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$\mathcal{D}$-stable C^*-algebras, the ideal property and real rank zero

by
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Abstract

Let $\mathcal{D}$ be a strongly self-absorbing, K_1-injective C^*-algebra (e.g., the Jiang-Su algebra $\mathcal{Z}$ and $\mathcal{O}_\infty$). We characterize, in particular, when $A \otimes \mathcal{D}$ has the ideal property, where A is a separable, purely infinite C^*-algebra. Answering a natural question, we prove that there is a separable, nuclear C^*-algebra B such that $\mathrm{RR}(B) = \mathrm{RR}(B \otimes \mathcal{Z}) = \mathrm{sr}(B) = \mathrm{sr}(B \otimes \mathcal{Z}) = 1$ and $\mathrm{Prim}(B)$ has two elements (in particular, $\mathrm{Prim}(B)$ has a basis consisting of compact-open sets) but $B \otimes \mathcal{Z}$ does *not* have the ideal property. We also study some (permanence) properties of large classes of separable, $\mathcal{D}$-stable C^*-algebras with the ideal property. For "many" separable C^*-algebras C we characterize when $\mathrm{RR}(C \otimes \mathcal{Z}) = 0$.

Key Words: C^*-algebra, minimal tensor product of C^*-algebras, ideal property, strongly self-absorbing, the Jiang-Su algebra, weakly purely infinite, purely infinite, strongly purely infinite, exact C^*-algebra, nuclear C^*-algebra, primitive ideal spectrum, real rank, stable rank, K-theory groups, K_1-injective, $\mathcal{D}$-stable, $C(X)$-algebra, K_0-liftable.
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1 Introduction

A unital and separable C^*-algebra $\mathcal{D} \not\cong \mathbb{C}$ is *strongly self-absorbing* if there is a *-isomorphism $\mathcal{D} \xrightarrow{\sim} \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the inclusion map $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, $d \mapsto d \otimes 1_{\mathcal{D}}$ ([37]). It is known that strongly self-absorbing C^*-algebras are simple and nuclear; moreover, they are either purely infinite or stably finite. The only known examples of strongly self-absorbing C^*-algebras are the *UHF* algebras of infinite type (i.e., every prime number that occurs in the respective supernatural number occurs with infinite multiplicity), the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the Jiang-Su algebra $\mathcal{Z}$ ([19]) and the tensor products of $\mathcal{O}_\infty$ with *UHF* algebras of infinite type (see [37]). All these examples are K_1-injective, where a unital C^*-algebra A is said to be K_1-*injective* if the canonical homomorphism $\mathcal{U}(\mathcal{A})/\mathcal{U}_0(A) \to K_1(A)$ is injective. For a strongly self-absorbing C^*-algebra $\mathcal{D}$, we say that a second C^*-algebra A is $\mathcal{D}$-*stable* if $A \otimes \mathcal{D}$ is *-isomorphic to A. Recently, a lot of interesting results about $\mathcal{D}$-stable C^*-algebras have been proved by several authors (see, e.g., the excellent survey paper [11]).

Elliott's classification program for separable, nuclear C^*-algebras by discrete invariants including the K-theory is one of the most successful research directions in Operator Algebras ([9]; see also [34]). While it is clear that not all the separable, nuclear C^*-algebras can be classified, very large classes of C^*-algebras are known to be classifiable (see e.g. [34]). The Jiang-Su algebra $\mathcal{Z}$ is a unital, separable, simple, stably finite, projectionless, nuclear, infinite dimensional C^*-algebra with the same Elliott invariant as the complex numbers, $\mathbb{C}$ (in particular, $K_0(\mathcal{Z}) = \mathbb{Z}$, $K_1(\mathcal{Z}) = 0$ and $\mathcal{Z}$ has a unique tracial state) ([19]). It became clear that $\mathcal{Z}$-stability is an important regularity property for nuclear C^*-algebras ([11]). Jiang and Su showed in [19] that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \cong \otimes_{n=1}^{\infty} \mathcal{Z}$ and that A is $\mathcal{Z}$-stable if A is a unital, simple, infinite dimensional AF algebra or if A is a Kirchberg algebra. Toms and Winter proved in [38] that the separable, approximately divisible C^*-algebras are $\mathcal{Z}$-stable, generalizing, in particular, the latter result. Based on a result of Gong, Jiang and Su in [16], it follows that the Elliott invariant of an arbitrary simple, unital C^*-algebra A and of $A \otimes \mathcal{Z}$ are isomorphic if $K_0(A)$ is weakly unperforated. "Most" of the classifiable C^*-algebras are known to be $\mathcal{Z}$-stable. On the other hand, there are recent strong results by Winter (and other authors) showing that a C^*-algebra is classifiable if it is $\mathcal{Z}$-stable and satisfies also some extra conditions. All known counterexamples to Elliott's classification conjecture fail to be $\mathcal{Z}$-stable.

The *real rank* (denoted by RR(.)) is an invariant for C^*-algebras that can be seen as a non-commutative notion of dimension and it was introduced by Brown and Pedersen in [4]). A C^*-algebra A is said to have real rank zero (written $\mathrm{RR}(A) = 0$) if its unitization $\tilde{A} := A + \mathbb{C} \cdot 1$ has the property that the set of all invertible, selfadjoint elements of $\tilde{A}$ is dense in the set of all selfadjoint elements of $\tilde{A}$ ([4]). Note that many interesting C^*-algebras have real rank zero and that Elliott's conjecture has been verified for large classes of C^*-algebras of real rank zero (see, e.g., [34]).

A C^*-algebra A is said to have the *ideal property* if each of its ideals is generated (as an ideal) by its projections (*in this paper, by an ideal we mean a closed, two-sided ideal*). Every simple C^*-algebra with an approximate unit of projections and every C^*-algebra of real rank zero has the ideal property. The ideal property has been studied extensively by us (alone or in collaboration), for example in [28-32] and [14-15]. The ideal property is important in Elliott's classification program. In [36], K. Stevens classified by a K-theoretical invariant a certain class of (non-simple) AI algebras with the ideal property; this result was recently generalized in [18]. In [28] we classified the AH algebras with the ideal property and with slow dimension growth up to a shape equivalence and we gave several characterizations of when an arbitrary AH algebra has the ideal property. In [14-15], jointly with Gong, Jiang and Li, we proved a reduction theorem saying that every AH algebra with the ideal property and with the dimensions of the local spectra uniformly bounded can be written as an AH algebra with the ideal property with (special) local spectra of dimensions ≤ 3. This result generalizes similar and strong reduction theorems for real rank zero AH algebras-proved by Dadarlat ([7]) and Gong ([12])-and also for simple AH algebras-proved by Gong ([13])-which have been major steps in the classification of the corresponding classes of AH algebras. Also, in [28-29], we proved several nonstable K-theoretical results for a large class of C^*-algebras with the ideal property. Indeed, if A is an AH algebra with the ideal property and with slow dimension growth, we proved in [28]) that A has *stable rank* one ($\mathrm{sr}(A) = 1$) (that means, in the unital case, that the set of the invertible elements in A is dense in A (see [44] for more on the stable rank)), that $K_0(A)$ is weakly unperforated in the sense of Elliott and is also a Riesz group ([28-29]) and that the strict comparability of the projections in A is determined by the tracial states of A, when A is unital ([28]). Also, jointly with Rørdam, we proved in [31] that the ideal property is *not* preserved by taking minimal tensor products (even in the separable case).

The *purely infinite* C^*-algebras have been introduced by Kirchberg and Rørdam in [25], extending the definition in the simple case given by Cuntz [6]. A C^*-algebra A is said to be purely infinite if A has no characters (or, equivalently, no non-zero abelian quotients) and if for every $a, b \in A^+$ such that a belongs to the ideal of A generated by b, it follows that there is a sequence (x_n) of elements in A such that $\|a - x_n^* b x_n\| \to 0$ as $n \to \infty$ ([25]). The study of purely infinite C^*-algebras was motivated by Kirchberg's classification of the separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebras up to stable isomorphism by an ideal related KK-theory. Kirchberg and Rørdam introduced in [26] two more important notions of being purely infinite: *weakly purely infinite* and *strongly purely infinite* (we refer to [26] for their definitions). It was observed in [26] that in general we have: strongly purely infinite $\Rightarrow$ purely infinite $\Rightarrow$ weakly purely infinite. Jointly with Rørdam, we characterized when a separable, purely infinite C^*-algebra has real rank zero and also when it has the ideal property ([32]).

If B is an arbitrary C^*-algebra, its *primitive ideal spectrum*, denoted $\mathrm{Prim}(B)$, is the set of all the primitive ideals in B (i.e., the kernels of the irreducible representations) equipped with the Jacobson topology. In Section 2 we obtain, in particular, several characterizations of the ideal property for $A \otimes \mathcal{D}$, where A is a separable, purely infinite C^*-algebra and $\mathcal{D}$ is a separable, unital, strongly self-absorbing C^*-algebra (Corollary 2.5) (for a more general result see Proposition 2.1). One of these characterizations is that $\mathrm{Prim}(A)$ has a basis consisting of compact-open sets. Answering a natural question suggested by this result (Question 2.7), we prove that there is a separable, nuclear C^*-algebra B such that $\mathrm{RR}(B) = \mathrm{RR}(B \otimes \mathcal{Z}) = \mathrm{sr}(B) = \mathrm{sr}(B \otimes \mathcal{Z}) = 1$ and $\mathrm{Prim}(B)$ has two elements (in particular, $\mathrm{Prim}(B)$ has a basis consisting of compact-open sets) but $B \otimes \mathcal{Z}$

does *not* have the ideal property (Theorem 2.9).

In Section 3 we study some (permanence) properties of large classes of separable, $\mathcal{D}$-stable C^*-algebras with the ideal property, where $\mathcal{D}$ is a strongly self-absorbing, K_1-injective C^*-algebra. Let $\mathcal{C}(\mathcal{D})$ be the class of all the separable, purely infinite C^*-algebras with the ideal property that are $\mathcal{D}$-stable (Definition 3.1). We prove that $\mathcal{C}(\mathcal{D})$ is stable under taking hereditary sub-C^*-algebras (in particular ideals), stable isomorphisms, quotients, countable inductive limits and extensions but *not* under taking minimal tensor products (see Proposition 3.2). Also, we show that if A is a $C(X)$-algebra, where X is a compact, Hausdorff topological space, and if A *locally* belongs to $\mathcal{C}(\mathcal{D})$, then A belongs to $\mathcal{C}(\mathcal{D})$ (see Proposition 3.2). If $\mathcal{C}_{nuc}(\mathcal{D}) := \mathcal{C}(\mathcal{D}) \bigcap \mathcal{N}$, where $\mathcal{N}$ denotes the class of all the nuclear C^*-algebras (see Definition 3.6), then $\mathcal{C}_{nuc}(\mathcal{D})$ has all the above mentioned properties of $\mathcal{C}(\mathcal{D})$, except that $\mathcal{C}_{nuc}(\mathcal{D})$ *is* stable under taking tensor products (see Proposition 3.7). If $\mathcal{C}$ is the class of all the separable, purely infinite, nuclear C^*-algebras with the ideal property (see Definition 3.6), then we also show that $\mathcal{C} = \mathcal{C}_{nuc}(\mathcal{O}_\infty) = \mathcal{C}_{nuc}(\mathcal{Z})$ and that $\mathcal{C}_{nuc}(\mathcal{D}) = \mathcal{C} \otimes \mathcal{D}$, where $\mathcal{C} \otimes \mathcal{D} := \{A \otimes \mathcal{D} : A \in \mathcal{C}\}$ (see Proposition 3.7).

In Section 4, we prove that for a separable C^*-algebra A, $\mathrm{RR}(A \otimes \mathcal{Z}) = 0 \Rightarrow \mathrm{RR}(A \otimes \mathcal{O}_\infty) = 0$ (Proposition 4.1) and we observe that $\mathrm{RR}(A \otimes \mathcal{O}_\infty) = 0 \not\Rightarrow \mathrm{RR}(A \otimes \mathcal{Z}) = 0$ (Remark 4.3). However, we prove that for every separable, weakly purely infinite C^*-algebra A, the following four properties are equivalent: (1) $\mathrm{RR}(A \otimes \mathcal{Z}) = 0$; (2) $\mathrm{RR}(A \otimes \mathcal{O}_\infty) = 0$; (3) $\mathrm{Prim}(A)$ has a basis consisting of compact-open sets and A is K_0-liftable; (4) $A \otimes \mathcal{Z}$ has the ideal property and A is K_0-liftable (Theorem 4.4). Recall that a C^*-algebra B is said to be K_0-*liftable* if for every pair of ideals $I \subseteq J$ in B the natural map $K_0(J) \to K_0(J/I)$ is surjective ([32, Definition 3.1]). Based on [32], we characterize when a separable, weakly purely infinite ($\mathcal{Z}$-stable) C^*-algebra has real rank zero (Proposition 4.6 and Corollary 4.7).

If A is a C^*-algebra, the fact that I is an ideal of A will be denoted $I \lhd A$. The symbol $\otimes$ will mean the minimal tensor product of C^*-algebras. We shall denote by $\mathcal{K}$ the C^*-algebra of the compact operators on $\ell^2(\mathbb{N})$. If X is a locally compact, Hausdorff space, $C_0(X)$ will denote the C^*-algebra of the continuous functions on X with values in $\mathbb{C}$ and which vanish at infinity.

2 The ideal property

We are interested in characterizing when a C^*-algebra of the form $A \otimes \mathcal{D}$ has the ideal property, where A is a separable C^*-algebra and $\mathcal{D}$ is a strongly self-absorbing C^*-algebra (specifically, $\mathcal{Z}$). A particular answer follows from Corollary 2.5 below, which is a consequence of the following result:

Proposition 2.1. *Let A be a separable, purely infinite C^*-algebra and let B be a separable, simple, exact C^*-algebra with the ideal property. Then, the following are equivalent:*

(1) *$A \otimes B$ has the ideal property;*
(2) *A has the ideal property;*
(3) *A has the ideal property and $A \otimes B$ is strongly purely infinite.*

The proof of the above proposition will use [32], [31] and, among other things, the following results of Blanchard-Kirchberg and Kirchberg:

Theorem 2.2 ([2]). *If A and B are separable C^*-algebras and either A or B is exact, then $\mathrm{Prim}(A \otimes B)$ and $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ are homeomorphic.*

Theorem 2.3 ([22]). *If A and B are C^*-algebras such that one of A or B is exact and the other one is strongly purely infinite, then $A \otimes B$ is strongly purely infinite.*

Proof of Proposition 2.1. (1) $\Rightarrow$ (2). Assume now that $A \otimes B$ has the ideal property. Since A and B are separable and B is exact, Theorem 2.2 implies that $\mathrm{Prim}(A \otimes B)$ is homeomorphic to $\mathrm{Prim}(A)$ (since B being simple, $\mathrm{Prim}(B)$ consists of only one element). But since $A \otimes B$ has the ideal property, it follows by [3, page 76] that $\mathrm{Prim}(A \otimes B)$ has a basis consisting of compact-open sets, and hence, by the above fact, so does $\mathrm{Prim}(A)$. Using this, the fact that A is separable and purely infinite and [32, Proposition 2.11] we conclude that A has the ideal property.

(2) $\Rightarrow$ (3). Assume that (2) is true. Since A is purely infinite and it has the ideal property, it follows from [32, Proposition 2.14] that A is strongly purely infinite. Now, since B is exact, it follows from Theorem 2.3 that $A \otimes B$ is strongly purely infinite.

(3) $\Rightarrow$ (1). Assume that (3) is true. Since both C^*-algebras A and B have the ideal property and B is exact, then by [31, Corollary 1.3] (which is based on a result of Kirchberg in [21]) it follows that (1) is true.

Remark 2.4. Observe that since A is separable and purely infinite, it follows by [32, Proposition 2.11] that the condition (2), and therefore the conditions (1), (2) and (3) in Proposition 2.1 are equivalent to:

(4) $\mathrm{Prim}(A)$ *has a basis consisting of compact-open sets.*

Note also that the equivalence between the conditions (1) and (2) in Proposition 2.1 and the condition (4) above was proved in [32, Corollary 4.3(ii)] in the particular case when $B = \mathcal{O}_2$.

Corollary 2.5. *Let A be a separable, purely infinite C^*-algebra and let $\mathcal{D}$ be a separable, unital, strongly self-absorbing C^*-algebra. Then, the following are equivalent:*
(1) *$A \otimes \mathcal{D}$ has the ideal property;*
(2) *A has the ideal property;*
(3) *A has the ideal property and $A \otimes \mathcal{D}$ is strongly purely infinite.*

Proof. It follows from Proposition 2.1 since $\mathcal{D}$ is separable, simple, unital and nuclear by [37] (based on results in [8] and [24]) (and every nuclear C^*-algebra is exact).

Let A be a separable C^*-algebra. Consider the following conditions:
(a) $A \otimes \mathcal{Z}$ has the ideal property;
(b) A has the ideal property;
(c) $\mathrm{Prim}(A)$ has a basis consisting of compact-open sets.

Question 2.6: *Is the equivalence* (a) $\Leftrightarrow$ (b) *true*?

Question 2.7: *Is the equivalence* (a) $\Leftrightarrow$ (c) *true*?

Remark 2.8. Clearly, (b) $\not\Leftrightarrow$ (c), even if $\mathrm{Prim}(A)$ has one element, i.e. A is simple. Indeed, one can take A to be a non-zero, separable, simple, nonunital, projectionless C^*-algebra. (One could also show that (b) $\not\Leftrightarrow$ (c) observing that the class $\mathcal{C}_3$ of the separable C^*-algebras satisfying (c) is closed under stable isomorphism while the class $\mathcal{C}_2$ of the separable C^*-algebras satisfying (b) is not ([30]), or, one could observe that $\mathcal{C}_3$ is closed under extensions ([32, Corollary 4.4 (i)]) while $\mathcal{C}_2$ is not ([29]).

Theorem 2.9. *There is a separable, nuclear C^*-algebra A such that* $\mathrm{RR}(A) = \mathrm{RR}(A \otimes \mathcal{Z}) = \mathrm{sr}(A) = \mathrm{sr}(A \otimes \mathcal{Z}) = 1$ *and* $\mathrm{Prim}(A)$ *has two elements* (*in particular,* $\mathrm{Prim}(A)$ *has a basis consisting of compact-open sets*) *but $A \otimes \mathcal{Z}$ does not have the ideal property* (*and, implicitly, A does not have the ideal property*).

Proof. Let D be a Bunce-Deddens algebra [5]. We showed in the proof of [29, Theorem 5.1], jointly with Dadarlat, that there is an extension of C^*-algebras:

$$0 \longrightarrow D \otimes \mathcal{K} \stackrel{\iota}{\longrightarrow} A \stackrel{\pi}{\longrightarrow} \mathbb{C} \longrightarrow 0 \qquad (2.1)$$

such that A does not have the ideal property, the corresponding exponential map $\delta_0 : K_0(\mathbb{C}) \to K_1(D \otimes \mathcal{K})$ is injective and $\mathrm{RR}(A) = \mathrm{sr}(A) = 1$. Since $\mathcal{Z}$ is exact (being nuclear), tensoring the extension (2.1) with $\mathcal{Z}$, we obtain the following short exact sequence of C^*-algebras:

$$0 \longrightarrow D \otimes \mathcal{K} \otimes \mathcal{Z} \stackrel{\iota \otimes id}{\longrightarrow} A \otimes \mathcal{Z} \stackrel{\pi \otimes id}{\longrightarrow} \mathbb{C} \otimes \mathcal{Z} \cong \mathcal{Z} \longrightarrow 0 \qquad (2.2)$$

We have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D \otimes \mathcal{K} & \stackrel{\iota}{\longrightarrow} & A & \stackrel{\pi}{\longrightarrow} & \mathbb{C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D \otimes \mathcal{K} \otimes \mathcal{Z} & \underset{\iota \otimes id}{\longrightarrow} & A \otimes \mathcal{Z} & \underset{\pi \otimes id}{\longrightarrow} & \mathbb{C} \otimes \mathcal{Z} \cong \mathcal{Z} & \longrightarrow & 0
\end{array}$$

where the vertical maps are of the form $x \mapsto x \otimes 1$. Using the naturality of the exponential map, we get a commutative diagram at the level of K_0 groups involving the associated exponential maps δ_0 and δ_0', of the form:

$$\begin{array}{ccc} K_0(\mathbb{C}) & \xrightarrow{\delta_0} & K_1(D \otimes \mathcal{K}) \\ \downarrow & & \downarrow \\ K_0(\mathcal{Z}) & \xrightarrow[\delta_0']{} & K_1(D \otimes \mathcal{K} \otimes \mathcal{Z}) \end{array}$$

where the vertical maps are group isomorphisms (use [19, Lemma 2.11] and the fact that $D \otimes \mathcal{K}$ has a countable, increasing, approximate unit of projections (because D has such an approximate unit of projections)). Then, since δ_0 is injective, it follows that δ_0' is also injective. Since also $(A \otimes \mathcal{Z})/(\iota \otimes id)\ (D \otimes \mathcal{K} \otimes \mathcal{Z}) \cong \mathcal{Z}$ is non-zero and stably finite, it follows from one of our joint results with Dadarlat (see [29, Lemma 5.2]) that $A \otimes \mathcal{Z}$ is *not* generated (as an ideal of $A \otimes \mathcal{Z}$) by its projections. Hence, $A \otimes \mathcal{Z}$ does not have the ideal property. The short exact sequence (2.1) implies that A is separable and nuclear (as an extension of the separable and nuclear C^*-algebras $D \otimes \mathcal{K}$ and $\mathbb{C}$) and Prim(A) has two elements (since, by (2.1), A is an extension of two simple C^*-algebras). Note that the separable C^*-algebra D is approximately divisible by [10], since D is a simple, unital, infinite dimensional inductive limit of a sequence of circle algebras, and hence, by [38, Theorem 2.3] we have that $D \cong D \otimes \mathcal{Z}$. Therefore, $D \otimes \mathcal{K} \otimes \mathcal{Z} \cong D \otimes \mathcal{K}$ and since $\mathrm{sr}(D \otimes \mathcal{K}) = 1$ ($D \otimes \mathcal{K}$ is an inductive limit of circle algebras that have stable rank one and now one can use [33]), we deduce that $\mathrm{sr}(D \otimes \mathcal{K} \otimes \mathcal{Z}) = 1$. Also, $\mathrm{sr}(\mathcal{Z}) = 1$ (use e.g. [35, Theorem 6.7]). But the index map $\delta_1 : K_1(\mathcal{Z}) = 0 \to K_0(D \otimes \mathcal{K} \otimes \mathcal{Z})$ associated to the short exact sequence (2.2) is zero. These things imply that $\mathrm{sr}(A \otimes \mathcal{Z}) = 1$ (see e.g. [27]). But since $A \otimes \mathcal{Z}$ does not have the ideal property (as noticed before), it follows that $\mathrm{RR}(A \otimes \mathcal{Z}) \neq 0$ (the same conclusion can be obtained using the fact that by (2.2), $(A \otimes \mathcal{Z})/(\iota \otimes id)\ (D \otimes \mathcal{K} \otimes \mathcal{Z}) \cong \mathcal{Z}$ is *not* of real rank zero (see [4])). By [4, Proposition 1.2] it follows that $\mathrm{RR}(A \otimes \mathcal{Z}) \leq 2\ \mathrm{sr}(A \otimes \mathcal{Z}) - 1 = 1$, and in conclusion, $\mathrm{RR}(A \otimes \mathcal{Z}) = 1$. (Finally, note that if B is a C^*-algebra such that $B \otimes \mathcal{Z}$ does not have the ideal property, then by [31, Corollary 1.3] it follows that B does not have the ideal property).

Remark 2.10. The above theorem proves that the answer to Question 2.7 is negative even if A is nuclear, $\mathrm{RR}(A) = \mathrm{RR}(A \otimes \mathcal{Z}) = \mathrm{sr}(A) = \mathrm{sr}(A \otimes \mathcal{Z}) = 1$ and Prim(A) has two elements.

3 Permanence properties

We will study some (permanence) properties of large classes of separable, $\mathcal{D}$-stable C^*-algebras with the ideal property, where $\mathcal{D}$ is a strongly self-absorbing, K_1-injective C^*-algebra.

Definition 3.1. Let $\mathcal{D}$ be a strongly self-absorbing, K_1-injective C^*-algebra. Let $\mathcal{C}(\mathcal{D})$ be the class of all the separable, purely infinite C^*-algebras with the ideal property that are $\mathcal{D}$-stable.

Recall from [20] that if X is a compact, Hausdorff topological space, then a $C(X)$-*algebra* is a C^*-algebra A endowed with a unital *-homomorphism from $C(X)$ in the center $\mathcal{Z}(\mathcal{M}(A))$ of the multiplier C^*-algebra $\mathcal{M}(A)$ of A. If $F \subseteq X$ is a closed subset we denote by A_F the quotient of A by the ideal $C_0(X \setminus F)A$.

Proposition 3.2. *Let $\mathcal{D}$ be as in Definition 3.1.*

(1) *$\mathcal{C}(\mathcal{D})$ is stable under taking hereditary sub-C^*-algebras (in particular ideals), stable isomorphisms, quotients, countable inductive limits and extensions but not under taking minimal tensor products.*

(2) *Let A be a $C(X)$-algebra, where X is a compact, Hausdorff topological space. Then, if every $x \in X$ has a compact neighborhood $V_x \subseteq X$ such that $A_{V_x} \in \mathcal{C}(\mathcal{D})$, then $A \in \mathcal{C}(\mathcal{D})$.*

The proof of the above proposition will need the following three preliminary results:

Proposition 3.3. *Let $\mathcal{D}$ be a strongly self-absorbing C^*-algebra. Then, the class of all the separable, purely infinite, $\mathcal{D}$-stable C^*-algebras with the ideal property is not stable under taking minimal tensor products.*

Proof. Let A and C be the separable C^*-algebras from [32, Proposition 4.5]. Then, $A \otimes \mathcal{O}_2$ and $C \otimes \mathcal{O}_2$ have the ideal property and $(A \otimes \mathcal{O}_2) \otimes (C \otimes \mathcal{O}_2) \cong A \otimes C \otimes \mathcal{O}_2$ does not have the ideal property (by [32, Proposition 4.5]). Define $E := A \otimes \mathcal{O}_2 \otimes \mathcal{D}$ and $F := C \otimes \mathcal{O}_2 \otimes \mathcal{D}$. Then E and F are separable, $\mathcal{D}$-stable, purely infinite C^*-algebras (since $M \otimes \mathcal{O}_2$ is purely infinite for any C^*-algebra M, by [25, Proposition 4.5]). Note also that E and F have the ideal property by [31, Corollary 1.3], since, as noticed above, $\mathcal{D}$ is simple, unital and nuclear. On the other hand, $E \otimes F \cong (A \otimes C \otimes \mathcal{O}_2) \otimes \mathcal{D}$ does not have the ideal property, by Corollary 2.5 (because $A \otimes C \otimes \mathcal{O}_2$ is separable, purely infinite (again by [25, Proposition 4.5]) and it does not have the ideal property). The proof is over.

Proposition 3.4. *Suppose that we have a pullback diagram of C^*-algebras:*

$$\begin{array}{ccc} C & \xrightarrow{\pi_1} & A_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \varphi_1} \\ A_2 & \xrightarrow[\varphi_2]{} & B \end{array}$$

where C is the pullback of (A_1, A_2) along (φ_1, φ_2) and at least one of the maps φ_1, φ_2 is surjective.

(1) *If A_1 and A_2 are separable, purely infinite and with the ideal property, then so is C.*

(2) *If A_1 and A_2 are nuclear, then so is C.*

Proof. (1). It is inspired by the proof of [17, Proposition 4.9]. Let us assume that φ_2 is surjective. Therefore, π_1 is also surjective an we have the following short exact sequence of C^*-algebras:

$$0 \longrightarrow \ker(\pi_1) \xrightarrow{\iota} C \xrightarrow{\pi_1} A_1 \longrightarrow 0 \tag{3.1}$$

where ι is the inclusion. We may identify:

$$C = \{(a_1, a_2) \in A_1 \oplus A_2 : \varphi_1(a_1) = \varphi_2(a_2)\} \subseteq A_1 \oplus A_2.$$

Under this identification, we clearly have:

$$\ker(\pi_1) = \{(0, a) : \varphi_2(a) = 0\} = 0 \oplus \ker(\varphi_2).$$

Since the separability, the ideal property and the property of being purely infinite pass to ideals (for the last assertion see [25, Proposition 4.3]) and since A_2 is separable, purely infinite and with the ideal property, it follows that $\ker(\varphi_2)$ ($\lhd A_2$) is separable, purely infinite and with the ideal property and hence, so is $\ker(\pi_1)$. Finally, since the class of the separable, purely infinite C^*-algebras with the ideal property is closed under extensions (by [32, Corollary 4.4 (ii)]), it follows from (3.1) that C is separable, purely infinite and with the ideal property.

(2). It is similar with the above proof of (1) and uses the well known facts that ideals and extensions of nuclear C^*-algebras are also nuclear.

Corollary 3.5. *Let X be a compact, Hausdorff topological space and let A be a $C(X)$-algebra. Suppose that each $x \in X$ has a compact neighborhood $V_x \subseteq X$ such that A_{V_x} is separable, purely infinite and with the ideal property* (*respectively, A_{V_x} is nuclear*). *Then, A is separable, purely infinite and with the ideal property* (*respectively, A is nuclear*).

Proof. It follows from the above Proposition 3.4 as in the proof of [17, Proposition 4.11].

Proof of Proposition 3.2. (1). The class of the separable, $\mathcal{D}$-stable C^*-algebras is stable under taking hereditary sub-C^*-algebras, stable isomorphisms, quotients, countable inductive limits and extensions (see [37]). The class of the C^*-algebras with the ideal property is closed to inductive limits ([29, Proposition 2.3]) and quotients, the class of the purely infinite C^*-algebras with the ideal property is stable under taking hereditary sub-C^*-algebras and stable isomorphisms (see [25, Proposition 4.17], [32, Proposition 2.10], [29, Proposition 2.4] and [25, Theorem 4.23]), and the class of the separable, purely infinite C^*-algebras with the ideal property is stable under extensions ([32, Corollary 4.4 (ii)]). Note also that the class of the purely infinite C^*-algebras is stable under taking quotients ([25, Proposition 4.3]) and inductive limits ([25, Proposition 4.18]). Finally, the fact that $\mathcal{C}(\mathcal{D})$ is *not* closed under taking minimal tensor products follows from Proposition 3.3. This ends the proof.

(2). It follows from the proof of [17, Proposition 4.11] that A is $\mathcal{D}$-stable. Now, to end the proof, we use also the above Corollary 3.5.

Definition 3.6. Let $\mathcal{D}$ and $\mathcal{C}(\mathcal{D})$ be as in Definition 3.1. Let $\mathcal{N}$ denote the class of all the nuclear C^*-algebras. Define $\mathcal{C}_{nuc}(\mathcal{D}) := \mathcal{C}(\mathcal{D}) \bigcap \mathcal{N}$ and define $\mathcal{C}$ to be the class of all the separable, purely infinite, nuclear C^*-algebras with the ideal property.

Proposition 3.7. *Let $\mathcal{D}$ be as in Definition 3.6.*

(1) $\mathcal{C} = \mathcal{C}_{nuc}(\mathcal{O}_\infty) = \mathcal{C}_{nuc}(\mathcal{Z})$;

(2) *$\mathcal{C}_{nuc}(\mathcal{D})$ has the same properties as $\mathcal{C}(\mathcal{D})$ in Proposition 3.2, except that it is stable under taking tensor products.*

(3) $\mathcal{C}_{nuc}(\mathcal{D}) = \mathcal{C} \otimes \mathcal{D}$, where $\mathcal{C} \otimes \mathcal{D} := \{A \otimes \mathcal{D} : A \in \mathcal{C}\}$.

Proof. (1). We will prove first that:

$$\mathcal{C} \subseteq \mathcal{C}_{nuc}(\mathcal{O}_\infty) \tag{3.2}$$

Indeed, let $A \in \mathcal{C}$, that is let A be a separable, purely infinite, nuclear C^*-algebra with the ideal property. Since A is purely infinite and with the ideal property, it follows by [32, Proposition 2.14] that A is strongly purely infinite. Since the property of being strongly purely infinite is preserved by stable isomorphisms (by [26, Proposition 5.11 (iii)]), one obtains the $A \otimes \mathcal{K}$ is strongly purely infinite. Then, [26, Theorem 8.6] implies that the separable, nuclear, strongly purely infinite, stable C^*-algebra $A \otimes \mathcal{K}$ is $\mathcal{O}_\infty$-stable. Using now [37, Corollary 3.2], we can conclude that A is $\mathcal{O}_\infty$-stable. Hence, (3.2) is proved.

Observe also that since O_∞ is $\mathcal{Z}$-stable (by [19, Corollary 2.13]), it follows that the following inclusion is true:

$$\mathcal{C}_{nuc}(\mathcal{O}_\infty) \subseteq \mathcal{C}_{nuc}(\mathcal{Z}) \tag{3.3}$$

Finally observe that (3.2), (3.3) and the obvious inclusion $\mathcal{C}_{nuc}(\mathcal{Z}) \subseteq \mathcal{C}$ imply that (1) is true.

(2). It is known that $\mathcal{N}$ is stable under taking hereditary sub-C^*-algebras, stable isomorphisms, quotients, inductive limits, extensions and tensor products. Now, the proof of (2) ends using Proposition 3.2, Corollary 3.5 and [32, Proposition 4.6].

(3). Observe that $\mathcal{C}_{nuc}(\mathcal{D}) \subseteq \mathcal{C} \otimes \mathcal{D}$ is obvious (since any element of $\mathcal{C}_{nuc}(\mathcal{D})$ is $\mathcal{D}$-stable and $\mathcal{C}_{nuc}(\mathcal{D}) \subseteq \mathcal{C}$). Conversely, let us prove now that $\mathcal{C} \otimes \mathcal{D} \subseteq \mathcal{C}_{nuc}(\mathcal{D})$. For this, let $A \in \mathcal{C}$. Since A and $\mathcal{D}$ have the ideal property ($\mathcal{D}$ is simple and unital), A is purely infinite and $\mathcal{D}$ is exact (since $\mathcal{D}$ is nuclear), [32, Proposition 4.6] implies that $A \otimes \mathcal{D}$ is purely infinite and with the ideal property. This ends the proof of the inclusion $\mathcal{C} \otimes \mathcal{D} \subseteq \mathcal{C}_{nuc}(\mathcal{D})$, since the fact that $A \otimes \mathcal{D}$ is separable, $\mathcal{D}$-stable and nuclear is obvious.

4 Real rank zero

We will characterize when $\mathrm{RR}(A \otimes \mathcal{Z}) = 0$, for "many" separable C^*-algebras A. We begin with the following:

Proposition 4.1. *Let A be a separable C^*-algebra. If $\mathrm{RR}(A \otimes \mathcal{Z}) = 0$, then $\mathrm{RR}(A \otimes \mathcal{O}_\infty) = 0$.*

The proof of the above result uses, among other things, the following:

Lemma 4.2. *Let A be a C^*-algebra. Then, the following are equivalent:*
(1) A is K_0-liftable;
(2) $A \otimes \mathcal{Z}$ is K_0-liftable.

Proof. It is inspired by the proof of [32, Lemma 3.4]. Observe first that since $\mathcal{Z}$ is nuclear and simple, any ideal of $A \otimes \mathcal{Z}$ is of the form $I \otimes \mathcal{Z}$, where $I \lhd A$ (by [1], see also [2, Proposition 2.16]). Also, if $I \lhd A$, $J \lhd A$ such that $I \subseteq J$, then $(J \otimes \mathcal{Z})/(I \otimes \mathcal{Z}) \cong (J/I) \otimes Z$, since $\mathcal{Z}$ is exact (being nuclear).

Now, let $I \lhd A$, $J \lhd A$ such that $I \subseteq J$. Consider the commutative diagram:

$$\begin{array}{ccc} J & \longrightarrow & J/I \\ \downarrow & & \downarrow \\ J \otimes \mathcal{Z} & \longrightarrow & (J/I) \otimes \mathcal{Z} \end{array}$$

where the vertical maps are of the form $x \mapsto x \otimes 1$ and the horizontal maps are the canonical surjections. Note that the vertical maps above induce isomorphisms at the level of K_0. This follows from the fact that if D is an arbitrary C^*-algebra, then the canonical embedding of D into $D \otimes \mathcal{Z}$ induces a group isomorphism at the level of K_0; in the unital case this was proved in [16, Corollary 1.3] and in the general case one needs to adjoin a unit and to use a standard argument involving the unital case and the six-term exact sequence in K-theory (notice that $K_1(\mathcal{Z}) = 0$). It is now obvious that A is K_0-liftable if and only if $A \otimes \mathcal{Z}$ is K_0-liftable. Hence, (1) $\Leftrightarrow$ (2).

Proof of Proposition 4.1. Assume that $\mathrm{RR}(A \otimes \mathcal{Z}) = 0$. Then, $A \otimes \mathcal{Z}$ has the ideal property and, by [3, page 76] it follows that $\mathrm{Prim}(A \otimes \mathcal{Z})$ has a basis consisting of compact-open sets. Since $\mathcal{Z}$ is simple and exact, by Theorem 2.2 it follows that $\mathrm{Prim}(A)$ has a basis consisting of compact-open sets ($\mathcal{Z}$ being simple, $\mathrm{Prim}(\mathcal{Z})$ has only one element). Now, by [4] any C^*-algebra of real rank zero is K_0-liftable. Hence, $A \otimes \mathcal{Z}$ is K_0-liftable, and, by Lemma 4.2, it follows that A is K_0-liftable. Now, since A is separable, [32, Corollary 4.3 (i)] implies that $\mathrm{RR}(A \otimes \mathcal{O}_\infty) = 0$.

Remark 4.3. Observe that the converse of the implication in Proposition 4.1 is not true. Indeed, if $A = \mathbb{C}$, then $\mathrm{RR}(A \otimes \mathcal{O}_\infty) = \mathrm{RR}(\mathcal{O}_\infty) = 0$ by [39], since $\mathcal{O}_\infty$

is purely infinite and simple ([34]), while $\mathrm{RR}(A \otimes \mathcal{Z}) = \mathrm{RR}(\mathcal{Z}) \neq 0$.

We show that in spite of Remark 4.3, there are "many" separable C^*-algebras A for which $\mathrm{RR}(A \otimes \mathcal{Z}) = 0 \Leftrightarrow \mathrm{RR}(A \otimes \mathcal{O}_\infty) = 0$:

Theorem 4.4. *Let A be a separable, weakly purely infinite C^*-algebra. Then, the following are equivalent*:
(1) $\mathrm{RR}(A \otimes \mathcal{Z}) = 0$;
(2) $\mathrm{RR}(A \otimes \mathcal{O}_\infty) = 0$;
(3) $\mathrm{Prim}(A)$ *has a basis consisting of compact-open sets and A is K_0-liftable*;
(4) *$A \otimes \mathcal{Z}$ has the ideal property and A is K_0-liftable.*

Proof. Observe first that since A is weakly purely infinite, [26, Theorem 4.8 (ii)] implies that A is traceless, and hence, by [23, Corollary 3.12] it follows that $A \otimes \mathcal{Z}$ is strongly purely infinite. Therefore, by [26, Proposition 5.4], it follows that $A \otimes \mathcal{Z}$ is purely infinite.
(1) $\Rightarrow$ (2) follows from Proposition 4.1.
(2) $\Rightarrow$ (3) follows from [32, Corollary 4.3 (i)].
Using the fact that the separable C^*-algebra $A \otimes \mathcal{Z}$ is purely infinite, (3) $\Rightarrow$ (1) follows from [32, Theorem 4.2], Lemma 4.2 and Theorem 2.2 ($\mathrm{Prim}(A \otimes \mathcal{Z})$ is homeomorphic to $\mathrm{Prim}(A)$, since $\mathcal{Z}$ is exact and simple).
Since, as proved above, the separable C^*-algebra $A \otimes \mathcal{Z}$ is purely infinite, (3) $\Leftrightarrow$ (4) follows from [32, Proposition 2.11] and Theorem 2.2.

Remark 4.5. The proof of Theorem 4.4 shows in fact that if A is a separable C^*-algebra such that $A \otimes \mathcal{Z}$ is purely infinite, then the conditions (1), (2), (3) and (4) in Theorem 4.4 are equivalent.

Finally, let us observe that using [32], one could easily characterize when a separable, weakly purely infinite ($\mathcal{Z}$-stable) C^*-algebra has real rank zero:

Proposition 4.6. *Let A be a separable, weakly purely infinite C^*-algebra. Then, the following are equivalent*:
(1) $\mathrm{RR}(A) = 0$;
(2) *A has the ideal property and A is K_0-liftable.*

Proof. (1) $\Rightarrow$ (2). This is true for every C^*-algebra A, since any C^*-algebra of real rank zero has the ideal property and and it is also K_0-liftable (by [4]).
(2) $\Rightarrow$ (1). Assume now that A has the ideal property and A is K_0-liftable. Since A is weakly purely infinite and with the ideal property, it follows from [32, Proposition 2.14] that A is purely infinite. Also, since A has the ideal property, then by a general argument ([3, page 76]) it follows that $\mathrm{Prim}(A)$ has a basis consisting of compact-open sets. Finally, [32, Theorem 4.2] implies that $\mathrm{RR}(A) = 0$.

Corollary 4.7. *Let A be a separable C^*-algebra and assume that $A \otimes \mathcal{Z}$ is a*

weakly purely infinite C^*-algebra. Then, the following are equivalent:
(1) $\mathrm{RR}(A \otimes \mathcal{Z}) = 0$;
(2) *$A \otimes \mathcal{Z}$ has the ideal property and A is K_0-liftable.*

Proof. It follows from Proposition 4.2 and Proposition 4.6.

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