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Minimizers of a generalized Yang-Mills functional

by
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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

We prove that any absolute minimum of the Yang-Mills functional in a certain space of connections is also a minimizer of the generalized gauge invariant functional defined in [2].

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Let E be a vector bundle with structure group G over a compact Riemannian manifold (M, g). We assume that $G \subset O(m)$ is a compact Lie group and that E carries an inner product compatible with G. A connection ∇ on the vector bundle E is called a G-connection if the natural extension of ∇ to tensor bundles of E annihilates the tensors which define the G-structure. We denote by $\mathcal{C}(E)$ the space of all smooth G-connections ∇ on E.

To each connection $\nabla \in \mathcal{C}(E)$ a curvature 2-form R^∇ is associated, and at each point $x \in M$ we can take its norm

$$\|R^\nabla\|_x^2 = \sum_{i<j} \|R^\nabla_{e_i,e_j}\|_x^2.$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_xM and the norm of $R^\nabla_{e_i,e_j}$ is the usual one, namely $<A, B>= \mathrm{tr}(A^t \circ B)$. For more detailes see [1].

Definition 1. *For any function $f : [0,\infty) \to [0,\infty)$ of class C^2, such that $f'(t) > 0$ for any $t \geq 0$, a generalized Yang-Mills functional is the mapping $YM_f : \mathcal{C}(E) \to \mathbb{R}$ given by (see [2])*

$$YM_f(\nabla) = \int_M f(\frac{1}{2}\|R^\nabla\|^2)\vartheta_g.$$

We note that if $f(t) = t$ the functional above is the clasical Yang-Mills functional and if $f(t) = \exp(t)$ the functional is the exponential Yang-Mills (see [3]). A critical point of YM_f will be called an f-Yang-Mills connection.

In [2] the following result is proven:

Theorem 1. *Let (M, g) be an n-dimensional compact Riemannian manifold, G a compact Lie group, and E a smooth G-vector bundle over M. Assume that $n \geq 5$ and $f''(0) \neq 0$ and let ∇ be a Yang-Mills connection. Then there exists a Riemannian metric $\tilde{g}$ on M conformally equivalent to g such that ∇ is a critical point of the functional YM_f.*

Now we look for minimizers of the functional YM_f.

A function $f : \mathbb{R}^p \to \mathbb{R}$ is called convex if for all $x, y \in \mathbb{R}^p$ and $0 \le \lambda \le 1$,

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y).$$

The following Jensen's inequality is wellknown:

Proposition 1. *Let f be a convex function on $\mathbb{R}^p$, S a set with $\mu(S) < \infty$, μ a non-negative bounded measure on S and $\mathcal{L}^1(S, \mu)$ the space of all integral measurable functions on S with respect to μ. Then for any $\xi_i \in \mathcal{L}^1(S, \mu)$, $1 \le i \le p$,*

$$f(\zeta^1, \ldots, \zeta^p) \le \frac{1}{\mu(S)} \int_S f(\xi_1(x), \ldots, \xi_p(x)) d\mu,$$

where

$$\zeta^i := \frac{1}{\mu(S)} \int_S \xi_i(x) d\mu.$$

The equality holds if and only if ξ_i is constant almost everywhere.

If we fix a G-connection ∇^0 of E, we define for any $1 < p < \infty$ the Sobolev space of $\mathcal{L}_1^p$ G-connections by

$$\mathcal{L}_1^p(E) = \{\nabla = \nabla^0 + A | A \in \mathcal{L}_1^p(T^\star M \otimes g_E)\},$$

where $\mathcal{L}_1^p(T^\star M \otimes g_E)$ is the completion of $\Omega^1(g_E)$ with respect to the norm

$$\|A\|_{1,p} = \left(\int_M \|\nabla A\|^p \vartheta_g \right)^{1/p} + \left(\int_M \|A\|^p \vartheta_g \right)^{1/p}.$$

Define also the $\mathcal{L}^p$ space of G-connections of E by

$$\mathcal{L}^p(E) = \{\nabla = \nabla^0 + A | A \in \mathcal{L}^p(T^\star M \otimes g_E)\},$$

where $\mathcal{L}^p(T^\star M \otimes g_E)$ is the completion of $\Omega^1(g_E)$ with respect to the norm

$$\|A\|_p = \left(\int_M \|A\|^p \vartheta_g \right)^{1/p}.$$

Finally we define the space

$$\mathcal{W}(E) = \bigcap_{p \ge 1} \mathcal{L}_1^p(E) \cap \{\nabla | YM_f(\nabla) < \infty\}.$$

Theorem 2. *Let ∇^0 be a minimizer in $\mathcal{W}(E)$ of the Yang-Mills functional YM such that the norm of the curvature $\|R^{\nabla^0}\|$ is almost everywhere constant. If we suppose that the function f is convex, then ∇^0 is also a minimizer of the functional YM_f and for any minimizer ∇' of the functional YM_f in $\mathcal{W}(E)$, the norm $\|R^{\nabla'}\|$ is almost everywhere constant.*

Proof: First we notice that for any $\nabla \in \mathcal{W}(E)$ we have

$$f\left(\frac{1}{\mathrm{vol}(M,g)} YM(\nabla) \right) \le \frac{1}{\mathrm{vol}(M,g)} YM_f(\nabla),$$

and the equality holds if and only if $\|R^\nabla\|$ is almost everywhere constant. Indeed as the connection ∇ satisfies $YM_f(\nabla) < \infty$ then $\frac{1}{2}\|R^\nabla\|$ belongs to the space $\mathcal{L}^p(M)$ of all integrable functions on M

with respect to the canonical volume element ϑ_g. Since the function f is convex we can use Jensen's inequality.

Now for any $\nabla' \in \mathcal{W}(E)$, from the monotonicity of the function f and using the previous remark we get

$$f\left(\frac{1}{\mathrm{vol}(M,g)} YM(\nabla^0)\right) \leq f\left(\frac{1}{\mathrm{vol}(M,g)} YM(\nabla')\right) \leq \frac{1}{\mathrm{vol}(M,g)} YM_f(\nabla'). \qquad (1)$$

Then we obtain

$$f\left(\frac{1}{\mathrm{vol}(M,g)} YM(\nabla^0)\right) \leq \inf_{\nabla' \in \mathcal{W}(E)} \frac{1}{\mathrm{vol}(M,g)} YM_f(\nabla').$$

On the other hand, since $\|R^{\nabla^0}\|$ is almost everywhere constant, we obtain:

$$\frac{1}{\mathrm{vol}(M,g)} YM_f(\nabla^0) = \frac{1}{\mathrm{vol}(M,g)} \int_M f(\frac{1}{2}\|R^{\nabla^0}\|^2)\vartheta_g =$$

$$= f(\frac{1}{2}\|R^{\nabla^0}\|^2) = f\left(\frac{1}{\mathrm{vol}(M,g)} YM(\nabla^0)\right),$$

and thus ∇^0 is also a minimizer of the functional YM_f.

On the other hand if we assume that ∇' is any minimizer of the functional YM_f then the second inequality of (1) is in fact equality and thus $\frac{1}{2}\|R^{\nabla'}\|$ is constant almost everywhere. □

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