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Remarks on Weyl Curvature in Almost Kähler Geometry

by

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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

We give a slightly different presentation of some recent results of Kirchberg on the integrability of compact almost Kähler manifolds and obtain an extension of a theoreom of Kirchberg in dimension 4.

Key Words: Almost Kähler manifold, conformal scalar curvature, Goldberg conjecture, Weyl curvature tensor.
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It is great pleasure for us to dedicate this article to our good friend Professor Stere Ianuş. Professor Ianuş' enthusiasm and dedication to mathematics has been an inspiration to us for many years.

1 Introduction and the Goldberg Conjecture

In 1969 S. I. Goldberg proved that if the curvature operator of an almost Kähler manifold commutes with the almost complex structure, then the manifold is Kähler [13] and conjectured that a compact almost Kähler Einstein manifold is Kähler. In full generality the conjecture is still open, but many partial results have been achieved. Under the additional assumption of non-negative scalar curvature the Goldberg conjecture was proved by K. Sekigawa in 1987 [24]. Without the assumption of compactness the conjecture is false; this already follows from a 1970 paper of Alekseevskii [1], but this went unnoticed for a long time. More recent non-compact counterexamples have been obtained: Ricci flat 4-dimensional examples in [19], [7], [2], and homogeneous examples in all dimensions $2n \geq 6$ [4].

Several works have considered natural ways of extending the Goldberg conjecture by relaxing the Einstein condition. One such extension was suggested by Professor Ianus and the first author [9]: instead of the Einstein assumption, ask only that the Ricci tensor commute with the almost complex structure. There is still not known any 4-dimensional example of a compact almost Kähler, non-Kähler manifold with J-invariant Ricci tensor. In dimension 6 and higher such examples do exist, as follows from the work of Davidov and Muškarov [12].

Recently there has been some interest in other curvature conditions that imply that an almost Kähler structure is Kählerian, especially questions involving the Weyl conformal curvature tensor. This is natural in view of the fact that Sekigawa's proof of the Goldberg conjecture in the case of non-negative scalar curvature involves the Pontrjagin classes and the Pontrjagin classes are conformal invariants [14]. In particular Satoh [23] and Kirchberg [16] have observed that Sekigawa's theorem can be generalized by replacing the Einstein condition with weaker assumptions on the Weyl and Ricci curvature of the almost Kähler structure. The goal of this note is to give a slightly different presentation of some results of Kirchberg from [16]. As a byproduct, we will obtain in dimension 4 an extension of Theorem 3.7 of [16] (see Theorem 3.9 in Section 3).

In contrast to the fact that there are no non-Kähler almost Kähler manifolds of constant curvature (see [22], [20], [7]), there are non-compact conformally flat non-Kähler almost Kähler manifolds, in particular $H^3 \times \mathbb{R}$, H^3 being hyperbolic 3-space [21]. Further non-compact examples and discussion can be found in [11].

2 Notations and Preliminaries

Let (M^{2n}, g, J, ω) be an almost Hermitian manifold with almost complex structure J, compatible metric g and fundamental 2-form $\omega(X,Y) = g(JX,Y)$. The manifold is *almost Kähler* if ω is closed (hence symplectic). Our convention for the curvature tensor is

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \ .$$

Using the metric to raise or lower indices, instead of the classical $(1,3)$ curvature tensor, we'll think of R as a $(0,4)$ tensor, or, most often, as a section of the bundle $S^2(\Lambda^2 M)$.

The Ricci tensor and $*$-Ricci tensor are then given by

$$\mathrm{Ric}(X,Y) = < R_{Xe_k}Y, e_k >, \quad \mathrm{Ric}_*(X,Y) = < R_{JXJe_k}Y, e_k >$$

where the e_k's denote a local orthonormal basis and the repeated index indicates a sum over the basis. The traces of these Ricci tensors define the scalar curvature s and the $*$-scalar curvature s^*, respectively. The Ricci tensor admits a decomposition into J-invariant and J-anti-invariant parts given by

$$\mathrm{Ric}'(X,Y) = \frac{1}{2}\Big(\mathrm{Ric}(X,Y) + \mathrm{Ric}(JX,JY)\Big),$$

$$\mathrm{Ric}''(X,Y) = \frac{1}{2}\Big(\mathrm{Ric}(X,Y) - \mathrm{Ric}(JX,JY)\Big) \ .$$

The $*$-Ricci tensor is, in general, not symmetric, but satisfies the relation

$$Ric_*(JX,JY) = Ric_*(Y,X) \ .$$

This allows the definition of a 2-form ρ_* by

$$\rho_*(X,Y) = \mathrm{Ric}_*(JX,Y) \ ,$$

which is called the $*$-Ricci form of the almost Hermitian structure. We also define the Ricci form ρ by

$$\rho(X,Y) = \mathrm{Ric}'(JX,Y) \ .$$

The Ricci form ρ is, by definition, J-invariant, the $*$-Ricci form ρ_* in general is not, and we denote its J-invariant and J-anti-invariant parts by the use of $'$ and $''$, respectively. The Weitzenböck formula applied to the harmonic form ω implies the well known relation between the Ricci forms of an almost Kähler structure

$$\rho_* - \rho = \frac{1}{2}\nabla^*\nabla\omega \ . \tag{1}$$

Taking the trace of (1), one gets the difference of the two types of scalar curvatures

$$s^* - s = |\nabla\omega|^2 = \frac{1}{2}|\nabla J|^2 \ . \tag{2}$$

Note that we use the convention that the point-wise norm on the bundle $\Lambda^2 M$ has a factor of $1/2$ compared with the norm on $\Lambda^1 M \otimes \Lambda^1 M$. Consequently, as we most often think of the curvature tensor R as a section of the bundle $S^2(\Lambda^2 M)$, its point-wise norm is defined by

$$|R|^2 = \frac{1}{4}\sum (R_{ijkl})(R_{ijkl}) \ .$$

The well known orthogonal decomposition of the curvature (for dimension $2n$) is given by (see for instance [8])

$$R = \frac{s}{4n(2n-1)} g \owedge g + \frac{1}{(2n-2)}(\mathrm{Ric}_0 \owedge g) + W \ , \tag{3}$$

where the notations are the traditional ones, that is:

• W denotes the Weyl curvature tensor and relation (3) can be used as its definition (as a (0,4) tensor).

• $Ric_0 = Ric - \frac{s}{2n} g$ is the trace free part of the Ricci tensor;

• $\owedge$ denotes the Kulkarni-Nomizu product; if h and k are symmetric 2-tensors, then $h \owedge k$ is the curvature type 4-tensor given by

$$(h \owedge k)_{XYZT} = h_{XZ}k_{YT} + h_{YT}k_{XZ} - h_{YZ}k_{XT} - h_{XT}k_{YZ}.$$

The existence of the compatible almost complex structure greatly refines the decomposition (3). The reader can consult [25]. One particular $U(n)$-component of the curvature will appear in our formulae, so we define it below.

$$W''_{XYZT} = \frac{1}{8}\Big(W_{XYZT} - W_{JXJYZT} - W_{XYJZJT} + W_{JXJYJZJT}$$
$$-W_{XJYZJT} - W_{JXYZJT} - W_{XJYJZT} + W_{JXYJZT}\Big). \tag{4}$$

It is the part of the Weyl tensor which acts like an endomorphism on the bundle of J-anti-invariant 2-forms and anti-commutes with the action of J on this bundle, hence the notation.

We denote by $d^\nabla : \Lambda^k M \otimes \Lambda^j M \to \Lambda^{k+1} M \otimes \Lambda^j M$ the differential on $\Lambda^j M$-valued k-forms, $j = 1, 2$, defined using the Levi-Civita connection ∇; δ^∇ denotes the adjoint of d^∇.

Thinking of the curvature and the Weyl tensors R and W as $\Lambda^2 M$-valued 2-forms, the differential Bianchi relation can be written as

$$\delta^\nabla R = -d^\nabla Ric \ , \qquad \delta^\nabla W = -\frac{2n-3}{2n-2} d^\nabla (Ric - \frac{1}{4n-2} sg) \ , \tag{5}$$

where in the right hand-side a symmetric 2-tensor b is seen as a $\Lambda^1 M$-valued 1-form and $d^\nabla b \in \Lambda^2 M \otimes \Lambda^1 M \simeq \Lambda^1 M \otimes \Lambda^2 M$ is defined by

$$(d^\nabla b)_{AB}(X) = (\nabla_A b)(B, X) - (\nabla_B b)(A, X) \ , \forall A, B, X \in TM.$$

The tensor $d^\nabla \delta^\nabla R$ will be important in what follows. Applying the differential $d^\nabla : \Lambda^1 M \otimes \Lambda^2 M \to \Lambda^2 M \otimes \Lambda^2 M$ in (5), we get different equivalent expressions for it

$$d^\nabla \delta^\nabla R = -d^\nabla (d^\nabla Ric) = \frac{2n-2}{2n-3} d^\nabla \delta^\nabla W - \frac{1}{4n-2} d^\nabla (d^\nabla (sg)) \ . \tag{6}$$

Additional matters of terminology will be discussed as they occur.

3 Sekigawa's integral formula

The original approach of Sekigawa [24] was based on Chern-Weil theory. On any almost Kähler manifold (M^{2n}, g, J, ω), one has two natural connections: the Levi-Civita connection ∇ and the canonical Hermitian connection, also called the Chern connection

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y$$

although it was originally defined by Licherowicz [17, 18]. By Chern-Weil theory, the first Pontrjagin class $p_1(M)$ has 4-forms representatives $p_1(\nabla)$, $p_1(\widetilde{\nabla})$ with respect to either of the two connections. Then the difference $p_1(\nabla) - p_1(\widetilde{\nabla})$ is an exact 4-form, so by Stokes theorem, it follows that

$$\int_M \left(p_1(\nabla) - p_1(\widetilde{\nabla}) \right) \wedge \omega^{n-2} = 0.$$

By explicitly computing the terms involved in the difference $p_1(\nabla) - p_1(\widetilde{\nabla})$, Sekigawa obtained an integral formula which yielded the result.

A point-wise version of Sekigawa's formula has been obtained in [4], Proposition 2.1. We restate below this result, pointing out the interesting fact, first noted by Kirchberg, that there are actually two potentially useful formulae whose combination yields the pointwise version of Sekigawa's formula.

Proposition 3.1. *On any almost Kähler manifold (M^{2n}, g, J, ω), the following formulae hold:*

$$\begin{aligned} \langle d^\nabla \delta^\nabla R, \omega \otimes \omega \rangle \quad &= \quad \frac{1}{2}\Delta s - 2\delta(J\delta(J\mathrm{Ric}'')) + |\mathrm{Ric}''|^2 \\ &\quad + 2\langle \rho, \phi \rangle - \langle \rho, \nabla^* \nabla \omega \rangle \ ; \end{aligned} \tag{7}$$

$$\begin{aligned} \langle d^\nabla \delta^\nabla R, \omega \otimes \omega \rangle \quad &= \quad \frac{1}{2}\Delta s^* - 4\delta(\langle \rho''_*, \nabla . \omega \rangle) + 4|W''|^2 \\ &\quad + \frac{1}{2}|\phi|^2 + \frac{1}{2}|\nabla^* \nabla \omega|^2 + \langle \rho, \nabla^* \nabla \omega \rangle \ , \end{aligned} \tag{8}$$

where ϕ denotes the J-invariant 2-form $\phi(X, Y) = \langle \nabla_{JX}\omega, \nabla_Y \omega \rangle$.

Note that the equality of the right hand-sides of (7) and (8) gives the formula in the statement of Proposition 2.1 in [4], which is the point-wise version of Sekigawa's formula.

Sketch of the Proof: The two formulae correspond to different ways of computing the the quantity $< d^\nabla \delta^\nabla R, \omega \otimes \omega >$. For the first, we use the second Bianchi identity to express $d^\nabla \delta^\nabla R$ in terms of the Ricci tensor (first equality of (6)). This gives

$$< d^\nabla \delta^\nabla R, \omega \otimes \omega >= -(\nabla^2_{ik} Ric_{jl}) J_{ij} J_{kl},$$

where, as usual the repeated indicies sum. After a relatively long computation of getting the J's inside the derivatives, one eventually obtains (7).

For the second formula, one uses the Weitzenböck formula for the curvature tensor due to Bourguignon [10]

$$d^\nabla \delta^\nabla R = \nabla^* \nabla R + c(R, R), \tag{9}$$

and then takes the inner product of both sides with $\omega \otimes \omega$. For the explicit expression of the quadratic in curvature $c(R, R)$ and further details on computation see [4]. □

On a compact almost Kähler manifold Kirchberg [16] introduces the quantity

$$Q(J) = -\int_M \langle d^\nabla \delta^\nabla R, \omega \otimes \omega \rangle \, dV. \tag{10}$$

Note that the definition of $Q(J)$ can be given in terms of the second derivatives of the Ricci tensor (as Kirchberg originally did)

$$Q(J) = \int_M \langle d^\nabla (d^\nabla Ric), \omega \otimes \omega \rangle \, dV, \tag{11}$$

or in terms of the Weyl tensor

$$Q(J) = -\int_M \frac{2(n-1)}{2n-3} \langle d^\nabla \delta^\nabla W, \omega \otimes \omega \rangle \, dV. \tag{12}$$

From relation (6), we have

$$\langle d^\nabla \delta^\nabla R, \omega \otimes \omega \rangle = \frac{2n-2}{2n-3} \langle d^\nabla \delta^\nabla W, \omega \otimes \omega \rangle + \frac{1}{2(2n-1)} \Delta s \ , \tag{13}$$

so, via (6) and (13), it is clear that the three definitions above for $Q(J)$ are equivalent. Integrating relations (7) and (8), we get the following integral formulae:

Proposition 3.2. *On any compact almost Kähler manifold (M^{2n}, g, J, ω), the following formulae hold:*

$$-Q(J) = \int_M \Big(|\mathrm{Ric}''|^2 + 2\langle \rho, \phi \rangle - \langle \rho, \nabla^* \nabla \omega \rangle \Big) \, d\mu_g; \tag{14}$$

$$-Q(J) = \int_M \Big(4|W''|^2 + \frac{1}{2}|\phi|^2 + \frac{1}{2}|\nabla^* \nabla \omega|^2 + \langle \rho, \nabla^* \nabla \omega \rangle \Big) \, d\mu_g. \tag{15}$$

The formulae above are equivalent with relations (73) and (74) from Proposition 2.5 of [16]. Indeed, the difference of (14) and (15) is formula (73) of [16] (and, in effect, Sekigawa's integral formula), whereas (15) is equivalent to formula (74) of Kirchberg. We now state one of the main results of Kirchberg in [16] (Theorem 3.6), extending an earlier result of Satoh [23]:

Theorem 3.3. *(Kirchberg, [16]) Let (M^{2n}, g, J, ω) be a compact almost Kähler manifold. If $Q(J) = 0$ and* Ric *is non-negative definite, then J is integrable.*

Proof: Summing relations (14) and (15) and using the assumption $Q(J) = 0$, we get

$$0 = \int_M \Big(|\mathrm{Ric}''|^2 + 2\langle \rho, \phi \rangle + 4|W''|^2 + \frac{1}{2}|\phi|^2 + \frac{1}{2}|\nabla^* \nabla \omega|^2 \Big) \, d\mu_g.$$

Now notice that all the terms in the right hand-side are non-negative. Indeed, the only term with a possible sign ambiguity is $\langle \rho, \phi \rangle$. But *Ric* is non-negative definite by assumption and so is ϕ by its definition, thus $\langle \rho, \phi \rangle \geq 0$. $\square$

Kirchberg goes on to describe a variety of cases when $Q(J) = 0$.

Proposition 3.4. *(Kirchberg, [16]) Let (M^{2n}, g, J, ω) be a compact almost Kähler manifold. Then $Q(J) = 0$ is a consequence of any one of the following conditions:*

- $[\nabla \mathrm{Ric}, J] = 0$
- $[\nabla^2_{X,Y} \mathrm{Ric} + \nabla^2_{Y,X} \mathrm{Ric}, J] = 0$
- $\delta W = 0$*, i.e. harmonic Weyl tensor.*

Note that the condition $[\nabla \mathrm{Ric}, J] = 0$ holds not only when the metric is Einstein, but also when the structure (g, J) is Kähler.

We next observe that formula (14) can be used to obtain a characterization of compact Einstein almost Kähler manifolds in terms of, a priori, weaker conditions.

Proposition 3.5. *Let (M^{2n}, g, J, ω) be a compact almost Kähler manifold. Then the following are equivalent: (a) The metric is Einstein; (b) $Q(J) = 0$ and $Ric'_0 = 0$ (i.e. the J-invariant part of Ricci is, at each point, a multiple of the metric).*

Proof: The implication (a) $\Rightarrow$ (b) is clear and (b) $\Rightarrow$ (a) follows from relation (14) which can be rewritten as

$$-Q(J) = \int_M \Big(|\mathrm{Ric}''|^2 + 2\langle \rho_0, \phi \rangle - \langle \rho_0, \nabla^* \nabla \omega \rangle \Big) \, d\mu_g \, . \tag{16}$$

This holds because

$$2 < \phi, \omega > = < \nabla^* \nabla \omega, \omega > = |\nabla \omega|^2 \, .$$

□

For the rest of the paper we specialize to dimension $2n = 4$. There are several particularities of this dimension. First of all, the trace-free parts of the Ricci and the *-Ricci forms coincide. In view of (1), this can be written as

$$(\rho_*)' - \rho = \frac{1}{2} (\nabla^* \nabla \omega)' = \frac{s^* - s}{4} \omega = \frac{1}{4} |\nabla \omega|^2 \omega \, . \tag{17}$$

A consequence of (17) is that

$$|\nabla^* \nabla \omega|^2 = |(\nabla^* \nabla \omega)'|^2 + |(\nabla^* \nabla \omega)''|^2 = \frac{1}{2} |\nabla \omega|^4 + 4 |\rho''_*|^2 \, . \tag{18}$$

Secondly, in dimension 4, the Kähler nullity $\mathcal{D}_p = \{X \in T_p M \mid \nabla_X \omega = 0\}$ has dimension 2 or 4 at all points $p \in M$. This implies that the form ϕ has a double eigenvalue 0 and a double eigenvalue $|\nabla \omega|^2$. In particular,

$$|\phi|^2 = |\nabla \omega|^4 \, . \tag{19}$$

Finally, in dimension 4, we have the self-dual, anti-self-dual splitting of the bundle of 2-forms

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M \, ,$$

and the Weyl tensor splits accordingly as $W = W^+ + W^-$. The anti-self-dual part W^- is $U(2)$-irreducible, whereas the self-dual part W^+ splits in three components W_i^+, $i = 1, 2, 3$: W_1^+ is determined by the conformal scalar curvature

$$\kappa = \frac{1}{3} < W^+(\omega), \omega > ;$$

W_2^+ is determined by the J-anti-invariant part of the *-Ricci form, ρ''_*; W_3^+ is just the component W'' defined earlier.

The point that interests us is that g is anti-self-dual, i.e. $W^+ \equiv 0$, if and only if $\kappa \equiv 0$, $\rho''_* \equiv 0$ and $W'' \equiv 0$. For more details on 4-dimensional almost Kähler geometry one can consult, for instance, [3].

Using the above 4-dimensional features, the integral formulae of Proposition 3.2 can be rewritten as follows

Proposition 3.6. *On a 4-dimensional almost Kähler manifold (M^4, g, J, ω), the following formulae hold:*

$$-Q(J) = \int_M \Big(|\mathrm{Ric}''|^2 + 2\langle \rho_0, \phi \rangle \Big) \, d\mu_g ; \tag{20}$$

$$-Q(J) = \int_M \left(4|W''|^2 + 2|\rho''_*|^2 + \frac{\kappa}{4}|\nabla\omega|^2\right) d\mu_g. \tag{21}$$

Proof: Relation (20) is immediate from (16) and (17). Formula (21) follows from (15), taking into account (17), (18), (19) and the fact that the conformal scalar curvature κ is related to s^* and s by

$$\kappa = \frac{3s^* - s}{2} = s^* + \frac{s^* - s}{2} = s^* + \frac{|\nabla\omega|^2}{2}\ . \tag{22}$$

□

From (21) and (22), we get the following 4-dimensional result of Kirchberg ([16] Theorem 3.7):

Theorem 3.7. *(Kirchberg) Let (M^4, g, J, ω) be a compact 4-dimensional almost Kähler manifold. If $Q(J) = 0$ and $s^* \geq 0$, then J is integrable.*

Our form of the integral formula (21) suggests that it may be possible to replace the condition $s^* \geq 0$ with the weaker one $\kappa \geq 0$. Indeed, under the assumption $Q(J) = 0$, we still get $\kappa|\nabla\omega|^2 \equiv 0$ on M. We would like to conclude that either $\kappa \equiv 0$ or $\nabla\omega \equiv 0$ on M. However, for arbitrary almost Kähler structures, it may happen that $\nabla\omega = 0$ on large open sets without having the same property on the whole M. In other words, there is no unique continuation property for $\nabla\omega$ for arbitrary almost Kähler structures. Recall that a map $u : M \longrightarrow E$ between connected Riemannian manifolds is said to have the *weak unique continuation property* if the constancy of u on an open subset of M implies the constancy on all of M. The map u has the *strong unique continuation property* if, instead of local constancy, u has contact of infinite order with the constant map at a given point (cf. [15]). A classical result on unique continuation is the following theorem of Aronszajn:

Theorem 3.8. *(Aronszajn, [5]) If M and E are Riemannian manifolds, then a a point-wise estimate*

$$|\nabla^*\nabla u|^2 \leq K\left(|\nabla u|^2 + |u|^2\right), \tag{23}$$

for some constant K, implies that the map $u : M \longrightarrow E$ satisfies the strong unique continuation property.

We can then state the following slight improvement of Theorem 3.7:

Theorem 3.9. *Let (M^4, g, J, ω) be a compact 4-dimensional almost Kähler manifold. If $Q(J) = 0$ and $\kappa \geq 0$, then either J is integrable or g is anti-self-dual.*

Proof: From the relation (21) the assumptions $\kappa \geq 0$ and $Q(J) = 0$ imply not only $\kappa|\nabla\omega|^2 \equiv 0$, but also $W'' \equiv 0$ and $\rho''_* \equiv 0$ on M. For almost Kähler 4-manifolds with $W'' \equiv 0$ and $\rho''_* \equiv 0$, it was shown in [3] (see Remark 1(i)) that an estimate of the form (23) does hold for $u = \nabla\omega$, so the unique continuation property for $\nabla\omega$ does hold in this case. It follows that either $\nabla\omega \equiv 0$, or $\kappa \equiv 0$ on M. Since $W'' \equiv 0$ and $\rho''_* \equiv 0$, the second condition corresponds to g being anti-self-dual. □

Remark: In his thesis [6], John Armstrong showed that there exist compact strictly almost Kähler, anti-self-dual metrics on certain ruled surfaces. Thus, from the point of view of the conclusion, the result of Theorem 3.9 is optimal. It remains to be seen if the condition $Q(J) = 0$ can be replaced by a conformally invariant assumption to obtain a fully conformal extension of Theorem 3.9.

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