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Functional-differential equations with "maxima" via weakly Picard operators theory

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Abstract

The purpose of this paper is to present a differential equation with "maxima". Existence, uniqueness, inequalities of Čaplygin type and data dependence (monotony, continuity) results for the solution of the Cauchy problem of this equation are obtained using weakly Picard operators theory.

Key Words: Picard operators, weakly Picard operators, functional-differential equations with "maxima", fixed points, data dependence.
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1 Introduction

Differential equations with maximum arise naturally when solving practical problems, in particular, in those which appear in the study of systems with automatic regulation. The existence and uniqueness of solutions of equation with maxima is considered in [1], [3]-[5], [8], [13]. The asymptotic stability of the solution of this equations and other problems concerning equations with maxima are investigated in [2], [6], [7], [9], [14].

The purpose of this paper is to study the following Cauchy problem

$$x'(t) = f(t, x(t), \max_{a \leq \xi \leq t} x(\xi)), \; t \in [a, b] \tag{1}$$

$$x(a) = \alpha \tag{2}$$

where

(C_1) $\alpha \in \mathbb{R}$ and $f \in C([a, b] \times \mathbb{R}^2)$ are given;

(C_2) there exists $L_f > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f \max(|u_1 - v_1|, |u_2 - v_2|)$$

for all $t \in [a, b]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$.

In the condition (C_1) the problem (1)–(2), $x \in C^1[a,b]$ is equivalent with the fixed point equation

$$x(t) = \alpha + \int_a^t f(s, x(s), \max_{a \le \xi \le s} x(\xi))ds,\ t \in [a,b], \tag{3}$$

$x \in C[a,b]$, and the equation (1) is equivalent with

$$x(t) = x(a) + \int_a^t f(s, x(s), \max_{a \le \xi \le s} x(\xi))ds,\ t \in [a,b], \tag{4}$$

$x \in C[a,b]$.

Let us consider the following operators:

$$B_f, E_f : C[a,b] \to C[a,b]$$

defined by

$$B_f(x)(t) := \text{second part of (3)}$$

and

$$E_f(x)(t) := \text{second part of (4)}.$$

For $\alpha \in \mathbb{R}$, we consider $X_\alpha := \{x \in C[a,b] |\ x(a) = \alpha\}$.

We remark that

$$C[a,b] = \underset{\alpha \in \mathbb{R}}{\cup} X_\alpha$$

is a partition of $C[a,b]$.

We have

Lemma 1.1. *If (C_1) is satisfied, then*

(a) $B_f(C[a,b]) \subset X_\alpha$ *and* $E_f(X_\alpha) \subset X_\alpha,\ \forall \alpha \in \mathbb{R}$;

(b) $B_f|_{X_\alpha} = E_f|_{X_\alpha},\ \forall \alpha \in \mathbb{R}$.

In this paper we shall prove that if (C_1) and (C_2) are satisfied and if L_f is small enough, then the operator E_f is weakly Picard operator ([11]), in $(C[a,b], \|\cdot\|)$ where $\|x\| := \max\limits_{a \le t \le b} x(t)$, and we study the equation (1) in the terms of the weakly Picard operator theory.

2 Weakly Picard operators

Let (X,d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of the nonempty invariant subsets of A;

By H we denote the Pompeiu-Housdorff functional, $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by:

$$H(Y, Z) := \max\{\sup_{y\in Y}\inf_{z\in Z} d(y,z), \sup_{z\in Z}\inf_{y\in Y} d(y,z)\}$$

Definition 2.1. ([*11*], [*12*]) *Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;

(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.2. ([*11*], [*12*]) *Let (X, d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 2.3. ([*11*], [*12*]) *If A is weakly Picard operator then we consider the operator A^∞ defined by*

$$A^\infty : X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 2.4. ([*11*], [*12*]) *It is clear that $A^\infty(X) = F_A$.*

Definition 2.5. ([*11*], [*12*]) *Let A be a weakly Picard operator and $c > 0$. The operator A is c -weakly Picard operator if*

$$d(x, A^\infty(x)) \le c d(x, A(x)),\ \forall x \in X.$$

For some examples of WPOs see [10], [11], [12].

3 Cauchy problem

Relative to problem (1)–(2) we have

Theorem 3.1. *We suppose that:*

(a) the condition (C_1) and (C_2) are satisfied;

(C_3) $L_f(b - a) < 1$.

Then the problem (1)–(2) has, in $C[a, b]$, a unique solution and this solution is the uniform limit of the successive approximations.

Proof: The problem (1)–(2) is equivalent with the fixed point equation

$$B_f(x) = x,\ x \in C[a, b].$$

On the other hand we have that

$$|B_f(x)(t)-B_f(y)(t)|\leq L_f\int_a^t \max\left(|x(s)-y(s)|\,,\left|\max_{a\leq\xi\leq s}x(\xi)-\max_{a\leq\xi\leq s}y(\xi)\right|\right)ds.$$

But

$$\max_{a\leq s\leq b}\left|\max_{a\leq\xi\leq s}x(\xi)-\max_{a\leq\xi\leq s}y(\xi)\right|\leq \max_{a\leq s\leq b}|x(s)-y(s)|\,.$$

So,

$$\|B_f(x)-B_f(y)\|\leq L_f(b-a)\,\|x-y\|\,,\ \forall x,y\in C[a,b],$$

i.e., B_f is a contraction w.r.t. Chebyshev norm on $C[a,b]$. The proof follows from the contraction principle. □

Remark 3.2. *In the conditions of Theorem 3.1, the operator B_f is PO. But*

$$B_f|_{X_\alpha}=E_f|_{X_\alpha},\ \forall\alpha\in\mathbb{R}.$$

Hence, the operator E_f is WPO and $F_{E_f}\cap X_\alpha=\{x_\alpha^*\},\forall\alpha\in\mathbb{R}$, where x_α^* is the unique solution of the problem (1)–(2).

4 Inequalities of Čaplygin type

We have

Theorem 4.1. *We suppose that:*

(a) the conditions (C_1), (C_2) and (C_3) are satisfied;

(b) $f(x,\cdot,\cdot):\mathbb{R}^2\to\mathbb{R}^2$ is increasing, i.e., $u_1\leq v_1,u_2\leq v_2\Rightarrow f(x,u_1,u_2)\leq f(x,v_1,v_2)$.

Let x be a solution of equation (1) and y a solution of the inequality

$$y'(t)\leq f(t,y(t),\max_{a\leq\xi\leq t}y(\xi)),\ t\in[a,b].$$

Then

$$y(a)\leq x(a)\text{ implies that }y\leq x.$$

Proof: In the terms of the operator E_f, we have

$$x=E_f(x)\text{ and }y\leq E_f(y),$$

and $x(a)\leq y(a)$.

From the conditions (C_1), (C_2) and (C_3) we have that the operator E_f is WPO. From the condition (b), E_f^∞ is increasing ([11]). If $\alpha \in \mathbb{R}$, then we denote by $\widetilde{\alpha}$ the following function

$$\widetilde{\alpha} : [a,b] \to \mathbb{R},\ \widetilde{\alpha}(t) = \alpha,\ \forall t \in [a,b].$$

We have

$$y \le E_f(y) \le \ldots \le E_f^\infty(y) = E_f^\infty(\widetilde{y}(a)) \le E_f^\infty(\widetilde{x}(a)) = x.$$

□

5 Data dependence: monotony

In this section we need the following abstract result.

Lemma 5.1. *(Comparison principle, [12]) Let $(X, d, \le)$ an ordered metric space and $A, B, C : X \to X$ be such that:*

(a) $A \le B \le C$;

(b) the operator A, B, C, are WPOs;

(c) the operator B is increasing.

Then $x \le y \le z$ imply that $A^\infty(x) \le B^\infty(y) \le C^\infty(z)$.

From this abstract result we have

Theorem 5.2. *Let $f_i \in C([a,b]\times\mathbb{R}^2), i = 1,2,$ be as in Theorem 3.1. We suppose that:*

(i) $f_1 \le f_2 \le f_3$;

(ii) $f_2(t,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ *is increasing;*

Let $x_i \in C^1[a,b]$ be a solution of the equation

$$x_i'(t) = f_i(t, x(t), \max_{a\le\xi\le t} x(\xi)),\ t \in [a,b] \text{ and } i = 1,2,3.$$

If $x_1(a) \le x_2(a) \le x_3(a)$, then $x_1 \le x_2 \le x_3$.

Proof: From Theorem 3.1 we have that the operator $E_{f_i}, i = 1,2,3$, are WPOs. From the condition (ii) the operator E_{f_2} is monotone increasing. From the condition (i) it follows that

$$E_{f_1} \le E_{f_2} \le E_{f_3}.$$

Let $\widetilde{x}_i(a) \in C[a,b]$ be defined by $\widetilde{x}_i(a)(t) = x_i(a),\ \forall t \in [a,b]$. It is clear that

$$\widetilde{x}_1(a)(t) \leq \widetilde{x}_2(a)(t) \leq \widetilde{x}_3(a)(t),\ \forall t \in [a,b].$$

From Lemma 5.1 we have that

$$E_{f_1}^{\infty}(\widetilde{x}_1(a)) \leq E_{f_2}^{\infty}(\widetilde{x}_2(a)) \leq E_{f_3}^{\infty}(\widetilde{x}_3(a)).$$

But $x_i = E_{fi}^{\infty}(\widetilde{x}_i(a))$, and $x_1 \leq x_2 \leq x_3$. □

6 Data dependence: continuity

Consider the Cauchy problem (1)–(2) and suppose the conditions of the Theorem 3.1 are satisfied. Denote by $x^*(\cdot;\alpha,f)$ the solution of this problem.

We need the following well known result (see [11]).

Theorem 6.1. *Let (X,d) be a complete metric space and $A,B : X \to X$ two operators. We suppose that*

(i) the operator A is a α -contraction;

(ii) $F_B \neq \emptyset$;

(iii) there exists $\eta > 0$ such that

$$d(A(x),B(x)) \leq \eta,\ \forall x \in X.$$

Then, if $F_A = \{x_A^*\}$ and $x_B^* \in F_B$, we have

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

We can state the following result:

Theorem 6.2. *Let $\alpha_i, f_i, i = 1,2$ be as in the Theorem 3.1. Furthermore, we suppose that there exists $\eta_i > 0, i = 1,2$ such that*

(i) $|\alpha_1(t) - \alpha_2(t)| \leq \eta_1, \forall t \in [a,b]$;

(ii) $|f_1(t,u_1,u_2) - f_2(t,u_1,u_2)| \leq \eta_2, \forall t \in [a,b], u_i \in \mathbb{R}, i = 1,2.$

Then

$$\|x_1^*(t;\alpha_1,f_1) - x_2^*(t;\alpha_2,f_2)\| \leq \frac{\eta_1 + (b-a)\eta_2}{1 - L_f(b-a)},$$

where $x_i^*(t;\alpha_i,f_i), i = 1,2$ are the solution of the problem (1)–(2) with respect to α_i, f_i and $L_f = \max(L_{f_1}, L_{f_2})$.

Proof: Consider the operators B_{α_i,f_i}, $i = 1, 2$. From Theorem 3.1 these operators are contractions.

Additionally

$$\|B_{\alpha_1,f_1}(x) - B_{\alpha_2,f_2}(x)\| \leq \eta_1 + (b-a)\eta_2,$$

$\forall x \in C[a, b]$.

Now the proof follows from the Theorem 6.1, with $A := B_{\alpha_1,f_1}$, $B = B_{\alpha_2,f_2}$, $\eta = \eta_1 + (b-a)\eta_2$ and $\alpha := L_f(b - t_0)$, where $L_f = \max(L_{f_1}, L_{f_2})$. □

In what follow we shall use the c-WPOs techniques to give some data dependence results.

Theorem 6.3. ([10], [12]) *Let (X, d) be a metric space and $A_i : X \to X$, $i = 1, 2$. Suppose that*

(i) the operator A_i is c_i-weakly Picard operator, $i = 1, 2$;

(ii) there exists $\eta > 0$ such that

$$d(A_1(x), A_2(x)) \leq \eta, \ \forall x \in X.$$

Then $H(F_{A_1}, F_{A_2}) \leq \eta \max(c_1, c_2)$.

We have

Theorem 6.4. *Let f_1 and f_2 be as in the Theorem 3.1. Let $S_{E_{f_1}}, S_{E_{f_2}}$ be the solution set of system (1) corresponding to f_1 and f_2. Suppose that there exists $\eta > 0$, such that*

$$|f_1(t, u_1, u_2) - f_2(t, u_1, u_2)| \leq \eta \tag{5}$$

for all $t \in [a, b], u_i \in \mathbb{R}, i = 1, 2$.

Then

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \leq \frac{(b-a)\eta}{1 - L_f(b-a)},$$

where $L_f = \max(L_{f_1}, L_{f_2})$ and $H_{\|\cdot\|_C}$ denotes the Pompeiu-Housdorff functional with respect to $\|\cdot\|_C$ on $C[a, b]$.

Proof: In the condition of Theorem 3.1, the operators E_{f_1} and E_{f_2} are c_i-weakly Picard operators, $i = 1, 2$.

Let

$$X_\alpha := \{x \in C[a, b] |\ x(a) = \alpha\}.$$

It is clear that $E_{f_1}|_{X_\alpha} = B_{f_1}$, $E_{f_2}|_{X_\alpha} = B_{f_2}$. Therefore,

$$\left|E_{f_1}^2(x) - E_{f_1}(x)\right| \leq L_{f_1}(b-a) \left|E_{f_1}(x) - x\right|,$$

$$\left|E_{f_2}^2(x) - E_{f_2}(x)\right| \leq L_{f_2}(b-a)\left|E_{f_2}(x) - x\right|,$$

for all $x \in C[a,b]$.

Now, choosing

$$\alpha_1 = L_{f_1}(b-a) \text{ and } \alpha_2 = L_{f_2}(b-a),$$

we get that E_{f_1} and E_{f_2} are c_i-weakly Picard operators, $i = 1,2$ with $c_1 = (1-\alpha_1)^{-1}$ and $c_2 = (1-\alpha_2)^{-1}$. From (5) we obtain that

$$\|E_{f_1}(x) - E_{f_2}(x)\|_C \leq (b-a)\eta,$$

$\forall x \in C[a,b]$. Applying Theorem 6.3 we have that

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \leq \frac{(b-a)\eta}{1 - L_f(b-a)},$$

where $L_f = \max(L_{f_1}, L_{f_2})$ and $H_{\|\cdot\|_C}$ is the Pompeiu-Housdorff functional with respect to $\|\cdot\|_C$ on
$C[a,b]$. □

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