Brieskorn [5], who showed that the inclusion of the subalgebra generated by the closed logarithmic 1-forms df_i/f_i into $\Omega_{\text{dR}}^*(M)$ induces an isomorphism in cohomology. We mention that the above question is open even for hyperplane arrangements.

6 Group presentations and 1-formality

In this section, we discuss the 1-formality property of several classes of groups, as well as the behavior of this property under standard group-theoretic constructions.

6.1 Commutator relators and vanishing cup products

Suppose G is a finitely presented group. Then, as shown in [33], one may find a finite presentation for the corresponding Malcev Lie algebra, $\mathfrak{m}(G)$. In favorable cases, this latter presentation is quadratic, and thus one may apply Lemma 3.3 to conclude that G is 1-formal.

As we saw in Example 3.4, the free group F_n has vanishing cup-product map \cup_{F_n} , and is 1-formal. Here is a partial converse.

Proposition 6.1 ([17]). *Let G be a group admitting a finite presentation with only commutator relators. If G is 1-formal and $\cup_G = 0$, then G is a free group.*

The proposition shows that, in some sense “generically,” 1-formality does not hold, at least not among commutator-relators groups.

Remark 6.2. The above result explains—without resorting to Massey products—why the Heisenberg group $G = G_{\mathbb{Z}}$ from Example 2.4 cannot be 1-formal. Indeed, the group has commutator-relators presentation $G = \langle x, y \mid [x, [x, y]], [y, [x, y]] \rangle$, and $\cup_G = 0$.

Remark 6.3. Let L be a link in S^3 obtained by closing up a pure braid, and let $M = S^3 \setminus L$ be its complement. It is readily seen that the group $G = \pi_1(M)$ admits a commutator-relations presentation. Assuming $\cup_G = 0$, it follows from Proposition 6.1 that G is 1-formal if and only if L is a trivial link. Therefore, we cannot expect to obtain much information from the vanishing of \cup_G , except, of course, the vanishing of all linking numbers.

6.2 Products, coproducts, and extensions

The 1-formality property behaves well with respect to (finite) direct products and coproducts.

Theorem 6.4 ([17]). *Let G_1 and G_2 be finitely presented, 1-formal groups. Then $G_1 \times G_2$ and $G_1 * G_2$ are also 1-formal.*

By contrast, 1-formality behaves rather badly with respect to quotients and subgroups. Of course, any group is a quotient of a free group, and thus free groups of rank at least 2 possess plenty of non-1-formal quotients. The next example shows that (finitely generated) subgroups of 1-formal groups need not be 1-formal.

Example 6.5. Let $G = G_{\mathbb{Z}}$ be the Heisenberg group, with presentation as in Remark 6.2. Consider the semidirect product $H = G \rtimes_{\varphi} \mathbb{Z}$, defined by the automorphism $\varphi: G \rightarrow G$ given by $x \mapsto y, y \mapsto xy$. Since clearly $b_1(H) = 1$, the group H is 1-formal. Yet, the normal subgroup $G = H^{(1)}$ is not 1-formal.

6.3 Artin groups

Let $\Gamma = (\mathbf{V}, \mathbf{E}, \ell)$ be a labeled finite simplicial graph, with vertex set \mathbf{V} , edge set \mathbf{E} , and labeling function $\ell: \mathbf{E} \rightarrow \mathbb{Z}_{\geq 2}$. The corresponding *Artin group* has one generator for each vertex $v \in \mathbf{V}$ and one defining relation

$$\underbrace{v w v \cdots}_{\ell(e)} = \underbrace{w v w \cdots}_{\ell(e)}$$

for each edge $e = \{v, w\}$ in \mathbf{E} . For example, if $\Gamma = K_{n-1}$ is the complete graph on vertices 1 through $n - 1$, with labels $\ell(\{i, j\}) = 2$ if $|i - j| > 1$ and $\ell(\{i, j\}) = 3$ if $|i - j| = 1$, then the corresponding Artin group is the braid group on n strings, B_n .

If $\Gamma = (\mathbf{V}, \mathbf{E})$ is unlabeled, then G_{Γ} is called a *right-angled Artin group*, and is defined by commutation relations $vw = wv$, one for each edge $\{v, w\} \in \mathbf{E}$. These groups interpolate between \mathbb{Z}^n (for $\Gamma = K_n$) and F_n (for $\Gamma = \overline{K}_n$), and behave nicely with respect to the join operation for graphs: $G_{\Gamma * \Gamma'} = G_{\Gamma} \times G_{\Gamma'}$.

Using the defining presentations, Kapovich and Millson proved the following theorem.

Theorem 6.6 ([27]). *All Artin groups are 1-formal.*

By combining Theorems 6.6 and 4.4, we showed in [35] that the classifying spaces of right-angled Artin groups are formal spaces. More generally, Notbohm and Ray showed that, for each finite simplicial complex L , the corresponding toric complex T_L is formal; see [32, Remark 5.7].

6.4 Bestvina–Brady groups

Given a finite simple graph Γ , let

$$N_{\Gamma} = \ker(\nu: G_{\Gamma} \rightarrow \mathbb{Z})$$

be the kernel of the “diagonal” epimorphism, sending each generator v to 1. As shown by Bestvina and Brady in [3], the homological finiteness properties of the group N_{Γ} are intimately connected to the topology of the flag complex Δ_{Γ} , that is, the maximal simplicial complex with 1-skeleton Γ . For

example, N_Γ is finitely generated if and only if Γ is connected; and N_Γ is finitely presented if and only if Δ_Γ is simply-connected.

Using the presentation of N_Γ derived by Dicks and Leary in [12], we proved in [36] the following result.

Theorem 6.7 ([36]). *All finitely presented Bestvina–Brady groups N_Γ are 1-formal.*

6.5 Welded braid groups

Let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank $n \geq 1$. The *welded braid group* on n strands is the subgroup of $\text{Aut}(F_n)$ consisting of those group automorphisms which send each generator x_i to a conjugate of another generator. The elements for which the associated permutation of the generators is the identity form the *pure* welded braid subgroup.

Making use of the presentation given by McCool in [29], the following result was obtained by Berceanu and Papadima.

Theorem 6.8 ([2]). *Pure welded braid groups are 1-formal.*

This theorem extends the well-known 1-formality of pure braid groups (which are fundamental groups of hyperplane arrangement complements), and leads to an explicit construction of the Kontsevich integral for the welded braid groups, described in [2].

7 Cohomology jump loci and the BNS invariant

In this section, we describe some implications of the 1-formality property on the structure of the cohomology jump loci and the Bieri–Neumann–Strebel invariant of a space (or a group).

7.1 Jump loci

As before, let X be a connected CW-complex with finite 1-skeleton, with fundamental group $G = \pi_1(X)$. Consider the algebraic group $\mathbb{T}(X) = \text{Hom}(G, \mathbb{C}^*)$. Each character $\rho \in \mathbb{T}(X)$ determines a rank 1 local system (or, a rank 1 complex flat bundle) on X , which we denote by \mathbb{C}_ρ . The *characteristic varieties* of X are the jumping loci for cohomology with coefficients in such local systems:

$$\mathcal{V}_d(X) = \{\rho \in \mathbb{T}(X) \mid \dim H^1(X, \mathbb{C}_\rho) \geq d\}, \quad \text{for } d > 0.$$

The characteristic varieties are Zariski closed subsets of $\mathbb{T}(X)$. These varieties depend only on the maximal metabelian quotient of the fundamental group, $G/G^{(2)}$, so we sometimes denote them as $\mathcal{V}_d(G)$. An irreducible component of $\mathcal{V}_d(X)$ is called *non-translated* if it contains the origin 1 of the algebraic group $\mathbb{T}(X)$.

Consider now the cohomology algebra $H^*(X, \mathbb{C})$. Left-multiplication by an element $x \in H = H^1(X, \mathbb{C})$ yields a cochain complex $(H^*(X, \mathbb{C}), \lambda_x)$. The *resonance varieties* of X are the jumping loci for the homology of this complex:

$$\mathcal{R}_d(X) = \{x \in H \mid \dim H^1(H^*(X, \mathbb{C}), \lambda_x) \geq d\}, \quad \text{for } d > 0.$$

The resonance varieties are homogeneous, Zariski closed subsets of the affine space $H^1(X, \mathbb{C}) = \text{Hom}(G, \mathbb{C})$. These varieties depend only on the co-restriction of the cup-product map, μ_G , so we sometimes denote them by $\mathcal{R}_d(G)$.

7.2 The exponential map

The usual exponential map, $\exp: \mathbb{C} \rightarrow \mathbb{C}^*$, induces a coefficient homomorphism, $\exp: H^1(G, \mathbb{C}) \rightarrow H^1(G, \mathbb{C}^*)$. The next result describes the behavior of this complex analytic map with respect to the cohomology jump loci of G , and some of the qualitative properties of these loci, under a formality assumption.

Theorem 7.1 ([17]). *Let G be a 1-formal group. For each $d > 0$,*

1. *The irreducible components of $\mathcal{R}_d(G)$ are all linear subspaces of $H^1(G, \mathbb{C})$, defined over \mathbb{Q} .*
2. *The non-translated components of $\mathcal{V}_d(G)$ are all subtori of the form $\exp(L)$, with L running through the irreducible components of $\mathcal{R}_d(G)$.*

The 1-formality hypothesis in the above theorem is crucial.

Example 7.2. Let K be the finite, 2-dimensional CW-complex defined in [17, Example 4.6]. In this case, the rationality property from (1) is violated for $\mathcal{R}_1(K)$. Consequently, the ring $H^*(K, \mathbb{Q})$ cannot be realized as $H^*(X, \mathbb{Q})$, for any formal space X .

This example stands in marked contrast with a basic result from simply-connected rational homotopy theory, due to Quillen [42] and Sullivan [45]: Any finite-dimensional, commutative graded algebra A^* defined over \mathbb{Q} , with $A^0 = \mathbb{Q}$ and $A^1 = 0$, can be realized as the cohomology algebra of a 1-connected, finite, formal CW-complex X .

7.3 The BNS invariant

Let G be a finitely generated group. The following definition was introduced by Bieri, Neumann, and Strebel, in their seminal paper [4].

Definition 7.3. Pick a finite generating set for G , and denote by \mathcal{C} the associated Cayley graph. The *BNS invariant*, $\Sigma^1(G)$, consists of those non-zero homomorphisms, $\varphi: G \rightarrow \mathbb{R}$, for which the full subgraph of \mathcal{C} on vertex set $\{g \in G \mid \varphi(g) \geq 0\}$ is connected. The definition is independent of the choice of generators for G .

If M is a compact manifold and $p: M \rightarrow S^1$ is a locally trivial fibration, then $p^*(\omega) \in \Sigma^1(\pi_1(M))$, where $\omega \in H^1(S^1, \mathbb{Z})$ is the generator given by the canonical orientation. In dimension 3, the elements of $\Sigma^1(\pi_1(M))$ coincide with the cohomology classes in $H^1(M, \mathbb{R})$ having closed, nowhere vanishing, de Rham representatives; see [4].

Theorem 7.4 ([39]). *If the group G is 1-formal, then $\Sigma^1(G) \subseteq H^1(G, \mathbb{R}) \setminus \mathcal{R}_1(G)$.*

For right-angled Artin groups G_Γ , the above inclusion becomes equality, see [35].

In the above theorem, the 1-formality hypothesis is again crucial, as the next example illustrates.

Example 7.5. Let $M = G_{\mathbb{R}}/G_{\mathbb{Z}}$ be the 3-dimensional Heisenberg nilmanifold from Example 2.4, with fundamental group $G = G_{\mathbb{Z}}$. Clearly, M is a torus bundle over S^1 , and thus the BNS invariant $\Sigma^1(G)$ is non-empty. On the other hand, $H^1(G, \mathbb{R}) \subseteq \mathcal{R}_1(G)$, since $\cup_G = 0$. Therefore, the above resonance upper bound for Σ^1 fails in this non-1-formal situation.

Here is a quick application of Theorem 7.4. For a closed, orientable 3-manifold M , it was shown in [18] that $\mathcal{R}_1(M) = H^1(M, \mathbb{C})$, provided $b_1(M)$ is even.

Corollary 7.6 ([39]). *Let M be a closed, orientable 3-manifold, with even first Betti number. If M is 1-formal, then M does not admit a smooth fibration over the circle.*

8 Serre's problem and classification results

We now turn to Kähler and quasi-Kähler manifolds, and the nature of their cohomology jumping loci. We illustrate the efficiency of our obstructions to 1-formality and (quasi-) Kählerianity with several classes of examples.

8.1 Serre's problem

A finitely presented group G is said to be a *Kähler group* if it can be realized as $G = \pi_1(M)$, where M is a compact Kähler manifold. If M can be chosen to be a smooth, irreducible, projective complex variety, then G is a *projective group*. The notions of quasi-Kähler and quasi-projective group are defined similarly.

Let \mathcal{K} , \mathcal{P} , \mathcal{QK} , and \mathcal{QP} be the respective classes of groups. Clearly, $\mathcal{P} \subseteq \mathcal{K}$ and $\mathcal{QP} \subseteq \mathcal{QK}$, though it is not known whether these inclusions are strict. Of course, $\mathcal{K} \subseteq \mathcal{QK}$ and $\mathcal{P} \subseteq \mathcal{QP}$, but both inclusions are strict: for example, $\mathbb{Z} = \pi_1(\mathbb{C}^*)$ is in \mathcal{QP} , but not in \mathcal{K} . It is readily seen that each of these classes of groups is closed under finite direct products.

Problem 8.1 (J.-P. Serre). Classify Kähler, projective, quasi-Kähler, and quasi-projective groups.

This appears to be a difficult problem. As shown by Serre [43], all finite groups are projective. From the above discussion, it follows that all finitely generated abelian groups are quasi-projective. To the best of our knowledge, the case of nilpotent groups is open.

8.2 Pencils

A good reason for grappling with Problem 8.1 is the fact that the fundamental group of a quasi-Kähler manifold X determines the *pencils* (or, admissible maps) on X .

Following Arapura [1, p. 590], we say that a map $f: X \rightarrow C$ to a connected, smooth complex curve C is *admissible* if f is holomorphic and surjective, and has a holomorphic, surjective extension with connected fibers to smooth compactifications, $\bar{f}: \bar{X} \rightarrow \bar{C}$, obtained by adding divisors with normal crossings. (In particular, the generic fiber of f is connected, and the induced homomorphism, $f_{\#}: \pi_1(X) \rightarrow \pi_1(C)$, is onto.) Two such maps, $f: X \rightarrow C$ and $f': X \rightarrow C'$, are said to be equivalent if there is an isomorphism $\psi: C \rightarrow C'$ such that $f' = \psi \circ f$. The pencil f is called of *general type* if $\chi(C) < 0$.

Theorem 8.2 (Arapura [1]). *Let X be a quasi-Kähler manifold, with fundamental group G . There is a bijection between the set of positive-dimensional, non-translated components of $\mathcal{V}_1(G)$, and the set of equivalence classes of pencils of general type, $f: X \rightarrow C$. This bijection associates to f the component $S_f = f^*(\mathbb{T}(C))$, a connected subtorus of $\mathbb{T}(G)$.*

The definition below, extracted from [17], plays an important role in the applications.

Definition 8.3. Let G be a finitely generated group. A vector subspace $U \subseteq H^1(G)$ is called *0-isotropic* if the restriction of \cup_G to $U \wedge U$ is trivial. Likewise, U is called *1-isotropic* if this restriction is a non-degenerate pairing, with 1-dimensional image.

Example 8.4. Let C be a complex curve with $\chi(C) < 0$. Then $H^1(C)$ is either 1- or 0-isotropic, according to whether C is compact or not.

Regarding the BNS invariant, here is a geometric counterpart to Theorem 7.4, based on Theorems 8.2 and 7.1, as well as a recent result of Delzant [11].

Theorem 8.5 ([39]). *Let X be a compact Kähler manifold with $b_1(X) > 0$, and let $G = \pi_1(X)$. Then $\Sigma^1(G) = H^1(G, \mathbb{R}) \setminus \mathcal{R}_1(G)$ if and only if there is no pencil on X onto an elliptic curve, having multiple fibers.*

8.3 Position obstructions

Theorems 7.1 and 8.2 lead to the following *position obstruction*, related to Serre’s problem. Note that this obstruction depends only on μ_G .

Theorem 8.6 ([17]). *Let G be a quasi-Kähler, 1-formal group. Then every positive-dimensional irreducible component of $\mathcal{R}_1(G)$ is p -isotropic with respect to \cup_G , and has dimension at least $2p + 2$, for some $p \in \{0, 1\}$.*

This theorem is a particular case of a more general result. In [17, Theorem C], two position obstructions were found, valid for an arbitrary quasi-Kähler group G , formulated in terms of the first characteristic variety $\mathcal{V}_1(G)$. The second obstruction leads to a powerful restriction on the *multivariable Alexander polynomial*, $\Delta^G \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

Theorem 8.7 ([14]). *If G is a quasi-Kähler group with $n := b_1(G) \neq 2$, then Δ^G has a single essential variable, that is, $\Delta^G(t_1, \dots, t_n) = P(t_1^{e_1} \cdots t_n^{e_n})$, for some polynomial $P(t) \in \mathbb{Z}[t^{\pm 1}]$.*

Theorem 8.6 leads to the following complete classification of (quasi-) Kähler groups within the classes of right-angled Artin groups and Bestvina–Brady groups introduced in §§6.3–6.4.

Theorem 8.8 ([17]). *Let Γ be a finite simple graph, and G_Γ the corresponding right-angled Artin group. Then:*

1. $G_\Gamma \in \mathcal{QK} \iff G_\Gamma \in \mathcal{QP} \iff \Gamma$ is a complete multipartite graph $K_{n_1, \dots, n_r} = \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$.
2. $G_\Gamma \in \mathcal{K} \iff G_\Gamma \in \mathcal{P} \iff \Gamma$ is a complete graph K_n , with n even.

Note that $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$ when $\Gamma = K_{n_1, \dots, n_r}$, and $G_\Gamma = \mathbb{Z}^n$ when $\Gamma = K_n$.

Theorem 8.9 ([13]). *Let Γ be a finite simple graph, and N_Γ the corresponding Bestvina–Brady group. Then:*

1. $N_\Gamma \in \mathcal{QK} \iff N_\Gamma \in \mathcal{QP} \iff \Gamma$ is either a tree, or $\Gamma = K_{n_1, \dots, n_r}$, with some $n_i = 1$, or all $n_i \geq 2$ and $r \geq 3$.
2. $N_\Gamma \in \mathcal{K} \iff N_\Gamma \in \mathcal{P} \iff \Gamma = K_n$, with n odd.

8.4 Applications to 3-manifolds

The position obstruction from Theorem 8.6 also turns out to be very efficient at determining which (quasi-) Kähler groups occur as fundamental groups of closed 3-manifolds.

Theorem 8.10 ([18]). *Let G be the fundamental group of a closed 3-manifold. Then G is a Kähler group if and only if G is a finite subgroup of $O(4)$, acting freely on S^3 .*

Theorem 8.11 ([16]). *Let G be the fundamental group of a closed, orientable 3-manifold. Then (up to Malcev completion) G is a quasi-Kähler, 1-formal group if and only if $G = F_n$, or $G = \mathbb{Z} \times \pi_1(\Sigma_g)$.*

In the case of boundary manifolds of line arrangements in \mathbb{CP}^2 , more can be said. Let \mathcal{A} be such an arrangement, and let M be the closed, orientable 3-manifold obtained by taking the boundary of a regular neighborhood of \mathcal{A} in \mathbb{CP}^2 . Theorem 7.1 was used in [9, Theorem 9.7] to classify those boundary manifolds which are formal, while Theorem 8.7 was used in [14, Proposition 4.7] to classify those boundary manifolds whose fundamental groups are quasi-projective. We summarize these results, as follows.

Theorem 8.12 ([9, 14]). *Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be an arrangement of lines in $\mathbb{C}\mathbb{P}^2$, and let M be the corresponding boundary manifold. The following are equivalent:*

1. *The manifold M is formal.*
2. *The group $G = \pi_1(M)$ is 1-formal.*
3. *The group G is quasi-projective.*
4. *\mathcal{A} is either a pencil or a near-pencil.*

The corresponding 3-manifolds are easy to describe: if \mathcal{A} is a pencil, then $M = \#^n S^1 \times S^2$, and if \mathcal{A} is a near-pencil, then $M = S^1 \times \Sigma_{n-1}$.

9 Formality and the monodromy action

We go on and examine the interplay between algebraic monodromy and 1-formality, with emphasis on extensions of \mathbb{Z} , and fibrations over S^1 . In this section, all manifolds are compact and connected.

9.1 Artin kernels

Let Γ be a finite simple graph. Every epimorphism $\chi: G_\Gamma \rightarrow \mathbb{Z}$ from the right-angled Artin group G_Γ to the integers gives rise to an *Artin kernel*, $N_\chi = \ker(\chi)$, generalizing the Bestvina–Brady group $N_\Gamma = \ker(\nu)$ from §6.4.

In [37], we found a combinatorial procedure which characterizes the triviality of the monodromy action of \mathbb{Z} on the homology groups of N_χ , up to a fixed degree q . As a result, we were able to establish the 1-formality of a large class of Artin kernels.

Theorem 9.1 ([37]). *If the monodromy action on $H_*(N_\chi)$ is trivial, up to degree 2, then the Artin kernel N_χ is a 1-formal group.*

For the Bestvina–Brady groups N_Γ , this triviality test boils down to verifying that $\tilde{H}_i(\Delta_\Gamma) = 0$, for $i \leq 1$. In the case when the flag complex Δ_Γ is simply-connected, we recover Theorem 6.7.

Remark 9.2. The above theorem may be regarded as a complement to §6.2, indicating a new formality-preserving construction: the passage from 1-formal groups to normal subgroups with infinite cyclic quotient, under certain triviality assumptions on the monodromy action. That the assumption on the monodromy is key to preserving formality is illustrated by the 1-formal group $H = G \rtimes_\varphi \mathbb{Z}$ from Example 6.5. In that case, we have an epimorphism $\chi: H \rightarrow \mathbb{Z}$, with kernel the (non-1-formal) Heisenberg group G , and non-trivial monodromy action on $H_1(G)$.

Theorem 9.1 enabled us to construct what seem to be the first instances of 1-formal groups which are *not* finitely presented. The example below is taken from [37, Example 10.3].

Example 9.3. Let $L = \Delta_\Gamma$ be a flag triangulation of the real projective plane, $\mathbb{R}\mathbb{P}^2$. Clearly, L is connected. On the other hand, $H_1(L, \mathbb{Z}) = \mathbb{Z}_2$, and so, by [3], N_Γ is not finitely presented. But $H_1(L, \mathbb{Q}) = 0$, and so, by Theorem 9.1, N_Γ is 1-formal.

9.2 Jordan blocks

We examine now the converse question: does 1-formality impose restrictions on the algebraic monodromy? In [38, Proposition 9.4], we obtained a general result, relating the monodromy action in extensions of \mathbb{Z} to the resonance varieties, without any formality assumptions. We used this to deduce in [40] the following implication of 1-formality on the algebraic monodromy.

Let U be a closed manifold, and let $h: U \rightarrow U$ be a diffeomorphism. Denote by U_h the mapping torus of h .

Theorem 9.4 ([40]). *Let $p: M \rightarrow U_h$ be a locally trivial smooth fibration. Assume M is 1-formal. Then, if the monodromy operator $h_*: H_1(U) \rightarrow H_1(U)$ has eigenvalue 1, all corresponding Jordan blocks must have size 1.*

This theorem substantially extends a result of Fernández, Gray, and Morgan [22], valid only for circle bundles over U_h . Those authors proved their result by a different method, which relies on Massey products.

Example 9.5. The 1-formality hypothesis is crucial here, even in the particular case when $M = U_h$ and $p = \text{id}$. For instance, the Heisenberg manifold from Example 2.4 is not 1-formal, yet it fibers over S^1 , with fiber the 2-torus and monodromy $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$.

Example 9.6. Let $f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of a reduced polynomial function, with associated singularity link K , and Milnor fibration $p: S^3 \setminus K \rightarrow S^1$. A basic fact in singularity theory is that the algebraic monodromy of p has no Jordan block of size greater than 1, for the eigenvalue 1. As shown by Durfee and Hain [19], the link complement is a formal space. Hence, the result on the algebraic monodromy follows from Theorem 9.4, applied to the link exterior.

10 Epilogue: beyond formality

We conclude with a class of examples inspired by the work of Geiges on 2-torus bundles over the 2-torus [26]. These examples illustrate some of the subtle interplay between monodromy and geometric structures on manifolds.

10.1 Mapping tori and Kähler metrics

Let $U = \Sigma_g$ be a compact Riemann surface of genus $g \geq 1$, and let $h: U \rightarrow U$ be an orientation-preserving diffeomorphism (which may be viewed as an element of the mapping class group \mathcal{M}_g).

As before, denote by U_h the mapping torus of h . By construction, this is a closed, orientable 3-manifold fibering over the circle, with fiber U and monodromy h .

Question 10.1. Suppose 1 is not an eigenvalue of the algebraic monodromy operator, $h_*: H_1(U) \rightarrow H_1(U)$. Does there exist a closed, connected manifold N such that $N \times U_h$ carries a Kähler metric?

In general, a question of this type has a negative answer. Using a deep classification result of Wall [47] concerning complex structures on 4-manifolds, Geiges showed in [26, p. 555] that the manifold $S^1 \times (T^2)_h$, where $h = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, supports no Kähler metric.

On the other hand, a negative answer to Question 10.1 cannot be obtained solely by rational homotopy methods. The next result makes it clear why.

Proposition 10.2. *With the setup from Question 10.1, suppose $N = S^1 \times M$, where M is a compact Kähler manifold. Then $N \times U_h$ has the same \mathbb{Q} -homotopy type as a Kähler manifold, namely, $M \times T^2 \times S^2$.*

Proof: Recall we are assuming 1 is not an eigenvalue of $h_*: H_1(U) \rightarrow H_1(U)$, that is, $\text{id} - h_*$ is an isomorphism. From the Wang sequence of the fibration $U \rightarrow U_h \rightarrow S^1$, we infer that $b_1(U_h) = 1$. Now, since U_h is a closed, orientable 3-manifold, Theorem 4.2 implies that U_h is formal, and has the same \mathbb{Q} -homotopy type as $S^1 \times S^2$. Our claim follows at once. \square

10.2 Eigenvalues of the monodromy operator

For a space X , recall $\mathbb{T}(X)$ denotes the character group $\text{Hom}(\pi_1(X), \mathbb{C}^*)$. Let $\mathbb{T}^0(X)$ be the connected component of $1 \in \mathbb{T}(X)$.

Returning to the setting of Question 10.1, let $p: U_h \rightarrow S^1$ be the canonical fibration of the mapping torus. Since $b_1(U_h) = 1$, the induced homomorphism, $p^*: H^1(S^1, \mathbb{C}^*) \rightarrow H^1(U_h, \mathbb{C}^*)$, may be used to identify $\mathbb{T}^0(U_h)$ with $\mathbb{T}(S^1) = \mathbb{C}^*$.

Lemma 10.3. *Under the above identification, the variety $\mathcal{V}_1(U_h) \cap \mathbb{T}^0(U_h) \subset \mathbb{C}^*$ consists of 1, together with the eigenvalues of h_* .*

Proof: Fix a character $\rho \in \mathbb{T}(S^1) = \mathbb{C}^*$. Consider the Leray-Serre spectral sequence of the fibration $U \hookrightarrow U_h \xrightarrow{p} S^1$, with coefficients in the local system determined by ρ :

$$E_{s,t}^2 = H_s(S^1, H_t(U, \mathbb{C})) \Rightarrow H_{s+t}(U_h, \mathbb{C}_{p^*(\rho)}),$$

where $\pi_1(S^1)$ acts on $H_t(U, \mathbb{C})$ by $\rho^{-1} \cdot h_*$. In total degree 1, we obtain an isomorphism

$$H_1(U_h, \mathbb{C}_{p^*(\rho)}) \cong H_1(S^1, \mathbb{C}_\rho) \oplus \text{coker}(h_* - \rho \cdot \text{id}).$$

Now, $H_1(S^1, \mathbb{C}_\rho) = 0$ or \mathbb{C} , according to whether $\rho \neq 1$ or $\rho = 1$. The conclusion follows. □

10.3 Quasi-Kähler manifolds and monodromy

For a quasi-Kähler manifold X , a powerful result, due to Arapura [1], guarantees that the isolated points of $\mathcal{V}_1(X)$ must be unitary characters. Within the realm of groups we consider here, this leads to a simple-to-verify obstruction for membership in \mathcal{QK} .

Proposition 10.4. *Let N be a compact, connected manifold, and suppose $\pi_1(N \times U_h)$ is a quasi-Kähler group. Then all eigenvalues of the monodromy operator, $h_*: H_1(U) \rightarrow H_1(U)$, have norm 1.*

Proof: By the Künneth formula,

$$\mathcal{V}_1(N \times U_h) = 1 \times \mathcal{V}_1(U_h) \cup \mathcal{V}_1(N) \times 1.$$

By Lemma 10.3, each eigenvalue ρ of h_* gives rise to the isolated point $1 \times \rho$ of $\mathcal{V}_1(N \times U_h)$. In view of the aforementioned result of Arapura, this finishes the proof. □

It is an easy matter to construct elements $h \in \mathcal{M}_g$ such that h_* has no eigenvalue of norm 1.

Example 10.5. Pick a matrix $A \in \text{SL}(2, \mathbb{Z})$ with $|\text{tr}(A)| \geq 3$. Then A has two distinct, non-unitary eigenvalues, say, λ_1 and λ_2 . Let B be the block-sum of g copies of A . Clearly, B belongs to $\text{Sp}(2g, \mathbb{Z})$, and has the same eigenvalues as A . By a classical result, there exists a diffeomorphism $h: U \rightarrow U$ (necessarily, orientation-preserving), such that $h_*: H_1(U) \rightarrow H_1(U)$ has matrix B .

Proposition 10.4 may be applied to give a negative answer to Question 10.1, even when rational homotopy methods are inconclusive.

Corollary 10.6. *Let $U = \Sigma_g$ and let $h: U \rightarrow U$ be a diffeomorphism as constructed in Example 10.5. Then, for any closed manifold N , the group $\pi_1(N \times U_h)$ is not a quasi-Kähler group. In particular, the manifold $N \times U_h$ does not carry any Kähler metric.*

Putting things together, we can now prove the result stated in the Introduction.

Proof: [Proof of Theorem 1.1] With notation as in the previous corollary, let $W = S^1 \times U_h$. Clearly, W is a closed, orientable, formal 4-manifold. Let M be an arbitrary compact Kähler manifold, and write $N = S^1 \times M$. By Proposition 10.2, $M \times W = N \times U_h$ has the \mathbb{Q} -homotopy type of a Kähler manifold, namely, $M \times T^2 \times S^2$. On the other hand, by Corollary 10.6, $M \times W$ admits no Kähler metric.

It remains to produce infinitely many manifolds W as above. For that, choose a surface $U = \Sigma_g$ and matrices $A_n = \begin{pmatrix} n+2 & -1 \\ 1 & 0 \end{pmatrix}$, with $n > 1$, as input for the construction. Denote by $B_{g,n}$ the block-sum of g copies of A_n , and let $W_{g,n}$ be the resulting 4-manifold. A straightforward computation shows that $H_1(W_{g,n}, \mathbb{Z}) = \mathbb{Z}^2 \oplus \bigoplus^g \mathbb{Z}/n\mathbb{Z}$. Hence, the manifolds $\{W_{g,n}\}_{g \geq 1, n > 1}$ are pairwise distinct. \square

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