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A survey on strong KT structures

by
Anna Fino and Adriano Tomassini *

Abstract

A Hermitian structure (J, g) on a manifold is called *strong KT* if its fundamental 2-form F is $\partial\overline{\partial}$-closed. We review some properties of strong KT metrics. Known examples of compact manifolds endowed with this type of Hermitian structures are also reviewed.

Key Words: Hermitian metric, torsion, Bismut connection, blow-up, resolution.
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1 Introduction

On every Hermitian manifold (M, J, g) of real dimension $2n$ there exists a unique connection ∇^B, called in the literature as *Bismut connection* or *KT connection*, satisfying the conditions

$$\nabla^B g = 0, \quad \nabla^B J = 0$$

and such that the tensor c defined by

$$c(X, Y, Z) = g(X, T^B(Y, Z))$$

is totally skew-symmetric, where T^B denotes the torsion of ∇^B. This Hermitian connection was used by Bismut [8] to prove a local index formula for the Dolbeault operator when the complex manifold is non-Kähler.

Considering, as usual the operator $d^c = -J^{-1}dJ$, it turns out that the 3-form c is related to the fundamental 2-form $F(X, Y) = g(JX, Y)$ by the relation $c = d^cF = JdJF$ and, if we denote by ∇^{LC} the Levi-Civita connection of g, we have that

$$g(\nabla^B_X Y, Z) = (\nabla^{LC}_X Y, Z) + \frac{1}{2}c(X, Y, Z),$$

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for any vector field X, Y, Z. Since ∇^B is a Hermitian connection, its (restricted) holonomy group $\mathrm{Hol}(\nabla^B)$ is in general contained in the unitary group $U(n)$.

By [20] the connection ∇^B belongs to a one-parameter family of canonical Hermitian connections

$$\nabla^t = t\nabla^C + (1-t)\nabla^0,$$

where ∇^C and ∇^0 denote the *Chern connection* and the *first canonical connection* respectively. For $t = -1$ one gets the Bismut connection ∇^B.

If $c = 0$, then the Bismut connection ∇^B coincides with the Levi-Civita ∇^{LC} connection and in this case the Hermitian manifold (M, J, g) is Kähler. In this note we are interested on non-Kähler geometry and we study Hermitian structures whose fundamental 2-form F satisfies weaker conditions than $dF = 0$. We recall the following (see [22])

Definition 1.1. *Let M be a manifold of real dimension $2n$. A Hermitian structure (J, g) on M or a J-Hermitian metric g is said to be* strong Kähler with torsion (strong KT) *if $dc = 0$, or, equivalently, if $\partial\overline{\partial}F = 0$ or $dd^cF = 0$.*

The strong KT metrics have been recently studied by many authors and they have also applications in type II string theory and in 2-dimensional supersymmetric σ-models [22, 44, 34]. Moreover, they have also relations with generalized Kähler geometry (see for instance [22, 24, 31, 3, 12, 18]). Indeed, by [24, 3] it follows that a generalized Kähler structure on a $2n$-dimensional manifold M is equivalent to a pair of strong KT structures (J_+, g) and (J_-, g), where $J_\pm$ are two integrable almost complex structures on M and g is a Hermitian metric with respect to $J_\pm$, such that the torsion 3-form $d^c_+F_+ = -d^c_-F_-$ is closed, where $F_\pm$ are the two fundamental 2-forms associated with the Hermitian structures $(J_\pm, g)$ and $d^c_\pm = i(\overline{\partial}_\pm - \partial_\pm)$.

By [43] any real compact semisimple Lie group G of even dimension has a natural strong KT metric and a twisted generalized Kähler structure (see [24]). For the strong KT structure, one may choose as Hermitian structure (J, g), with J a left-invariant complex structure and g a compatible bi-invariant metric. The associated Bismut connection ∇^B is the flat connection with skew-symmetric torsion $g(X, [Y, Z])$ corresponding to an invariant 3-form on the Lie algebra of G.

As far as we know, the only known solvmanifolds, i.e. compact quotients of solvable Lie groups by discrete subgroups, admitting a generalized Kähler structure are the Inoue surface of type S_M defined in [33] and the 6-dimensional one found in [18]. This solvmanifold turns out to be a $\mathbb{T}^2$-bundle over the Inoue surface of type S_M, whose construction we will review in the last section.

On a complex manifold of complex dimension n, if the $(n-1, n-1)$-form

$$F^{n-1} = \underbrace{F \wedge \ldots \wedge F}_{(n-1)-\text{times}},$$

is $\partial\overline{\partial}$-closed, then, according to [23], the Hermitian metric g is said to be *standard*. If we denote by δ the coderivative, then a Hermitian metric is standard if and only if the 1-form $\theta = J\delta F$ is co-closed. The 1-form θ is also called the *Lee form* and a Hermitian manifold is said to be *balanced* if $\theta = 0$, or equivalently if $dF^{n-1} = 0$ and *conformally balanced* if θ is exact. Consequently, on a complex surface a standard Hermitian metric coincides with a strong KT metric. If $n > 2$, then a direct computation shows that a strong KT g is standard if and only if $|dF|^2 = (n-1)^2|\theta \wedge F|^2$, where $|\cdot|$ denotes the norm on forms induced by g.

By [23], it follows that any conformal class of a Hermitian metric on a compact complex manifold can be represented by a standard metric. Thus, in particular, on a complex surface one can be found a strong KT metric in the conformal class of any given Hermitian metric. So it is interesting to study strong KT structures starting from real dimension at least six. Compact quotients of solvable (in particular nilpotent) Lie groups by uniform discrete subgroups provide a natural class of examples to be investigated. Indeed, in general, it is well known that a *nilmanifold*, i.e. a compact quotient of a nilpotent Lie group by a discrete subgroup, cannot admit any Kähler structure unless it is a torus (see for instance [7]). Moreover, in the case of solvmanifolds, Hasegawa proved in [27, 28] that a solvmanifold carries a Kähler metric if and only if it is covered by a finite quotient of a complex torus, which has the structure of a complex torus bundle over a complex torus.

This survey consists essentially of two parts. The first part collects some definitions and results on the holonomy of the Bismut connection, strong KT structures, in relation also with the blow-up of a complex manifold and with resolutions of orbifolds. The second part reviews the construction of the examples found in [16, 17, 18, 19].

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2 Holonomy of the Bismut connection in $SU(n)$

A $2n$-dimensional Hermitian manifold (M, J, g) is said to be *Calabi-Yau with torsion*, shortly CYT, if the restricted holonomy group of the Bismut connection $\mathrm{Hol}(\nabla^B)$ is contained in $SU(n)$. In such a case, we have that the first Chern class $c_1(M)$ of (M, J) is zero.

In [25] it was conjectured that any compact complex manifold (M, J) with $c_1(M) = 0$ admits a Hermitian metric such that $\mathrm{Hol}(\nabla^B) \subseteq SU(n)$. A counter-example to the above conjecture has been found, showing that the condition $\mathrm{Hol}(\nabla^B) \subseteq SU(n)$ is also not stable under small deformations of the complex structure in [16]. The example was constructed by using the following

Theorem 2.1. *[16] Let (M, J) be a compact complex manifold of complex dimension n with a holomorphic non-vanishing $(n, 0)$-form. If the Ricci tensor of the Bismut connection of some Hermitian metric g vanishes, then (M, J, g) is conformally balanced and in particular admits a balanced metric.*

The previous theorem can be applied for instance to any nilmanifold $\Gamma\backslash G$, endowed with a left-invariant complex structure. In general, by [39] any simply connected Lie group which admits a discrete subgroup with compact quotient is unimodular and in particular has a bi-invariant volume form $d\mu$. For the balanced condition, by using a "symmetrization" process, which is based on a previous idea of Belgun [6], one has the following

Theorem 2.2. *[16] Let M be the compact quotient $\Gamma\backslash G$ of a $2n$-dimensional Lie group G by a discrete subgroup Γ. If M admits a left-invariant complex structure J and F is the fundamental 2-form of a non-invariant J-Hermitian metric g, then*

$$\alpha(A_1, ..., A_{2n-2}) = \int_M F^{n-1}|_m(A_1|_m, ...A_{2n-2}|_m)d\mu,$$

is equal to $\tilde{F}^{n-1}$ for some fundamental 2-form $\tilde{F}$ of a left-invariant J-invariant Hermitian metric $\tilde{g}$. If $dF^{n-1} = 0$, then $d\tilde{F}^{n-1} = 0$.

Therefore, if g is balanced, then (M, J) admits a left-invariant balanced J-Hermitian metric $\tilde{g}$. By using essentially the same argument, Ugarte showed in [46] that if the Hermitian metric g is strong KT, then (M, J) admits a left-invariant invariant J-Hermitian metric $\tilde{g}$, which is strong KT.

By using Theorem 2.1 and 2.2 the first author with Grantcharov constructed in [16] a family of left-invariant complex structures on the Iwasawa manifold not admitting balanced metrics, except for the natural bi-invariant complex structure.

Example 2.3. The Iwasawa manifold is the compact quotient $\Gamma\backslash H^3_{\mathbb{C}}$, where

$$H_3^{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} : z^j \in \mathbb{C} \right\},$$

is the complex Heisenberg group and Γ is the uniform discrete subgroup of $H^3_{\mathbb{C}}$ defined by z^j, $j = 1, 2, 3$, Gaussian integers. Let $\mathfrak{g}_{s,t}$ be the family of 2-step nilpotent Lie algebras defined by:

$$\begin{cases} de^j = 0, \ j = 1, \ldots, 4, \\ de^5 = s(e^1 \wedge e^2 + 2e^3 \wedge e^4) + t(e^1 \wedge e^3 - e^2 \wedge e^4), \\ de^6 = t(e^1 \wedge e^4 + e^2 \wedge e^3), \end{cases}$$

where $t \neq 0$ and s are real numbers. For any $t \neq 0$ and s, the Lie algebra $\mathfrak{g}_{t,s}$ is isomorphic to the Lie algebra of $H^3_{\mathbb{C}}$. Therefore, by considering on $\mathfrak{g}_{t,s}$ the complex structure defined by the $(1,0)$-forms

$$\eta^1 = e^1 + ie^2, \, \eta^2 = e^3 + ie^4, \, \eta^3 = e^5 + ie^6,$$

one gets a family of left-invariant complex structures $J_{t,s}$ on the Iwasawa manifold. In [16] it was proved that the Iwasawa manifold $(\Gamma\backslash H^3_{\mathbb{C}}, J_{t,s})$ admits a metric with vanishing Ricci tensor for the Bismut connection if and only if $s = 0$.

Moreover, this example shows also that the property "vanishing Ricci tensor for the Bismut connection" is not stable under small deformations of the complex structure.

The balanced condition is complementary to the strong KT one. Indeed, by [4], a (non-Kähler) Hermitian metric g on a complex manifold (M, J) of complex dimension $n > 2$, can be strong KT, only if the Lee form θ does not vanish.

In [34], it was shown that a conformally balanced strong KT structure (J, g) on a compact manifold M of real dimension $2n$ with $\mathrm{Hol}(\nabla^B) \subseteq SU(n)$ is necessarily Kähler and therefore it gives rise to a Calabi-Yau structure.

It is an interesting problem to find examples of complex manifolds which admit strong CYT structures. Sufficient conditions for principal toric bundles over compact Kähler manifolds to admit Calabi-Yau with torsion structures and as well strong KT metrics have been found in [21].

We will provide a compact example of strong CYT manifold, constructed as a $\mathbb{T}^2$-bundle over the Hopf surface. This complex manifold is *locally conformally balanced*, i.e. its Lee form θ is closed and non-exact.

Example 2.4. We recall that a Hopf surface is diffeomorphic to a fiber bundle $X = S^1 \times_{\mathbb{Z}_m} S^3/K$ over S^1 with fiber S^3/K, where K is a finite subgroup of $U(2)$ acting freely on X [29]. Then, the Hopf surface can be also viewed as a compact quotient $X = L/\Theta$, where the Lie algebra of L is $\mathfrak{l} = \mathfrak{su}(2) \oplus \mathbb{R}$ and Θ is a uniform discrete subgroup of L.

Consider the 6-dimensional Lie algebra $\mathfrak{g}$ with structure equations:

$$\begin{cases} de^1 = e^2 \wedge e^3, \\ de^2 = e^3 \wedge e^1, \\ de^3 = e^1 \wedge e^2, \\ de^4 = 0, \\ de^5 = e^6 \wedge e^4, \\ de^6 = e^4 \wedge e^5. \end{cases} \tag{1}$$

The Lie algebra $\mathfrak{g}$ is the direct sum $\mathfrak{su}(2) \oplus \mathfrak{h}$, where $\mathfrak{h}$ is a 3-dimensional non-completely solvable Lie algebra. Moreover, $\mathfrak{l} = \mathrm{span} < e_1, e_2, e_3, e_4 >$ is a Lie subalgebra of $\mathfrak{g}$, where we denote by $\{e_1, \ldots, e_6\}$ the dual basis of $\{e^1, \ldots, e^6\}$. Define the almost complex structures J on $\mathfrak{g}$, by setting

$$\eta^1 = e^1 + ie^4, \quad \eta^2 = e^2 + ie^3, \quad \eta^3 = e^5 + ie^6.$$

Then by definition (η^1, η^2, η^3) are the $(1,0)$-forms associated with J. The almost complex structure J is integrable. Indeed:

$$\begin{aligned} d\eta^1 &= \tfrac{i}{2}\, \eta^2 \wedge \overline{\eta}^2, \\ d\eta^2 &= \tfrac{i}{2} (\eta^1 \wedge \eta^2 + \overline{\eta}^1 \wedge \eta^2), \\ d\eta^3 &= \tfrac{1}{2} (\eta^1 \wedge \eta^3 - \overline{\eta}^1 \wedge \eta^3). \end{aligned}$$

Consider the inner product g defined by

$$g = \sum_{j=1}^{6} e^j \otimes e^j . \tag{2}$$

Thus g is J-Hermitian. Denote by F the fundamental 2-form associated with the Hermitian structure (J, g). By a direct computation we have

$$JdF = -e^1 \wedge e^2 \wedge e^3,$$

with $e^1 \wedge e^2 \wedge e^3$ a closed and non-exact 3-form.
Therefore $\mathfrak{g}$ admits a strong KT structure defined by (J, g). The Bismut connection ∇^B in this case is flat. Indeed, the torsion of ∇^B is the 3-form

$$JdF = -e^1 \wedge e^2 \wedge e^3 .$$

Hence, the torsion forms $\tau^i\,, i = 1, \ldots, 6$ of ∇^B are given by

$$\tau^1 = e^2 \wedge e^3\,,\ \tau^2 = -e^2 \wedge e^3\,,\ \tau^3 = e^1 \wedge e^2\,,\ \tau^4 = 0\,,\ \tau^5 = 0\,,\ \tau^6 = 0\,.$$

Denoting by ω^i_j the connections 1-forms of ∇^B, by the Cartan structure equations

$$de^i + \sum_{j=1}^{6} \omega^i_j \wedge e^j = \tau^i\,, \quad \omega^i_j + \omega^j_i = 0\,, \quad i, j = 1, \ldots, 6\,, \tag{3}$$

we immediately get

$$\omega^5_6 = e^4\,,$$

the other ω^i_j being zero. Hence ∇^B is flat and consequently its holonomy algebra is trivial. Moreover, the Lee form, given by e^4 is closed and then the Hermitian metric is locally conformally balanced. The simply connected Lie group H with Lie algebra $\mathfrak{h}$ is a semi-direct product of the form $\mathbb{R} \ltimes \mathbb{R}^2$ and by considering the lattice Γ in H generated by $\frac{1}{2}$ and $\mathbb{Z}^2$, one has that $b_1(\Gamma\backslash H) = 1$. Therefore, the above Hermitian structure (J, g) on $\mathfrak{g}$ induces a left-invariant Hermitian structure (J, g) on the compact manifold $M = \Gamma\backslash H \times S^3/K$, which can be viewed as a non-trivial $\mathbb{T}^2$-bundle over the Hopf surface. Note that

$$\overline{\partial}\left(\eta^1 \wedge \eta^2 \wedge \eta^3\right) = \frac{1}{2}\,(1 - i)\,\eta^1 \wedge \overline{\eta}^1 \wedge \eta^2 \wedge \eta^3,$$

and thus there are no left-invariant holomorphic $(3, 0)$-forms on M. Since the holonomy of the Bismut connection is contained in $SU(3)$, then by [16, Theorem 4.1] (M, J) cannot admit any non-vanishing holomorphic $(3, 0)$-form. Indeed, if (M, J) admits such a form, then (J, g) has to be conformally balanced, but this is not possible since (J, g) is strong KT.

3 Strong KT metrics, currents, blow-ups and resolutions

On a compact complex manifold the Kähler, balanced and strong KT condition can be characterized by using conditions on the space of positive currents.

We start to review some known facts about positive currents. Let Ω be an open set of $\mathbb{C}^n$ and let $\Lambda^{p,q}(\Omega)$ (respectively by $\mathcal{D}^{p,q}(\Omega)$) be the space of (p,q)-forms (respectively (p,q)-forms with compact support) on Ω. On the space $\mathcal{D}^{p,q}(\Omega)$ one considers the $\mathcal{C}^\infty$-topology and one defines the *space of currents* of *bi-dimension* (p,q) or of *bi-degree* $(n-p, n-q)$ as the topological dual $\mathcal{D}'_{p,q}(\Omega)$ of $\mathcal{D}^{p,q}(\Omega)$. A current of bi-dimension (p,q) on Ω can be viewed as a $(n-p, n-q)$-form on Ω with coefficients distributions.

A current $T \in \mathcal{D}'_{p,q}(\Omega)$ is said to be of *order* 0 if its coefficients are measures and is called *normal* if T and dT are currents of order 0.
A current T of bi-dimension (p,p) is *real* if $T(\varphi) = T(\overline{\varphi})$, for any $\varphi \in \mathcal{D}^{p,q}(\Omega)$. Therefore, if $T \in \mathcal{D}'_{p,p}(\Omega)$ is real, then T can be expressed as

$$T = \sigma_{n-p} \sum_{I,\overline{J}} T_{I\overline{J}} dz_I \wedge d\overline{z}_J \,,$$

where $\sigma_{n-p} = \frac{i^{(n-p)^2}}{2^{(n-p)}}$, $T_{I\overline{J}}$ are distributions on Ω such that $T_{J\overline{I}} = \overline{T}_{I\overline{J}}$ and I, J are multi-indices of length $n-p$, $I = (i_1, \dots, i_{n-p})$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_{n-p}}$.

A real current $T \in \mathcal{D}'_{p,p}(\Omega)$ is *positive* if

$$T(\sigma_p \, \varphi^1 \wedge \cdots \wedge \varphi^p \wedge \overline{\varphi}^1 \wedge \cdots \wedge \overline{\varphi}^p) \geq 0$$

for any choice of $\varphi^1, \dots, \varphi^p \in \mathcal{D}^{1,0}(\Omega)$, where $\sigma_p = \frac{i^{p^2}}{2^p}$. Moreover, T is *strictly positive* if $\varphi^1 \wedge \cdots \wedge \varphi^p \neq 0$ implies $T(\sigma_p \, \varphi^1 \wedge \cdots \wedge \varphi^p \wedge \overline{\varphi}^1 \wedge \cdots \wedge \overline{\varphi}^p) > 0$.
If T is a positive current of bi-degree (p,p), then T is of order 0.
A real current T of bi-dimension (p,p) on Ω is said to be *negative* if the current $-T$ is positive and *plurisubharmonic* if $i\,\partial\overline{\partial} T$ is positive.

Given a Hermitian structure (J,g) on a complex manifold M, then the fundamental 2-form F corresponds to a real strictly positive current of bi-degree $(1,1)$. In particular, if the Hermitian structure (J,g) is strong KT, then the corresponding current is $\partial\overline{\partial}$-closed.

An important type of $\partial\overline{\partial}$-closed currents is given by the (p,p)-components of a boundary. We recall that a current T of bi-degree (p,p) is said to be the (p,p)-*component of a boundary* if there exists a real current S of bi-degree $(p, p-1)$ such that $T = \partial\overline{S} + \overline{\partial} S$.

Harvey and Lawson proved that a compact complex manifold admits a Kähler metric if and only if there is no non-zero positive current of bi-dimension $(1,1)$ which is the $(1,1)$-component of a boundary (see [26]). A first generalization of the previous result was obtained by Michelson in [40] and it is the following: a compact complex manifold has a balanced metric if and only if there is no non-zero positive current of bi-dimension $(n-1, n-1)$, which is the $(n-1, n-1)$-component of a boundary. In the case of strong KT metrics, Egidi proved in [14] that a compact complex manifold has a strong KT metric if and only if there is no non-zero positive current of bi-dimension $(1,1)$ which is dd^c-exact.

Miyaoka proved in [41] that, if a compact complex manifold has a Kähler metric in the complement of a point, then it admits itself a Kähler metric.

By using a deep extension and regularity result for positive or negative plurisubharmonic currents by Alessandrini and Bassanelli (see [2]) we get the following

Theorem 3.1. *[19] If $M^{2n} \setminus \{p\}$, $n \geq 2$, admits a strong KT metric, then there exists a strong KT metric on M^{2n}.*

This theorem is a generalization of Miyaoka's extension result (see [41]).
A classical result by Blanchard (see [9]) states that the blow-up of a Kähler manifold at a point or along a compact complex submanifold is still Kähler. For the strong KT case we have the same result (see [19]), namely

Theorem 3.2. *The blow-up of a strong KT manifold (M, J, g) at a point or along a compact complex submanifold of M is still strong KT.*

Then, as an application of the previous theorem we have that, if M is a complex manifold and $\tilde{M}_p$ is the blow-up of M at a point $p \in M$, then $\tilde{M}_p$ has a strong KT metric if and only if M admits a strong KT metric.

Theorem 3.1 can be applied to complex orbifolds endowed with a strong KT metric in a similar way as for the symplectic orbifolds (see [11]).
We start with recalling that a *complex orbifold* is a singular complex manifold M of dimension n such that each singularity p is locally isomorphic to U/G, where U is an open set of $\mathbb{C}^n$, G is a finite subgroup of $GL(n, \mathbb{C})$ acting linearly on U with the only one fixed point p. Moreover, the set S of singular points of M of the orbifold M has real codimension at least two.

Therefore, for instance, the quotient of a complex manifold by a holomorphic action of a finite group G with non-identity fixed point sets of real codimension at least two one is a complex orbifold.

The notions of smooth r-forms and (p,q)-forms make also sense on complex orbifolds and the differential d splits as usual as $d = \partial + \overline{\partial}$. A *Hermitian metric* g on a complex orbifold (M, J) is a J-Hermitian metric in the usual sense on the non-singular part of (M, J) and G-invariant in any chart U/G. In this case, for any chart U/G, we have $G \subset U(n)$. The Hermitian metric on the complex orbifold (M, J) is strong KT if $\partial\overline{\partial}F = 0$, where F is the fundamental 2-form associated to (J, g).

We recall that in general a resolution $(\tilde{M}, \pi)$ of a singular complex variety M is a normal, nonsingular complex variety $\tilde{M}$ with a proper surjective birational morphism $\pi : \tilde{M} \to M$. In [19] we studied the resolution of singularities of a complex orbifold endowed with a strong KT metric in order to obtain a smooth complex manifold admitting a strong KT metric. We set the following

Definition 3.3. *Let (M, J, g) be a complex orbifold endowed with a strong KT metric g . A* strong KT resolution *of a strong KT orbifold (M, J, g) is the datum* *of a smooth complex manifold $(\tilde{M}, \tilde{J})$ endowed with a $\tilde{J}$-Hermitian strong KT metric $\tilde{g}$ and of a map $\pi : \tilde{M} \to M$, such that*

i) $\pi : \tilde{M} \setminus E \to M \setminus S$ *is a biholomorphism, where S is the singular set of M and $E = \pi^{-1}(S)$ is the* exceptional set*;*

ii) $\tilde{g} = \pi^* g$ *on the complement of a neighborhood of E.*

By applying Hironaka Resolution of Singularities Theorem [32] to resolve the singularities of a complex algebraic variety by a finite number of blow-ups, in [19] we proved the following

Theorem 3.4. *Let (M, J) be a complex orbifold of complex dimension n endowed with a J-Hermitian strong KT metric g. Then there exists a strong KT resolution.*

4 Simply-connected examples

In [19] we applied the previous theorem to a quotient of a torus by the finite group $\mathbb{Z}_2$. More precisely let $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$ be the standard torus and let $(x^1, \ldots, x^6)$ be global coordinates on $\mathbb{R}^{2n}$.

Consider on $\mathbb{T}^6$ the involution σ induced by

$$\sigma\left((x^1, \ldots, x^6)\right) = (-x^1, \ldots, -x^6).$$

and the complex structure J defined by

$$\begin{cases} \eta^1 = dx^1 + i\left(f\,dx^3 + dx^4\right), \\ \eta^j = dx^j + i\,dx^{3+j}, \quad j = 2, 3, \end{cases}$$

where $f = f(x^3, x^6)$ is a $\mathcal{C}^\infty$, $\mathbb{Z}^6$-periodic and even function.

Then, as a consequence of Theorem 3.4, we can prove the following

Theorem 4.1. *[19] The quotient $(\mathbb{T}^6/\langle\sigma\rangle, J)$ is a complex orbifold with singular point set*

$$S = \left\{x + \mathbb{Z}^6 \mid x \in \frac{1}{2}\mathbb{Z}^6\right\}.$$

The J-Hermitian metric $g = \frac{1}{2}\sum_{j=1}^n \left(\eta^j \otimes \overline{\eta}^j + \overline{\eta}^j \otimes \eta^j\right)$ is strong KT. Moreover, $(\mathbb{T}^6/\langle\sigma\rangle, J)$ admits a strong KT resolution which is simply-connected.

Very interesting examples of 6-dimensional simply-connected strong KT manifolds were found in [21] by using torus bundles. They proved that for every positive integer $k \geq 1$, the manifold $(k-1)(S^2 \times S^4) \,\sharp\, k(S^3 \times S^3)$ admits a strong

KT structure, or alternatively, that any 6-dimensional compact simply-connected spin manifold with torsion free cohomology and free S^1-action has a strong KT metric. The previous manifold has been constructed as a real two-dimensional toric bundle over the complex surface obtained as blow up of the complex projective plane $\mathbb{CP}^2$ at k ($k \geq 2$) points on a smooth irreducible cubic.

Recently Swann in [45] reproduced the previous examples via the twist construction, extending them to higher dimensions, and finding further compact simply-connected strong KT manifolds. The basic idea of this construction is to consider a manifold M with an action of a n-dimensional torus $\mathbb{T}^n$. If $P \to M$ is is a principal $\mathbb{T}^n$-bundle with connection and if the $\mathbb{T}^n$-action lifts to P commuting with the principal action, then one can construct the quotient space (the "twist ") $P/\mathbb{T}^n$. Moreover, if the lifted $\mathbb{T}^n$-action preserves the principal connection, then tensors on M can be transferred to tensors on the quotient $P/\mathbb{T}^n$ by requiring their pull-backs to P to coincide on horizontal vectors. In this way, an invariant geometric structure on M determines a corresponding geometric structure on the twist $P/\mathbb{T}^n$.

5 6-dimensional strong KT nilmanifolds

Other examples, besides the compact semisimple Lie groups, of homogeneous strong KT manifolds have been found in [17] and they are 6-dimensional nilmanifolds $\Gamma\backslash G$, i.e. compact quotients of 6-dimensional nilpotent Lie groups G by discrete subgroups Γ.

By the classification obtained in [17] it turns out that only 4 classes of 6-dimensional nilpotent Lie algebras admit a strong KT structure and that the existence of a strong KT metric compatible with a left-invariant complex structure J on the nilmanifold $\Gamma\backslash G$ depends only on the complex structure J.

More precisely

Theorem 5.1. *[17] Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold, J be a left-invariant complex structure and g any J-Hermitian metric. Then the Hermitian structure (J, g) is strong KT if and only if there exists a basis $(\alpha^1, \alpha^2, \alpha^3)$ of left-invariant $(1,0)$-forms such that*

$$\begin{cases} d\alpha^1 = & d\alpha^2 = 0\,, \\ d\alpha^3 = & A\overline{\alpha}^1 \wedge \alpha^2 + B\overline{\alpha}^2 \wedge \alpha^2 + C\alpha^1 \wedge \overline{\alpha}^1 + \\ & D\alpha^1 \wedge \overline{\alpha}^2 + E\alpha^1 \wedge \alpha^2 \end{cases}$$

with $A, B, C, D, E \in \mathbb{C}$ such that

$$|A|^2 + |D|^2 + |E|^2 + 2Re(\overline{B}C) = 0\,.$$

As a consequence of the last Theorem, the nilpotent Lie group G has to be 2-step and the existence of a strong KT metric depends only on the complex structure. By applying Nomizu's result [37] about the de Rham cohomology of a nilmanifold, one obtains that the first Betti number $b_1(M)$ of M is at least 4.

The previous result has been used in [17] to classify explicitly the real 6-dimensional nilpotent Lie algebras admitting a strong KT structure. Indeed, one has that a 6-dimensional nilmanifold $M = \Gamma\backslash G$, endowed with a left-invariant complex structure J, has a J-Hermitian strong KT metric if and only if the Lie algebra $\mathfrak{g}$ of G is isomorphic to one of the following

$$
\begin{aligned}
&(0,0,0,0,13+42,14+23)\,,,\\
&(0,0,0,0,12,14+23)\,,\\
&(0,0,0,0,12,34)\,,\\
&(0,0,0,0,0,12)\,,
\end{aligned}
$$

where, for instance for $(0,0,0,0,0,12)$ we mean the Lie algebra with structure equations

$$
\begin{cases} de^i = 0\,, \quad i = 1,\dots,5\,,\\ de^6 = e^1 \wedge e^2\,. \end{cases}
$$

A detailed study of the strong KT structures, up to equivalence of the complex structure on 6-dimensional nilpotent Lie algebra, was also carried out in [46].

By [45] the strong KT nilmanifolds can be also obtained applying repeatedly the "twist construction "to a torus.

Since the condition strong KT on the 6-dimensional nilmanifolds depends only on the complex structure, in [17] we studied the strong KT equations when G is the complex Heisenberg group $H^3_{\mathbb{C}}$ and the compact quotient $M = \Gamma\backslash H^3_{\mathbb{C}}$ is the Iwasawa manifold. None of the standard complex structures (see [1]) on $H^3_{\mathbb{C}}$ are strong KT, so it was interesting to discover which ones are.
By the results of [38], the features of an invariant complex structure J on M depend on a matrix $\mathbf{X}\overline{\mathbf{X}}$, where $\mathbf{X}$ is a 2×2 matrix representing the induced action of J on $M/\mathbb{T}^2 \cong \mathbb{T}^4$, by viewing the Iwasawa manifold M as the total space of a $\mathbb{T}^2$-bundle over $\mathbb{T}^4$. In [17] it is showed that the strong KT condition constrains the eigenvalues of $\mathbf{X}\overline{\mathbf{X}}$ to be complex conjugates lying on the curve of equation

$$(1+|z|^2)\,|1+z|^2 = 8|z|^2$$

in the complex plane.

Moreover, by [19] the Iwasawa manifold $\Gamma\backslash H_3^{\mathbb{C}}$ is also an example for which the condition for a Hermitian metric to be strong KT is not stable under small deformations of the complex structure underlying the strong KT structure.

6 A 6-dimensional generalized Kähler structure solvmanifold

Generalized Kähler structures have been introduced by Gualtieri in [24] in the context of generalized geometries as generalization of Kähler structures:

Definition 6.1. *A* generalized Kähler structure *on a $2n$-dimensional manifold M is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of generalized complex structures on M such that*

1. *$\mathcal{J}_1$ and $\mathcal{J}_2$ commute;*
2. *$\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible with the indefinite metric $(\ ,\ )$ on $TM \oplus T^*M$;*
3. *the bilinear form $-(\mathcal{J}_1\mathcal{J}_2\,\cdot\,,\,\cdot)$ is positive definite.*

In terms of bi-Hermitian geometry, Apostolov and Gualtieri proved in [3] that a generalized Kähler structure on M is equivalent to a triple (g, J_+, J_-) where:

1. g is a Riemannian metric on M;
2. J_+ and J_- are two complex structures on M compatible with g such that

$$d^c_+ F_+ + d^c_- F_- = 0\,, \quad dd^c_+ F_+ = dd^c_- F_- = 0\,,$$

   where $d^c_\pm = i(\overline{\partial}_\pm - \partial_\pm)$ and $F_\pm$ is the fundamental form of the Hermitian structure $(J_\pm, g)$.

The 3-form $d^c_+ F_+$ is called the *torsion form* of the generalized Kähler structure and the generalized Kähler structure is said to be *untwisted* if the de Rham cohomology class $[d^c_+ F_+] \in H^3(M)$ vanishes and *twisted* if $[d^c_+ F_+] \neq 0$.

Constructions of non-trivial generalized Kähler structures are given for instance in [24, 3, 5, 31, 36, 35, 15]. For example in [36] the generalized Kähler quotient construction is considered in relation with the hyperkähler quotient construction and generalized Kähler structures are given on $\mathbb{CP}^n$, on some toric varieties and on the complex Grassmannian.

An interesting problem is thus to look for compact examples of generalized Kähler manifolds which do not admit any Kähler structure. A natural class of manifolds where to investigate the existence of these structures is provided by Lie groups.

By [24] any real compact semisimple Lie group G of even dimension admits a twisted generalized Kähler structure. Indeed, by [42] G has left and right invariant complex structures J_L and J_R, which can be choosen to be Hermitian with respect to the bi-invariant metric induced by the Killing form. Both (J_L, g) and (J_R, g) are strong KT structures and the Bismut connection ∇ is the flat connection with skew-symmetric torsion $g(X, [Y, Z])$. Moreover, (J_L, J_R, g) is a generalized Kähler structure.

In [10] Cavalcanti proved that there are no nilmanifolds, except tori, carrying an invariant generalized Kähler structure, since every generalized complex structure on a nilpotent Lie algebra has holomorphically trivial canonical bundle. About solvmanifolds no general results are known for the existence of generalized Kähler (non Kähler) structures.

In this section we review the construction of the 6-dimensional generalized Kähler solvmanifold obtained in [18] as $\mathbb{T}^2$-bundle over an Inoue surface of type S_M.

Let $\mathfrak{s}_{a,b}$ be the 2-step solvable Lie algebra with structure equations:

$$\begin{cases} de^1 = a\, e^1 \wedge e^2\,, \\ de^2 = 0\,, \\ de^3 = \frac{1}{2} a\, e^2 \wedge e^3\,, \\ de^4 = \frac{1}{2} a\, e^2 \wedge e^4\,, \\ de^5 = b\, e^2 \wedge e^6\,, \\ de^6 = -b\, e^2 \wedge e^5\,, \end{cases}$$

where a and b are non-zero real numbers.

If we denote by $S_{a,b}$ the simply-connected solvable Lie group with Lie algebra $\mathfrak{s}_{a,b}$, then the product on the Lie group, in terms of the global coordinates $(t, x_1, x_2, x_3, x_4, x_5)$ on $\mathbb{R}^6$, is given by:

$$\begin{aligned}(t, x_1, x_2, x_3, x_4, x_5) \cdot (t', x_1', x_2', x_3', x_4', x_5') = (t + t',\, e^{-a\,t} x_1' + x_1,\, e^{\frac{a}{2} t} x_2' + x_2,\\ e^{\frac{a}{2} t} x_3' + x_3, x_4' \cos(b\,t) - x_5' \sin(b\,t) + x_4, x_4' \sin(b\,t) + x_5' \cos(b\,t) + x_5).\end{aligned} \tag{4}$$

Since the trace of ad_X vanishes for any $X \in \mathfrak{s}_{a,b}$ and ad_{e_2} has complex eigenvalues, the Lie group $S_{a,b}$ is unimodular and it is non-completely solvable. Moreover, $S_{a,b}$ is as a semi-direct product of the form

$$\mathbb{R} \ltimes_\varphi (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2)\,,$$

where $\varphi = (\varphi_1, \varphi_2)$ is the diagonal action of $\mathbb{R}$ on $(\mathbb{R} \times \mathbb{R}^2) \times \mathbb{R}^2$ described by (4). In contrast with the nilpotent case, there are no existence theorems for uniform discrete subgroups of a solvable Lie group and, if the Lie group is non-completely solvable and admits a compact quotient, one cannot apply Hattori's theorem [30] to compute the de Rham cohomology of the compact quotient.

In [18] we showed that $S_{a,b}$ admits a compact quotient. Indeed one has the following

Theorem 6.2 ([18]). *Let $S_{1,\frac{\pi}{2}}$ be the simply-connected solvable Lie group with Lie algebra $\mathfrak{s}_{1,\frac{\pi}{2}}$. Then*

1. *$S_{1,\frac{\pi}{2}}$ has a compact quotient $M^6 = \Gamma \backslash S_{1,\frac{\pi}{2}}$ by a uniform discrete subgroup Γ.*

2. *The compact manifold* M^6 *is the total space of a* $\mathbb{T}^2$*-bundle over an Inoue surface* S_M.

3. M^6 *has first Betti number equal to* 1*, thus it has no Kähler structures.*

4. *The solvmanifold* M^6 *carries a left-invariant* (*non-trivial*) *twisted generalized Kähler structure.*

In order to obtain a uniform discrete subgroup Γ of $S_{a,b}$, in [18] we showed that the solvable Lie group $S_{1,\frac{\pi}{2}}$ is isomorphic to $(\mathbb{R}^6 = \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}), *)$ with product $*$ given by

$$(t, u, z, w) * (t', u', z', w') = (t + t', c^t u' + u, \alpha^t z' + z, e^{i\frac{\pi}{2}t} w' + w),$$

for any $t, t', u, u' \in \mathbb{R}$ and $z, z', w, w' \in \mathbb{C}$.

It turns out that the discrete subgroup Γ is isomorphic to $\mathbb{Z} \ltimes (\mathbb{Z}^3 \times \mathbb{Z}^2)$ and it is generated by the transformations

$$\begin{aligned}
&g_0 : (t, u, z, w) \mapsto (t + 1, cu, \alpha z, iw)\,, \\
&g_j : (t, u, z, w) \mapsto (t, u + c_j, z + \alpha_j, w)\,,\ j = 1, 2, 3, \\
&g_4 : (t, u, z, w) \mapsto (t, u, z, w + 1)\,, \\
&g_5 : (t, u, z, w) \mapsto (t, u, z, w + i)\,.
\end{aligned}$$

Moreover, the map

$$\begin{aligned}
\pi :\quad \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}) &\to \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C}), \\
(t, u, z, w) &\mapsto (t, u, z)
\end{aligned}$$

inherits to M^6 the structure of a $\mathbb{T}^2$-bundle over an Inoue surface of type S_M.

In [18] we proved that

$$\Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}$$

and consequently the first Betti number of M^6 is equal to 1.

The left-invariant generalized Kähler structure is defined by the pair of Hermitian structures $(J_\pm, g)$ on the Lie algebra $\mathfrak{s}_{1,\frac{\pi}{2}}$, given by the two integrable complex structures $J_\pm$ associated with the $(1,0)$-forms

$$\begin{aligned}
\omega^1_+ &= e^1 + ie^2, \quad \omega^2_+ = e^3 + ie^4, \quad \omega^3_+ = e^5 + ie^6, \\
\omega^1_- &= e^1 - ie^2, \quad \omega^2_- = e^3 + ie^4, \quad \omega^3_- = e^5 + ie^6.
\end{aligned}$$

and compatible with the inner product

$$g = \sum_{\alpha=1}^{6} e^\alpha \otimes e^\alpha.$$

By a direct computation we get that

$$d^c_+ F_+ = -d^c_- F_- = e^1 \wedge e^3 \wedge e^4,$$

which is a closed 3-form. Therefore, since the 3-form is non-exact, the corresponding left-invariant generalized Kähler structure on M^6 is twisted.

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Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10
10123 Torino
Italy
E-mail: `annamaria.fino@unito.it`

Dipartimento di Matematica
Università di Parma
Viale G.P. Usberti 53/A
43100 Parma
Italy
E-mail: `adriano.tomassini@unipr.it`