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Curvature of a class of indefinite globally framed f-manifolds

by
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Abstract

We present a compared analysis of some properties of indefinite almost $\mathcal{S}$-manifolds and indefinite $\mathcal{S}$-manifolds. We give some characterizations in terms of the Levi-Civita connection and of the characteristic vector fields. We study the sectional and φ-sectional curvature of indefinite almost $\mathcal{S}$-manifolds and state an expression of the curvature tensor field for the indefinite $\mathcal{S}$-space forms. We analyse the sectional curvature of indefinite $\mathcal{S}$-manifold in which the number of the spacelike characteristic vector fields is equal to that of the timelike characteristic vector fields. Some examples are also described.

Key Words: Semi-Riemannian manifolds, indefinite metrics, f-structures, sectional curvature, φ-sectional curvature.
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1 Introduction

In the framework of Riemannian geometry, almost S-manifolds and S-manifolds represent a natural generalization of contact and Sasaki manifolds, respectively. Such manifolds have been extensively studied by several authors and from different points of view ([2, 3, 4, 7, 8, 12]). On the other hand, also Sasakian manifolds with semi-Riemannian metric have been considered ([10, 6, 17]), and in recent works many authors, (for example, in [13], K.L. Duggal and B. Sahin) study lightlike submanifolds of indefinite Sasakian manifolds. Indefinite $\mathcal{S}$-manifolds are natural generalizations of indefinite Sasaki manifolds. Moreover many space-time manifolds can be endowed with f-structures ([9]).

After a first section on f-structures and indefinite metric $g.f.f$-structures, in section 3, we carry out an in-depth study of the indefinite (almost) $\mathcal{S}$-manifolds. In section 4 we describe two examples of 6-dimensional indefinite $\mathcal{S}$-manifolds having two characteristic vector fields which are both spacelike or both timelike. A third example is a Lorentzian indefinite $\mathcal{S}$-manifold of dimension 4 with two characteristic vector fields of different causal type. In section 5, after some Lemmas, we prove that the φ-sectional curvatures completely determine the sectional curvatures. Then, we find an expression of the curvature tensor field R which characterizes the indefinite $\mathcal{S}$-space forms, that is indefinite $\mathcal{S}$-manifolds with constant φ-sectional curvature. Then, in section 6, we consider the curvature of special indefinite $\mathcal{S}$-manifold in which the number of the characteristic vector fields is even with an equal number of spacelike and timelike characteristic vector fields; we prove that the special indefinite $\mathcal{S}$-manifold described in the third example in section 4 turns out to be an indefinite $\mathcal{S}$-space form whose φ-sectional curvature vanishes.

All manifolds and tensor fields are assumed to be smooth.

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2 Indefinite metric f-structure

We recall that an f-structure on a manifold M is a non null $(1,1)$-tensor field φ on M of constant rank such that $\varphi^3 + \varphi = 0$. A manifold M, provided with an f-structure, is said to be an *f-manifold*, and it is known that TM splits into two complementary subbundles $\operatorname{Im}\varphi$ and $\ker\varphi$ and that the restriction of φ to $\operatorname{Im}\varphi$ determines a complex structure on it and the rank of φ is even. An interesting case of f-structure occurs when $\ker\varphi$ is parallelizable for which there exist global vector fields ξ_α, $\alpha \in \{1, \dots, r\}$, with their dual 1-forms η^α, satisfying: $\varphi^2 = -I + \sum_{\alpha=1}^{r} \eta^\alpha \otimes \xi_\alpha$, and $\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta$. Such an f-structure is called an *f-structure with parallelizable kernel* or *globally framed f-structure*, briefly denoted *$g.f.f$-structure* ([14]). Moreover, a manifold M endowed with a $g.f.f$-structure is called a *$g.f.f$-manifold*, and it is denoted with $(M, \varphi, \xi_\alpha, \eta^\alpha)$; the vector fields ξ_α, $(\alpha = 1, ..., r)$, are called *characteristic vector fields*.

It is also known that an f-structure, on a manifold M, is called *normal* if the tensor field $N = N_\varphi + 2\sum_{\alpha=1}^{r} d\eta^\alpha \otimes \xi_\alpha$ vanishes, where N_φ is the Nijenhuis torsion of φ.

Definition 2.1. Let (M, φ) be a $(2n+r)$-dimensional f-manifold and g a semi-Riemannian metric on M with index ν, $0 < \nu < 2n + r$. Then, the pair (φ, g) is said to be an *indefinite metric f-structure*, and the triple (M, φ, g) is called an *indefinite metric f-manifold*, if φ is skew-symmetric with respect to g, that is, for any $X, Y \in \Gamma(TM)$:

$$g(\varphi X, Y) + g(X, \varphi Y) = 0.$$

Definition 2.2. Let $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha)$ be a $g.f.f$ -manifold, and g a semi-Riemannian metric on M with index ν, $0 < \nu < 2n + r$. Then, we say that the two structures are *compatible* if for any $X, Y \in \Gamma(TM)$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{r} \varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y), \quad \varepsilon_\alpha g(X, \xi_\alpha) = \eta^\alpha(X) \tag{1}$$

for any $\alpha \in \{1, \ldots, r\}$, where $\varepsilon_\alpha = \pm 1$ according to whether ξ_α is spacelike or timelike. Then $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is called an *indefinite metric g.f.f-manifold.*

We shall use the Einstein convention omitting the sum symbol for repeated indices above and below, writing, e.g., $\varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y)$ to mean $\sum_{\alpha=1}^{r} \varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y)$.

Observe that if g is a semi-Riemannian metric on a $g.f.f$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha)$ compatible with the f-structure φ, then the pair (φ, g) is necessarily an indefinite metric f-structure. The fundamental 2-form Φ is defined putting $\Phi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$. Let $(M, \varphi, \xi_\alpha, \eta^\alpha)$, with $\alpha = 1, \ldots, r$, be a $g.f.f$-manifold, and g a compatible semi-Riemannian metric on M. We know that the orthogonal decomposition $TM = \operatorname{Im}\varphi \oplus \ker\varphi$ holds, and that the induced structure J on $\operatorname{Im}\varphi$ is an almost complex structure; then $(\operatorname{Im}\varphi, g = g|_{\operatorname{Im}\varphi}, J)$ is a indefinite Hermitian distribution and the only possible signatures of g are $(2p, 2q)$ with $p + q = n$; therefore g cannot be a Lorentz metric, for $n > 1$. We shall denote $\operatorname{Im}\varphi$ and $\ker\varphi$ with $\mathfrak{D}$ and $\mathfrak{D}^\perp$ respectively and for a section of $\mathfrak{D}$ ($\mathfrak{D}^\perp$) we will write $X \in \mathfrak{D}$ or $X \in \Gamma(\mathfrak{D})$ ($X \in \mathfrak{D}^\perp$ or $X \in \Gamma(\mathfrak{D}^\perp)$).

We recall the following result due to A. Bejancu and K.L. Duggal ([10]).

Theorem 2.3. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha)$, $\alpha = 1, \ldots, r$, be a g.f.f.-manifold and h_0 a semi-Riemannian metric on M; we suppose that $\{\xi_\alpha\}_{1\le\alpha\le r}$ are h_0-orthonormal and that $h_0(\xi_\alpha, \xi_\alpha) = -\varepsilon_\alpha$, for any $\alpha \in \{1, \ldots, r\}$. Then there exists a symmetric tensor field g of type* $(0, 2)$ *on M satisfying* (1).

Now, with a standard computation as in the Riemannian setting ([2]), one can prove the following results.

Proposition 2.4. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric g.f.f-manifold. Then, the Levi-Civita connection satisfies the following equality, for any $X, Y, Z \in \Gamma(TM)$:*

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) &= 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + g(N(Y, Z), \varphi X) \\ &+ \varepsilon_\alpha N^{(2)}_\alpha(Y, Z)\eta^\alpha(X) + 2\varepsilon_\alpha d\eta^\alpha(\varphi Y, X)\eta^\alpha(Z) \\ &- 2\varepsilon_\alpha d\eta^\alpha(\varphi Z, X)\eta^\alpha(Y), \end{aligned} \tag{2}$$

where $N^{(2)}_\alpha(X, Y) = (\mathcal{L}_{\varphi X}\eta^\alpha)(Y) - (\mathcal{L}_{\varphi Y}\eta^\alpha)(X) = 2d\eta^\alpha(\varphi X, Y) - 2d\eta^\alpha(\varphi Y, X)$.

Proposition 2.5. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric g.f.f-manifold. Then the following statements hold:*

a) $(\mathcal{L}_{\xi_\alpha}\Phi)(X, Y) = (\mathcal{L}_{\xi_\alpha} g)(X, \varphi Y) + g(X, (\mathcal{L}_{\xi_\alpha}\varphi)Y)$, *for any* $\alpha \in \{1, \ldots, r\}$.

b) $(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z)$, *for any* $X, Y, Z \in \Gamma(TM)$.

c) If $\mathcal{L}_{\xi_\alpha}\varphi = 0$, then $\eta^\beta[\varphi Z, \xi_\alpha] = 0$, for any $\beta \in \{1, \dots, r\}$.

d) $N = 0 \Rightarrow N^{(2)}_\alpha = 0$, for any $\alpha \in \{1, \dots, r\}$.

Between the indefinite metric $g.f.f$-manifolds, we can define the following classes.

Definition 2.6. Let $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric $g.f.f$-manifold. M is called *indefinite $\mathcal{K}$-manifold* if it is normal and $d\Phi = 0$.

In this case $\mathcal{L}_{\xi_\alpha}\Phi = i_{\xi_\alpha} d\Phi + d i_{\xi_\alpha}\Phi = 0$, therefore, from a) of Proposition 2.5, we obtain that $\mathcal{L}_{\xi_\alpha}\varphi = 0$ if and only if the characteristic vector fields ξ_α are Killing. Two subclasses of indefinite $\mathcal{K}$-manifolds are those of indefinite $\mathcal{C}$-manifolds and indefinite $\mathcal{S}$-manifolds, that are defined as follows: an indefinite $\mathcal{K}$-manifold is called *indefinite $\mathcal{C}$-manifold* if $d\eta^\alpha = 0$ for any $\alpha \in \{1, \dots, r\}$, while it is called *indefinite $\mathcal{S}$-manifold* if $d\eta^\alpha = \Phi$ for any $\alpha \in \{1, \dots, r\}$.

3 Indefinite $\mathcal{S}$-manifolds

The properties of (almost) $\mathcal{S}$-manifolds (with Riemannian metric) are studied in [12] and in [2]. Now, we discuss indefinite (almost) $\mathcal{S}$-manifolds and their properties.

3.1 Indefinite almost $\mathcal{S}$-manifolds

Definition 3.1. Let $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric $g.f.f$-manifold. M is called *indefinite almost $\mathcal{S}$-manifold* if $d\eta^\alpha = \Phi$ for any $\alpha \in \{1, \dots, r\}$.

Lemma 3.2. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $\mathcal{S}$-manifold. Then the tensor fields $N^{(2)}_\alpha$ vanish and for any $X, Y \in \Gamma(\mathfrak{D})$ and $\alpha \in \{1, \dots, r\}$, we have*

$$\eta^\alpha[\varphi X, Y] = \eta^\alpha[\varphi Y, X]$$

Proof: For $\alpha \in \{1, \dots, r\}$, we have $N^{(2)}_\alpha(X, Y) = 2d\eta^\alpha(\varphi X, Y) - 2d\eta^\alpha(\varphi Y, X) = 2\Phi(\varphi X, Y) - 2\Phi(\varphi Y, X) = 0$. Then, for any $X, Y \in \Gamma(\mathfrak{D})$, $2d\eta^\alpha(\varphi X, Y) = -\eta^\alpha([\varphi X, Y])$ implies $\eta^\alpha[\varphi X, Y] = \eta^\alpha[\varphi Y, X]$. ∎

Proposition 3.3. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $\mathcal{S}$-manifold and $\bar\eta := \sum_{\alpha=1}^r \varepsilon_\alpha \eta^\alpha$. Then, the following statements hold:*

$$2g((\nabla_X\varphi)Y, Z) = g(N(Y, Z), \varphi X) + 2g(\varphi Y, \varphi X)\bar\eta(Z) - 2g(\varphi Z, \varphi X)\bar\eta(Y), \quad (3)$$

$$\nabla_{\xi_\alpha}\varphi = 0, \qquad \nabla_{\xi_\alpha}\xi_\beta = 0 \quad (4)$$

for all $\alpha, \beta \in \{1, \dots, r\}$.

Proof: Equation (3) follows from (2) using $d\Phi = 0$, $N_\alpha^{(2)} = 0$ and $d\eta^\alpha = \Phi$, for $\alpha \in \{1, \ldots, r\}$. Then, putting $X = \xi_\alpha$, we obtain $\nabla_{\xi_\alpha}\varphi = 0$.

Hence, we have $0 = (\nabla_{\xi_\alpha}\varphi)(\xi_\beta) = -\varphi(\nabla_{\xi_\alpha}\xi_\beta)$, therefore $\nabla_{\xi_\alpha}\xi_\beta \in \mathfrak{D}^\perp$, which implies that $[\xi_\alpha, \xi_\beta] \in \mathfrak{D}^\perp$. On the other hand, for any $\gamma \in \{1, \ldots, r\}$

$$0 = \Phi(\xi_\alpha, \xi_\beta) = d\eta^\gamma(\xi_\alpha, \xi_\beta) = -\frac{1}{2}\eta^\gamma[\xi_\alpha, \xi_\beta] = -\frac{1}{2}\varepsilon_\gamma g([\xi_\alpha, \xi_\beta], \xi_\gamma).$$

Therefore $[\xi_\alpha, \xi_\beta] \in \mathfrak{D} \cap \mathfrak{D}^\perp$ and we obtain $[\xi_\alpha, \xi_\beta] = 0$ and $\nabla_{\xi_\alpha}\xi_\beta = \nabla_{\xi_\beta}\xi_\alpha$. Now we check that $\nabla_{\xi_\alpha}\xi_\beta \in \mathfrak{D}$, that is, for any $\gamma \in \{1, \ldots, r\}$, $g(\nabla_{\xi_\alpha}\xi_\beta, \xi_\gamma) = 0$. Being $g(\xi_\beta, \xi_\gamma) = \varepsilon_\beta\delta_{\beta\gamma}$ and using the covariant derivative with respect to ξ_α, we find $g(\nabla_{\xi_\alpha}\xi_\beta, \xi_\gamma) + g(\xi_\beta, \nabla_{\xi_\alpha}\xi_\gamma) = 0$, and, covariantly differentiating $g(\xi_\alpha, \xi_\gamma) = \varepsilon_\alpha\delta_{\alpha\gamma}$ with respect to ξ_β, we obtain $g(\nabla_{\xi_\beta}\xi_\alpha, \xi_\gamma) + g(\xi_\alpha, \nabla_{\xi_\beta}\xi_\gamma) = 0$. From the last two equations, using $\nabla_{\xi_\alpha}\xi_\beta = \nabla_{\xi_\beta}\xi_\alpha$, we have $g(\xi_\beta, \nabla_{\xi_\alpha}\xi_\gamma) = g(\xi_\alpha, \nabla_{\xi_\beta}\xi_\gamma)$. Therefore,

$$g(\nabla_{\xi_\alpha}\xi_\beta, \xi_\gamma) = g(\xi_\alpha, \nabla_{\xi_\gamma}\xi_\beta) = g(\xi_\alpha, \nabla_{\xi_\beta}\xi_\gamma) = -g(\nabla_{\xi_\beta}\xi_\alpha, \xi_\gamma) = -g(\nabla_{\xi_\alpha}\xi_\beta, \xi_\gamma),$$

from which $g(\nabla_{\xi_\alpha}\xi_\beta, \xi_\gamma) = 0$ follows. This result and $\nabla_{\xi_\alpha}\xi_\beta \in \mathfrak{D}^\perp$ imply

$$\nabla_{\xi_\alpha}\xi_\beta = 0.$$

□

Proposition 3.4. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $\mathcal{S}$-manifold. Then*

a) for any $\alpha \in \{1, \ldots, r\}$ the operator $h_\alpha = \frac{1}{2}\mathcal{L}_{\xi_\alpha}\varphi$ is self-adjoint,

b) for any $\alpha, \beta \in \{1, \ldots, r\}$, $h_\alpha(\xi_\beta) = 0$,

c) for any $\alpha \in \{1, \ldots, r\}$, $h_\alpha \circ \varphi + \varphi \circ h_\alpha = 0$.

Proof: As first step, using (4), for any $X, Y \in \Gamma(TM)$ and any $\alpha \in \{1, \ldots, r\}$, we easily obtain,

$$g((\mathcal{L}_{\xi_\alpha}\varphi)X, Y) = \varepsilon_\alpha(-(\varphi X)(\eta^\alpha(Y)) + \eta^\alpha(\nabla_{\varphi X}Y + \nabla_X(\varphi Y))).$$

It follows that

$$\begin{aligned} 2g(h_\alpha(X), Y) - 2g(h_\alpha(Y), X) &= -\varepsilon_\alpha(\varphi X)(\eta^\alpha(Y)) + \varepsilon_\alpha\eta^\alpha[\varphi X, Y] \\ &\quad + \varepsilon_\alpha(\varphi Y)(\eta^\alpha(X)) - \varepsilon_\alpha\eta^\alpha[\varphi Y, X] \\ &= -\varepsilon_\alpha(\mathcal{L}_{\varphi X}\eta^\alpha)(Y) + \varepsilon_\alpha(\mathcal{L}_{\varphi Y}\eta^\alpha)(X) = 0. \end{aligned}$$

Obviously, for any $\alpha, \beta \in \{1, \ldots, r\}$ we have $h_\alpha(\xi_\beta) = 0$ and finally

$$\begin{aligned} 2(h_\alpha \circ \varphi + \varphi \circ h_\alpha)(X) &= \mathcal{L}_{\xi_\alpha}(\varphi^2 X) - \varphi(\mathcal{L}_{\xi_\alpha}(\varphi X)) + \varphi(\mathcal{L}_{\xi_\alpha}(\varphi X) - \varphi(\mathcal{L}_{\xi_\alpha}X)) \\ &= \xi_\alpha(\eta^\beta(X))\xi_\beta - \eta^\beta[\xi_\alpha, X]\xi_\beta = 0 \end{aligned}$$

for any $\alpha \in \{1, \ldots, r\}$ and any $X \in \Gamma(TM)$. □

Proposition 3.5. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $\mathcal{S}$-manifold. Then, for any $X, Y \in \Gamma(TM)$, the following properties hold:*

a) $\varphi(N(X, Y)) + N(\varphi X, Y) = 2\eta^\alpha(X)h_\alpha(Y)$,

b) $N(X, Y) \in \mathfrak{D}$.

Proof: Using Lemma 3.2, we obtain

$$\begin{aligned}\varphi(N(X, Y)) + N(\varphi X, Y) &= -(\mathcal{L}_{\varphi Y}\eta^\alpha)(X)\xi_\alpha + (\mathcal{L}_{\varphi X}\eta^\alpha)(Y)\xi_\alpha \\ &\quad + \eta^\alpha(X)(\mathcal{L}_{\xi_\alpha}\varphi)(Y) = 2\eta^\alpha(X)h_\alpha(Y).\end{aligned}$$

Now, we observe that for any $\alpha \in \{1, \ldots, r\}$ we have $[\xi_\alpha, \mathfrak{D}] \subset \mathfrak{D}$, in fact, if $\beta \in \{1, \ldots, r\}$ and $X \in \Gamma(TM)$, we have $\eta^\beta[\xi_\alpha, \varphi X] = -2d\eta^\beta(\xi_\alpha, \varphi X) = 0$ and in particular, if $X \in \mathfrak{D}$ and $\alpha = \beta$, we get $\eta^\alpha[\xi_\alpha, X] = 0$. So, if $Z \in \mathfrak{D}$ then $N(\xi_\alpha, Z) = -[\xi_\alpha, Z] - \varphi[\xi_\alpha, \varphi Z] \in \mathfrak{D}$. It is easy to check that $N(\xi_\alpha, \xi_\beta) = 0$ for any $\alpha, \beta \in \{1, \ldots, r\}$; therefore, we have that $N(\xi_\alpha, X) \in \mathfrak{D}$ for any $X \in \Gamma(TM)$. Finally, applying a), we have $g(N(\varphi X, Y), \xi_\alpha) = 2\eta^\beta(X)g(h_\beta(Y), \xi_\alpha) = 0$. Hence, if $X, Y \in \Gamma(TM)$, we get $N(X, Y) = -N(\varphi^2 X, Y) + \eta^\alpha(X)N(\xi_\alpha, Y)$, and being $N(\varphi^2 X, Y) \in \mathfrak{D}$ and $N(\xi_\alpha, Y) \in \mathfrak{D}$, we conclude that $N(X, Y) \in \mathfrak{D}$. □

Proposition 3.6. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $\mathcal{S}$-manifold. For any $X \in \Gamma(TM)$ and for any $\alpha \in \{1, \ldots, r\}$,*

$$\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi(X) - \varphi(h_\alpha X).$$

Proof: Putting $X = \xi_\alpha$ in a) of Proposition 3.5, we have that for any $Z, Y \in \Gamma(TM)$

$$g(N(\xi_\alpha, Y), \varphi Z) = -g(\varphi(N(\xi_\alpha, Y)), Z) = -2\eta^\beta(\xi_\alpha)g(h_\beta(Y), Z) = -2g(h_\alpha(Y), Z).$$

Moreover, applying (3) of Proposition 3.3, for any $\alpha \in \{1, \ldots, r\}$ we find:

$$\begin{aligned}g(-\varphi(\nabla_X \xi_\alpha), Z) &= \frac{1}{2}g(N(\xi_\alpha, Z), \varphi X) - g(\varphi Z, \varphi X)\eta(\xi_\alpha) \\ &= -g(h_\alpha(Z), X) - \varepsilon_\alpha g(Z, X) + \varepsilon_\alpha \varepsilon_\beta \eta^\beta(X)\eta^\beta(Z) \\ &= g(-h_\alpha(X) - \varepsilon_\alpha X + \varepsilon_\alpha \eta^\beta(X)\xi_\beta, Z),\end{aligned}$$

then $\varphi(\nabla_X \xi_\alpha) = h_\alpha(X) + \varepsilon_\alpha X - \varepsilon_\alpha \eta^\beta(X)\xi_\beta$, and, applying φ, we complete the proof. Note that $\nabla_X \xi_\alpha \in \mathfrak{D}$. □

Proposition 3.7. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $\mathcal{S}$-manifold. For $X, Y \in \Gamma(TM)$, we have*

$$(\nabla_X \varphi)(Y) + (\nabla_{\varphi X}\varphi)(\varphi Y) = 2g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X) - \eta^\alpha(Y)h_\alpha(X).$$

where $\bar{\xi} := \sum_{\alpha=1}^r \xi_\alpha$ and $\bar{\eta}(X) = g(X, \bar{\xi})$, for any $X \in \Gamma(TM)$.

Proof: Using (3), Proposition 3.5 and Proposition 3.6, for any $X, Y, Z \in \Gamma(TM)$ we have

$$\begin{aligned} 2g((\nabla_X\varphi)(Y), Z) + 2g((\nabla_{\varphi X}\varphi)(\varphi Y), Z) &= -g(\varphi(N(Y,Z)) + N(\varphi Y, Z), X) \\ &\quad + 4g(\varphi Y, \varphi X)\bar{\eta}(Z) - 2g(\varphi Z, \varphi X)\bar{\eta}(Y) \\ &= -2g(Z, \eta^\alpha(Y)h_\alpha(X)) + \\ &\quad + 4g(\varphi Y, \varphi X)g(Z, \bar{\xi}) + 2g(Z, \bar{\eta}(Y)\varphi^2 X). \end{aligned}$$

Then, we deduce

$$(\nabla_X\varphi)(Y) + (\nabla_{\varphi X}\varphi)(\varphi Y) = 2g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X) - \eta^\alpha(Y)h_\alpha(X).$$

Obviously, $\bar{\eta}(X) = \sum_{\alpha=1}^{r} \varepsilon_\alpha \eta^\alpha(X) = \sum_{\alpha=1}^{r} g(X, \xi_\alpha) = g(X, \bar{\xi})$. □

Corollary 3.8. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $\mathcal{S}$-manifold. Then, for any $X, Y \in \mathfrak{D}$:*

a) $(\nabla_X\varphi)(Y) + (\nabla_{\varphi X}\varphi)(\varphi Y) = 2g(X, Y)\bar{\xi}$,

b) $(\nabla_X\varphi)(\varphi X) = (\nabla_{\varphi X}\varphi)(X)$.

Proof: The first statement follows from the above proposition. Putting $Y := \varphi X$ in a), we have $(\nabla_X\varphi)(\varphi X) + (\nabla_{\varphi X}\varphi)(\varphi^2 X) = 2g(X, \varphi X)\bar{\xi} = 0$, therefore, being $\varphi^2 X = -X$, we obtain $(\nabla_X\varphi)(\varphi X) = (\nabla_{\varphi X}\varphi)(X)$. □

Remark 3.9. The statement b) can be written as $\nabla_X(\varphi^2 X) - \varphi(\nabla_X \varphi X) = \nabla_{\varphi X}(\varphi X) - \varphi(\nabla_{\varphi X} X)$, i.e. as $\nabla_X X + \nabla_{\varphi X}(\varphi X) = \varphi[\varphi X, X]$.

3.2 Indefinite $\mathcal{S}$-manifolds

Definition 3.10. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric $g.f.f$-manifold. M is said an *indefinite $\mathcal{S}$-manifold* if it is a normal indefinite almost $\mathcal{S}$-manifold.

Proposition 3.11. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $\mathcal{S}$-manifold. Then M is an indefinite $\mathcal{S}$-manifold if and only if, for any $X, Y \in \Gamma(TM)$, the Levi-Civita connection satisfies:*

$$(\nabla_X\varphi)Y = g(X, Y)\bar{\xi} - \bar{\eta}(Y)X - \varepsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y)\bar{\xi} + \bar{\eta}(Y)\eta^\alpha(X)\xi_\alpha,$$

or equivalently

$$(\nabla_X\varphi)Y = g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X). \tag{5}$$

Proof: Assuming that M is an indefinite $\mathcal{S}$-manifold, (3) becomes

$$g((\nabla_X\varphi)Y,Z) = g(\varphi Y,\varphi X)\bar{\eta}(Z) - g(\varphi Z,\varphi X)\bar{\eta}(Y) = g(Z, g(\varphi Y,\varphi X)\bar{\xi} + \bar{\eta}(Y)\varphi^2 X),$$

from which

$$\begin{aligned}(\nabla_X\varphi)Y &= g(\varphi X,\varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X)\\ &= g(X,Y)\bar{\xi} - \varepsilon_\alpha\eta^\alpha(X)\eta^\alpha(Y)\bar{\xi} - \bar{\eta}(Y)X + \bar{\eta}(Y)\eta^\alpha(X)\xi_\alpha.\end{aligned}$$

Vice versa, we suppose that ∇ satisfies (5). Then we obtain $g((\nabla_X\varphi)Y,Z) = g(\varphi Y,\varphi X)\bar{\eta}(Z) - g(\varphi Z,\varphi X)\bar{\eta}(Y)$, and comparing with (3), we deduce for any $X,Y \in \Gamma(TM)$, $g(N(Y,Z),\varphi X) = 0$. From Proposition 3.5, we obtain that $N(Y,Z) = 0$ for any $Y,Z \in \Gamma(TM)$, that is M is normal. □

Remark 3.12. In an indefinite $\mathcal{S}$-manifold $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$, the operators $\mathcal{L}_{\xi_\alpha}\varphi$, and then h_α, vanish. In fact, by direct computation for any $X \in \Gamma(TM)$ and for any $\alpha \in \{1,\dots,r\}$ we get $N(\varphi X,\xi_\alpha) = (\mathcal{L}_{\xi_\alpha}\varphi)X = 2h_\alpha(X)$, and the normality condition implies $h_\alpha = 0$. Using Proposition 3.6, we obtain, for any $\alpha \in \{1,\dots,r\}$, $\nabla_X\xi_\alpha = -\varepsilon_\alpha\varphi X$.

Now, we give the condition of indefinite $\mathcal{S}$-manifold in terms of the fundamental 2-form:

Proposition 3.13. *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ be an indefinite almost $\mathcal{S}$-manifold. Then M is an indefinite $\mathcal{S}$-manifold if and only if for any $X,Y,Z \in \Gamma(TM)$:*

$$(\nabla_X\Phi)(Y,Z) = \bar{\eta}(Y)g(\varphi X,\varphi Z) - \bar{\eta}(Z)g(\varphi X,\varphi Y). \tag{6}$$

Proof: One simply uses $(\nabla_X\Phi)(Y,Z) = g(Y,(\nabla_X\varphi)Z)$ in (5). □

Proposition 3.14. *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ be an indefinite metric g.f.f-manifold. If the vector fields ξ_α are Killing, $\mathcal{L}_{\xi_\alpha}\eta^\beta = 0$ for any $\alpha,\beta \in \{1,\dots,r\}$ and M satisfies* (5) *or equivalently* (6)*, then M is an indefinite $\mathcal{S}$-manifold.*

Proof: Being $3d\Phi(X,Y,Z) = \mathfrak{S}_{X,Y,Z}(\nabla_X\Phi)(Y,Z)$, from (6) we get $d\Phi = 0$ and $(\mathcal{L}_{\xi_\alpha}\Phi)(X,Y) = 0$, since $\mathcal{L}_{\xi_\alpha}\Phi = i_{\xi_\alpha}d\Phi + di_{\xi_\alpha}\Phi$. Proposition 2.5 implies $(\mathcal{L}_{\xi_\alpha}g)(X,\varphi Y) + g(X,(\mathcal{L}_{\xi_\alpha}\varphi)Y) = 0$, for any $\alpha \in \{1,\dots,r\}$ and $X,Y \in \Gamma(TM)$. Hence, being ξ_α a Killing vector field, we find $\mathcal{L}_{\xi_\alpha}\varphi = 0$ and then $\eta^\beta([\xi_\alpha,\varphi Y]) = 0$, for any $\alpha,\beta \in \{1,\dots,r\}$. In these hypotheses, (2) becomes

$$\begin{aligned}2g((\nabla_X\varphi)Y,Z) = g(N(Y,Z),\varphi X) &+ 2\varepsilon_\alpha[d\eta^\alpha(\varphi Y,Z)\eta^\alpha(X) - d\eta^\alpha(\varphi Z,Y)\eta^\alpha(X)\\ &+ d\eta^\alpha(\varphi Y,X)\eta^\alpha(Z) - d\eta^\alpha(\varphi Z,X)\eta^\alpha(Y)].\end{aligned}$$

On the other hand, (6) implies $g(Y,(\nabla_X\varphi)Z) = \bar{\eta}(Y)g(\varphi X,\varphi Z) - \bar{\eta}(Z)g(\varphi X,\varphi Y)$, therefore we deduce

$$\begin{aligned}g(N(Y,Z),\varphi X) = -2\varepsilon_\alpha[&(d\eta^\alpha(\varphi Y,Z) - d\eta^\alpha(\varphi Z,Y))\eta^\alpha(X)\\ &+ (d\eta^\alpha(\varphi Y,X) - g(\varphi X,\varphi Y))\eta^\alpha(Z)\\ &- (d\eta^\alpha(\varphi Z,X) - g(\varphi X,\varphi Z))\eta^\alpha(Y)].\end{aligned}$$

Putting $Y = \xi_\beta$ in the above equation, we get

$$g(N(\xi_\beta, Z), \varphi X) = 2\varepsilon_\beta(d\eta^\beta(\varphi Z, X) - g(\varphi X, \varphi Z)). \tag{7}$$

Since $N(\xi_\beta, Z) = -[\xi_\beta, Z] - \varphi[\xi_\beta, \varphi Z] + \xi_\beta(\eta^\alpha(Z))\xi_\alpha$, then $\varphi N(\xi_\beta, Z) = (\mathcal{L}_{\xi_\alpha}\varphi)Z - \eta^\alpha[\xi_\beta, \varphi Z]\xi_\alpha = 0$ and (7) gives $d\eta^\beta(\varphi Z, X) = g(\varphi X, \varphi Z) = \Phi(\varphi Z, X)$. Finally, $\mathcal{L}_{\xi_\alpha}\eta^\beta = 0$ implying $i_{\xi_\alpha} d\eta^\beta = 0$ and being $Y = -\varphi^2 Y + \eta^\alpha(Y)\xi_\alpha$, for any $Y \in \Gamma(TM)$, we obtain $d\eta^\beta(Y, X) = -d\eta^\beta(\varphi^2 Y, X) + \eta^\alpha(Y) d\eta^\beta(\xi_\alpha, X) = -\Phi(\varphi^2 Y, X) = \Phi(Y, X)$. Then M is an indefinite almost $\mathcal{S}$-manifold and we apply Proposition 3.11. □

4 Examples of indefinite $\mathcal{S}$-manifolds

We describe some examples of indefinite $\mathcal{S}$-manifolds, where the characteristic vector fields are either timelike or spacelike or of both types.

Example 4.1. We consider $\mathbb{R}^6$ with its standard coordinates $\{x^1, x^2, y^1, y^2, z^1, z^2\}$. We introduce on $\mathbb{R}^6$ an indefinite $g.f.f$-structure $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ by setting

$$\xi_\alpha = \frac{\partial}{\partial z^\alpha}, \qquad \eta^\alpha = dz^\alpha - \sum_{i=1}^{2} y^i dx^i, \qquad \alpha \in \{1, 2\},$$

$$g = -\sum_{\alpha=1}^{2} \eta^\alpha \otimes \eta^\alpha + \frac{1}{2}\sum_{i=1}^{2}((dx^i)^2 + (dy^i)^2),$$

and φ given, with respect to the frame $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \xi_1, \xi_2\}$, by the matrix

$$F = \begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}, \qquad \text{where} \quad Y = \begin{pmatrix} y^1 & y^2 \\ y^1 & y^2 \end{pmatrix}.$$

We put $M = (\mathbb{R}^6_2, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$. A straightforward computation shows that g is a metric tensor field. Firstly we check that g is non-degenerate and then we compute its index. The matrix G of g is given by

$$G = \begin{pmatrix} \frac{1}{2} - 2(y^1)^2 & -2y^1y^2 & 0 & 0 & y^1 & y^1 \\ -2y^1y^2 & \frac{1}{2} - 2(y^2)^2 & 0 & 0 & y^2 & y^2 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ y^1 & y^2 & 0 & 0 & -1 & 0 \\ y^1 & y^2 & 0 & 0 & 0 & -1 \end{pmatrix},$$

and $detG = \frac{1}{16} \neq 0$. Now, to determine the index of g, we look for the eigenvalues of G. Since

$$det(G - \lambda I) = -(\frac{1}{2} - \lambda)^3(1 + \lambda)(\lambda^2 + (2(y^1)^2 + 2(y^2)^2 + \frac{1}{2})\lambda - \frac{1}{2}),$$

we find that the index of g is two; therefore g is a semi-Riemannian metric of the index 2 on $\mathbb{R}^6$. We remark that ξ_1 and ξ_2 are timelike vector fields. It is easy to prove that M is an indefinite $\mathcal{S}$-manifold.

Example 4.2. The second example of an indefinite $\mathcal{S}$-manifold is $M = (\mathbb{R}^6_2, \varphi, \xi_\alpha, \eta^\alpha, g)$, where, for any $\alpha \in \{1, 2\}$, we put

$$\xi_\alpha := \frac{\partial}{\partial z^\alpha}, \qquad \eta^\alpha := dz^\alpha - \sum_{i=1}^{2} \tau_i y^i dx^i,$$

φ, g are given by

$$F = \begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}, \quad \text{where} \quad Y = \begin{pmatrix} -y^1 & y^2 \\ -y^1 & y^2 \end{pmatrix},$$

and

$$g = \sum\nolimits_{\alpha=1}^{2} \eta^\alpha \otimes \eta^\alpha + \frac{1}{2} \sum\nolimits_{i=1}^{2} \tau_i((dx^i)^2 + (dy^i)^2),$$

respectively, where $\tau_i = \mp 1$ according to whether $i = 1$ or $i = 2$. Moreover, the symmetric $(0, 2)$-type tensor field g is a semi-Riemannian metric because $detG = \frac{1}{16} \neq 0$. Therefore g is non degenerate, and

$$det(G - \lambda I) = -(\frac{1}{2} + \lambda)^2(\frac{1}{2} - \lambda)(\lambda - 1)(\lambda^2 - (\frac{3}{2} + 2(y^1)^2 + 2(y^2)^2)\lambda + \frac{1}{2}),$$

so, since the signs of eigenvalues are independent from the coordinates, the index of g is constant. We note that in this example ξ_1 and ξ_2 are spacelike. One proves that M is an indefinite $\mathcal{S}$-manifold.

Example 4.3. The third example is $M = (\mathbb{R}^4_1, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ constructed as follows. Denoting the standard coordinates with $\{x, y, z^1, z^2\}$, we endow $\mathbb{R}^4$ with the structure $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ where

$$\xi_\alpha = \frac{\partial}{\partial z^\alpha}, \quad \eta^\alpha = dz^\alpha + ydx,$$

for any $\alpha \in \{1, 2\}$ and where the tensor fields φ and g are given by

$$F := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix} \qquad G := \begin{pmatrix} \frac{1}{2} & 0 & y & -y \\ 0 & \frac{1}{2} & 0 & 0 \\ y & 0 & 1 & 0 \\ -y & 0 & 0 & -1 \end{pmatrix}$$

respectively. An immediate computation shows that g is non-degenerate and its index is constant. In fact, we have $detG = -\frac{1}{4}$, and

$$det(G - \lambda I) = (\frac{1}{2} - \lambda)(\lambda^3 - \frac{1}{2}\lambda^2 - (2y^2 + 1)\lambda + \frac{1}{2}),$$

hence $detG \neq 0$ and, using Cartesio's rule, we deduce that the index is 1. Therefore, the tensor field g is a Lorentzian metric. Now, we observe that ξ_1 is a spacelike vector field while ξ_2 is a timelike vector field. One can check that M is an indefinite $\mathcal{S}$-manifold.

5 Sectional curvature and φ-sectional curvature

In this section, we look for some results about the sectional curvature of indefinite $\mathcal{S}$-manifolds. Following the notations in ([15]), for the curvature tensor R we have $R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, and $R(X, Y, Z, W) = g(R(Z, W, Y), X)$, for any $X, Y, Z, W \in \Gamma(TM)$.

A two-dimensional subspace π of the tangent space T_pM is called *non-degenerate* if and only if we have $\Delta(\pi) = g_p(X, X)g_p(Y, Y)$-$g_p(X, Y)^2 \neq 0$ for any basis $\{X, Y\}$ of π. We know that if π is a non-degenerate 2-plane of T_pM then we can define the *sectional curvature* $K_p(\pi)$ at p with respect to the 2-plane π, putting

$$K_p(\pi) = \frac{R_p(X, Y, X, Y)}{\Delta(\pi)} = \frac{g_p(R_p(X, Y, Y), X)}{\Delta(\pi)},$$

where $\pi = span\{X, Y\}$. In the following we denote $K_p(\pi) = K_p(X, Y)$.

Proposition 5.1. *In an indefinite $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ one has:*

a) the distribution $\ker\varphi$ *is integrable and flat;*

b) the sectional curvatures $K(X, \xi_\alpha) = \varepsilon_\alpha$, for any $\alpha \in \{1, \ldots, r\}$, and non lightlike $X \in \operatorname{Im}\varphi$.

Proof: For $X, Y \in \ker\varphi$ we have $X = f^\alpha\xi_\alpha$, $Y = t^\beta\xi_\beta$ then $[X, Y] = [f^\alpha\xi_\alpha, t^\beta\xi_\beta] = f^\alpha\xi_\alpha(t^\beta)\xi_\beta - t^\beta\xi_\beta(f^\alpha)\xi_\alpha \in \ker\varphi$ and $\ker\varphi$ is integrable. Furthermore, since $\nabla_{\xi_\alpha}\xi_\beta = 0$ and $[\xi_\alpha, \xi_\beta] = 0$, we have $R(\xi_\alpha, \xi_\beta, \xi_\gamma) = 0$ and $\ker\varphi$ is flat. Note that a) holds also for indefinite almost $\mathcal{S}$-manifolds. Now, being M an indefinite $\mathcal{S}$-manifold, we know that $\nabla_X\xi_\alpha = -\varepsilon_\alpha\varphi X$, $\mathcal{L}_{\xi_\alpha}\varphi = 0$ and we have

$$\begin{aligned} R(\xi_\alpha, X, \xi_\beta) &= -\varepsilon_\beta\nabla_{\xi_\alpha}(\varphi X) + \varepsilon_\beta\varphi[\xi_\alpha, X] \\ &= \varepsilon_\beta(\varphi[\xi_\alpha, X] - [\xi_\alpha, \varphi X] - \nabla_{\varphi X}\xi_\alpha) = \varepsilon_\beta\varepsilon_\alpha\varphi^2 X. \end{aligned}$$

So, for $X \in \operatorname{Im}\varphi$, X non lightlike, we have $K(X, \xi_\alpha) = -\frac{\varepsilon_\alpha g(\varphi^2 X, X)}{g(X, X)} = \varepsilon_\alpha$.

□

As usual, we say that a 2-plane π in T_pM, $p \in M$, is a *φ-plane* if $\pi = span\{X, \varphi X\}$ with $X \in \mathfrak{D}_p$, and the sectional curvature at p of such a plane, with X a non lightlike vector, is said the *φ-sectional curvature* at p and is denoted by $H_p(X)$.

We shall prove that on an indefinite $\mathcal{S}$-manifold, as in the Sasakian case, the φ-sectional curvatures determine the sectional curvatures.

As in [3], we define a tensor field of type (0,4) given for any X, Y, Z, W in $\Gamma(TM)$ by

$$\begin{aligned} P(X, Y; Z, W) &= \Phi(X, Z)g(Y, W) - \Phi(X, W)g(Y, Z) \\ &\quad - \Phi(Y, Z)g(X, W) + \Phi(Y, W)g(X, Z). \end{aligned}$$

The following lemmas can be easily proved.

Lemma 5.2. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite $\mathcal{S}$-manifold. Then:*

a) $P(X, Y; Z, W) = -P(Z, W; X, Y)$, for any $X, Y, Z, W \in \Gamma(TM)$,

b) $P(X, Y; X, \varphi Y) = g(X, \varphi Y)^2 + g(X, Y)^2 - \varepsilon_X \varepsilon_Y$, where X, Y are unit vector fields in $\mathfrak{D}$ and $\varepsilon_X = g(X, X)$ and $\varepsilon_Y = g(Y, Y)$.

Proposition 5.3. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite $\mathcal{S}$-manifold. Then, putting $\varepsilon = \sum_{\alpha=1}^r \varepsilon_\alpha$, for any $X, Y, Z, W \in \Gamma(TM)$*

$$g(R(X, Y, \varphi Z), W) + g(R(X, Y, Z), \varphi W) = -\varepsilon P(X, Y; Z, W) - Q(X, Y; Z, W)$$

where

$$\begin{aligned} Q(X, Y; Z, W) &= g(W, \varphi Y)(\varepsilon(g(X, Z) - g(\varphi X, \varphi Z)) - \bar\eta(Z)\bar\eta(X)) \\ &\quad - g(W, \varphi X)(\varepsilon(g(Y, Z) - g(\varphi Y, \varphi Z)) - \bar\eta(Z)\bar\eta(Y)) \\ &\quad - g(Z, \varphi Y)(\varepsilon(g(X, W) - g(\varphi X, \varphi W)) - \bar\eta(X)\bar\eta(W)) \\ &\quad + g(Z, \varphi X)(\varepsilon(g(Y, W) - g(\varphi Y, \varphi W)) - \bar\eta(Y)\bar\eta(W)). \end{aligned}$$

Moreover if $X, Y, Z, W \in \mathfrak{D}$ then obviously $Q(X, Y; Z, W) = 0$ and the following statements hold:

a) $g(R(\varphi X, \varphi Y, \varphi Z), \varphi W) = g(R(X, Y, Z), W)$;

*b) $g(R(X, \varphi X, Y), \varphi Y) = g(R(X, Y, X), Y)+$
$g(R(X, \varphi Y, X), \varphi Y) - 2\varepsilon P(X, Y, X, \varphi Y)$;*

c) $g(R(\varphi X, Y, \varphi X), Y) = g(R(X, \varphi Y, X), \varphi Y)$.

Remark 5.4. We remark that ε can vanish only if r is an even number and the number of timelike characteristic vector fields is equal to the number of spacelike characteristic vector fields. Moreover, $\varepsilon = 0$ means that $g(\bar\xi, \bar\xi) = 0$, i.e. $\bar\xi = \sum_{\alpha=1}^r \xi_\alpha$ is a lightlike vector field.

We put

$$B(X,Y) = g(R(X,Y,X),Y), \qquad X,Y \in \Gamma(TM)$$

and

$$D(X) = B(X,\varphi X), \qquad X \in \Gamma(\mathfrak{D}).$$

The following Lemma, of which we omit the long proof, gives the useful expression of $B(X,Y)$, for any $X,Y \in \Gamma(\mathfrak{D})$.

Lemma 5.5. *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ be an indefinite $\mathcal{S}$-manifold. Then, for any $X,Y \in \Gamma(\mathfrak{D})$,*

$$B(X,Y) = \frac{1}{32}\{3D(X+\varphi Y) + 3D(X-\varphi Y) - D(X+Y) \\ - D(X-Y) - 4D(X) - 4D(Y) + 24\varepsilon P(X,Y;X,\varphi Y)\}. \tag{8}$$

Using the previous Lemmas it is possible to compute the sectional curvature of a non degenerate 2-plane $\pi = span\{X,Y\}$ of $\mathfrak{D}_p$, as follows.

Proposition 5.6. *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ be an indefinite $\mathcal{S}$-manifold and p in M. We consider a non degenerate 2-plane $\pi = span\{X,Y\}$ of $\mathfrak{D}_p$, where X and Y are unit vectors of $\mathfrak{D}_p$. Then the sectional curvature $K_p(X,Y)$ is given by*

$$K_p(X,Y) = \frac{1}{32(\varepsilon_X\varepsilon_Y - g(X,Y)^2)}\{3(\varepsilon_X+\varepsilon_Y+2g(X,\varphi Y))^2H_p(X+\varphi Y) \\ + 3(\varepsilon_X+\varepsilon_Y-2g(X,\varphi Y))^2H_p(X-\varphi Y) \\ - (\varepsilon_X+\varepsilon_Y+2g(X,Y))^2H_p(X+Y) \\ - (\varepsilon_X+\varepsilon_Y-2g(X,Y))^2H_p(X-Y) - 4H_p(X) - 4H_p(Y) \\ + 24\varepsilon(g(X,\varphi Y)^2 + g(X,Y)^2 - \varepsilon_X\varepsilon_Y)\}.$$

Proof: We note that if $X \in \mathfrak{D}_p$ we have

$$D_p(X) = B_p(X,\varphi X) = g_p(R_p(X,\varphi X,X),\varphi X) = -g_p(X,X)^2H_p(X)$$

and if X and Y are unit vectors of $\mathfrak{D}_p$, we find

$$g(X+\varphi Y,X+\varphi Y) = \varepsilon_X+\varepsilon_Y+2g(X,\varphi Y), \quad g(X+Y,X+Y) = \varepsilon_X+\varepsilon_Y+2g(X,Y).$$

Being $\Delta(\pi) = \varepsilon_X\varepsilon_Y - g_p(X,Y)^2$, we get $K_p(\pi) = -g_p(R_p(X,Y,X),Y)/\Delta(\pi) = -B_p(X,Y)/\Delta(\pi)$. Then, using (8) and Lemma 5.2, we get the required formula. □

Remark 5.7. We note that if $X \in \Gamma(\mathfrak{D})$ is a unit vector field we have

$$R(\xi_\alpha,X,\xi_\beta) = -\varepsilon_\beta\varepsilon_\alpha X, \qquad R(X,\xi_\alpha,X) = -\varepsilon_X\varepsilon_\alpha\bar{\xi}.$$

In fact, if $Y \in \Gamma(TM)$, for any $\alpha \in \{1, \ldots, r\}$, we have

$$\begin{aligned} g(R(X,\xi_\alpha,X),Y) &= -g(R(X,Y,\xi_\alpha),X) = \varepsilon_\alpha g(\nabla_X(\varphi Y) - \nabla_Y(\varphi X) - \varphi[X,Y],X) \\ &= \varepsilon_\alpha g((\nabla_X\varphi)Y - (\nabla_Y\varphi)X,X) = \varepsilon_\alpha g(-\bar\eta(Y)X - \bar\eta(X)\varphi^2 Y,X) \\ &= -\varepsilon_X\varepsilon_\alpha\bar\eta(Y) = -\varepsilon_X\varepsilon_\alpha g(\bar\xi,Y). \end{aligned}$$

Finally, if $X, Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(TM)$ then we get

$$g(R(X,\xi_\alpha,Y),Z) = -\varepsilon_\alpha g(Y,X)\bar\eta(Z) = -\varepsilon_\alpha g(Y,X)g(\bar\xi,Z).$$

Theorem 5.8. *The φ-sectional curvatures completely determine the sectional curvatures of an indefinite $\mathcal{S}$-manifold.*

Proof: We show that for any $p \in M$ and for any non degenerate 2-plane $\pi = span\{X,Y\}$ in $T_p(M)$ the sectional curvature $K_p(X,Y)$ is uniquely determined by the φ-sectional curvature. In the sequel of the proof we suppose that $p \in M$ is fixed. If $X, Y \in \mathfrak{D}_p$, then we apply the previous Proposition and if X or Y is ξ_α, for any $\alpha \in \{1, \ldots, r\}$, we have already seen that $K_p(X,Y) = \varepsilon_\alpha$. If $X, Y \in T_pM$, they can be written in the following way:

$$X = aZ + \eta^\alpha(X)\xi_\alpha, \quad Y = bW + \eta^\alpha(Y)\xi_\alpha,$$

where $Z, W \in \mathfrak{D}$, $g_p(Z,Z) = \varepsilon_Z$, $g_p(W,W) = \varepsilon_W$, and a and b must satisfy:

$$a^2\varepsilon_Z = \varepsilon_X - \varepsilon_\alpha(\eta^\alpha(X))^2, \quad b^2\varepsilon_W = \varepsilon_Y - \varepsilon_\alpha(\eta^\alpha(Y))^2.$$

Therefore, we compute

$$\begin{aligned} g_p(R_p(X,Y,X),Y) &= a^2b^2 g_p(R_p(Z,W,Z),W) + 2a^2b\,\eta^\beta(Y)g_p(R_p(Z,W,Z),\xi_\beta) \\ &+ 2ab^2\eta^\alpha(X)g_p(R_p(Z,W,\xi_\alpha),W) + 2ab\eta^\alpha(X)\eta^\beta(Y)g_p(R_p(Z,W,\xi_\alpha),\xi_\beta) \\ &+ a^2\eta^\beta(Y)\eta^\delta(Y)g_p(R_p(Z,\xi_\beta,Z),\xi_\delta) + 2ab\eta^\beta(Y)\eta^\alpha(X)g_p(R_p(Z,\xi_\beta,\xi_\alpha),W) \\ &+ 2a\eta^\beta(Y)\eta^\alpha(X)\eta^\delta(Y)g_p(R_p(Z,\xi_\beta,\xi_\alpha),\xi_\delta) \qquad (9) \\ &+ b^2\eta^\alpha(X)\eta^\gamma(X)g_p(R_p(\xi_\alpha,W,\xi_\gamma),W) \\ &+ 2b\eta^\alpha(X)\eta^\beta(Y)\eta^\gamma(X)g_p(R_p(\xi_\alpha,Z,\xi_\gamma),\xi_\beta) \\ &+ \eta^\alpha(X)\eta^\beta(Y)\eta^\gamma(X)\eta^\delta(Y)g_p(R_p(\xi_\alpha,\xi_\beta,\xi_\gamma),\xi_\delta). \end{aligned}$$

Now, separately we take the terms of previous expression into account, using

Remark 5.7 and the Bianchi identity, as follows:

$$\begin{aligned} g_p(R_p(Z,W,Z),\xi_\beta) &= g_p(R_p(Z,\xi_\beta,Z),W) = -\varepsilon_Z\varepsilon_\beta g_p(\bar{\xi},W) = 0,\\ g_p(R_p(Z,W,\xi_\alpha),W) &= g_p(R_p(\xi_\alpha,W,Z),W) = g_p(R_p(W,\xi_\alpha,W),Z)\\ &= -\varepsilon_W\varepsilon_\alpha g_p(\bar{\xi},Z) = 0,\\ g_p(R_p(Z,W,\xi_\alpha),\xi_\beta) &= -g_p(R_p(Z,\xi_\alpha,\xi_\beta),W) - g_p(R_p(Z,\xi_\beta),\xi_\alpha),W)\\ &= g_p(R_p(\xi_\alpha,Z,\xi_\beta),W) + \varepsilon_\beta g_p(Z,W)g_p(\bar{\xi}),\xi_\alpha)\\ &= -\varepsilon_\beta\varepsilon_\alpha g_p(Z,W) + \varepsilon_\beta\varepsilon_\alpha g_p(Z,W) = 0,\\ g_p(R_p(Z,\xi_\beta,\xi_\alpha),W) &= -g_p(R_p(Z,\xi_\beta,W)\xi_\alpha) = \varepsilon_\beta g_p(Z,W)g_p(\bar{\xi}),\xi_\alpha)\\ &= \varepsilon_\beta\varepsilon_\alpha g_p(Z,W),\\ g_p(R_p(Z,\xi_\beta,\xi_\alpha),\xi_\delta) &= -g_p(R_p(\xi_\beta,Z,\xi_\alpha),\xi_\delta) = \varepsilon_\beta\varepsilon_\alpha g_p(Z,\xi_\delta) = 0,\\ g_p(R_p(\xi_\alpha,W,\xi_\gamma),\xi_\beta) &= \varepsilon_\gamma\varepsilon_\alpha g_p(Z,\xi_\beta) = 0. \end{aligned}$$

Therefore, replacing the previous expressions in (9), we have:

$$\begin{aligned} g_p(R_p(X,Y,X),Y) = a^2b^2 g_p(R_p(Z,W,Z),W) - a^2\varepsilon_Z\bar{\eta}(Y)\bar{\eta}(Y)\\ + 2ab\bar{\eta}(Y)\bar{\eta}(X)g_p(Z,W) - b^2\varepsilon_W\bar{\eta}(X)\bar{\eta}(X). \end{aligned}$$

Hence, being $K_p(X,Y) = -\varepsilon_X\varepsilon_Y g_p(R_p(X,Y,X),Y)$, we deduce

$$\begin{aligned} K_p(X,Y) = \varepsilon_X\varepsilon_Y\{a^2b^2 g_p(R_p(Z,W,W),Z) - 2ab\bar{\eta}(Y)\bar{\eta}(X)g_p(Z,W)\\ + b^2\varepsilon_W\bar{\eta}(X)^2 + a^2\varepsilon_Z\bar{\eta}(Y)^2\}. \end{aligned} \tag{10}$$

Now, we note that

$$\begin{aligned} g_p(Z,W) &= \frac{1}{ab}g_p(X-\eta^\alpha(X)\xi_\alpha, Y-\eta^\beta(Y)\xi_\beta) + \eta^\alpha(X)\eta^\beta(Y)g_p(\xi_\alpha,\xi_\beta)\}\\ &= -\frac{1}{ab}\varepsilon_\alpha\eta^\alpha(X)\eta^\alpha(Y), \end{aligned}$$

$$\begin{aligned} g_p(R_p(Z,W,W),Z) &= [\varepsilon_Z\varepsilon_W - g_p(Z,W)^2]K_p(Z,W)\\ &= \frac{1}{a^2b^2}[a^2\varepsilon_Z b^2\varepsilon_W - (\varepsilon_\alpha\eta^\alpha(X)\eta^\alpha(Y))^2]K_p(Z,W)\\ &= \frac{1}{a^2b^2}[\left(\varepsilon_X - \varepsilon_\alpha\eta^\alpha(X)^2\right)\left(\varepsilon_Y - \varepsilon_\alpha\eta^\alpha(Y)^2\right)\\ &\quad - (\varepsilon_\alpha\eta^\alpha(X)\eta^\alpha(Y))^2]K_p(Z,W). \end{aligned}$$

Thus, (10) becomes

$$\begin{aligned} K_p(X,Y) = \varepsilon_X\varepsilon_Y\{&[(\varepsilon_X - \varepsilon_\alpha(\eta^\alpha(X))^2)(\varepsilon_Y - \varepsilon_\beta(\eta^\beta(Y))^2)\\ &- (\varepsilon_\alpha\eta^\alpha(X)\eta^\alpha(Y))^2]K_p(Z,W) + 2\bar{\eta}(Y)\bar{\eta}(X)\varepsilon_\alpha\eta^\alpha(X)\eta^\alpha(Y)\\ &+ (\varepsilon_Y - \varepsilon_\beta(\eta^\beta(Y))^2)\bar{\eta}(X)^2 + (\varepsilon_X - \varepsilon_\alpha(\eta^\alpha(X))^2)\bar{\eta}(Y)^2\}, \end{aligned}$$

and this completes the proof, since $K_p(Z,W)$ is given as in Proposition 5.6. ∎

We recall the following result.

Lemma 5.9 ([16]). *Let (V,g) be a semi-Euclidean vector space and R a $(0,4)$-type tensor on V such that for any $X,Y,Z,W \in V$ the following conditions hold:*

a) $R(X,Y,Z,W) = -R(Y,X,Z,W)$,

b) $R(X,Y,Z,W) = -R(X,Y,W,Z)$,

c) $R(X,Y,Z,W) = R(Z,W,X,Y)$,

d) $\mathfrak{S}_{Y,Z,W} R(X,Y,Z,W) = 0$.

If $R(X,Y,X,Y) = 0$ for any linearly independent and non lightlike vectors $X,Y \in V$, then $R = 0$. Moreover, if R and S are $(0,4)$-type tensors on V such that the conditions (a-d) are satisfied and $R(X,Y,X,Y) = S(X,Y,X,Y)$ for any $X,Y \in V$ linearly independent non lightlike vectors, then $R = S$.

Proposition 5.10. *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ be an indefinite $\mathcal{S}$-manifold, T and S be $(0,4)$-type tensor fields on M such that the following conditions hold:*

i) $T(X,Y,Z,W) = -T(Y,X,Z,W)$, $S(X,Y,Z,W) = -S(Y,X,Z,W)$, $X,Y,Z,W \in \Gamma(TM)$

ii) $T(X,Y,Z,W) = -T(X,Y,W,Z)$, $S(X,Y,Z,W) = -S(X,Y,W,Z)$, $X,Y,Z,W \in \Gamma(TM)$

iii) $T(X,Y,Z,W) = T(Z,W,X,Y)$, $S(X,Y,Z,W) = S(Z,W,X,Y)$, $X,Y,Z,W \in \Gamma(TM)$

iv) $\mathfrak{S}_{Y,Z,W} T(X,Y,Z,W) = 0$, $\mathfrak{S}_{Y,Z,W} S(X,Y,Z,W) = 0$, $X,Y,Z,W \in \Gamma(TM)$

v) for any $X,Y,Z,W \in \Gamma(\mathfrak{D})$

$$T(X,Y,\varphi Z,W) + T(X,Y,Z,\varphi W) = \varepsilon P(X,Y;Z,W)$$
$$S(X,Y,\varphi Z,W) + S(X,Y,Z,\varphi W) = \varepsilon P(X,Y;Z,W)$$

vi) for any $X,Y \in \Gamma(\mathfrak{D})$ *and for any* $\alpha,\beta,\gamma,\delta \in \{1,\ldots,r\}$

(a) $T(X,\xi_\alpha,X,Y) = S(X,\xi_\alpha,X,Y)$,

(b) $T(\xi_\alpha,X,\xi_\beta,Y) = S(\xi_\alpha,X,\xi_\beta,Y)$,

(c) $T(\xi_\alpha,X,\xi_\beta,\xi_\gamma) = S(\xi_\alpha,X,\xi_\beta,\xi_\gamma)$,

(d) $T(\xi_\alpha,\xi_\beta,\xi_\gamma,\xi_\delta) = S(\xi_\alpha,\xi_\beta,\xi_\gamma,\xi_\delta)$.

Then, if $T(X,\varphi X,X,\varphi X) = S(X,\varphi X,X,\varphi X)$ for any $X \in \Gamma(\mathfrak{D})$ non lightlike vector field, one has $T = S$.

Proof: It is easy to verify that *v)* implies that for any X', Y', Z', W' in $\Gamma(\mathfrak{D})$

$$T(\varphi X', \varphi Y', \varphi Z', \varphi W') = T(X', Y', Z', W'),$$

and, using the above formula, we obtain

$$T(\varphi X', \varphi Y', Z', W') = T(X', Y', \varphi Z', \varphi W').$$

Analogously, for the tensor field S we have

$$S(\varphi X', \varphi Y', Z', W') = S(X', Y', \varphi Z', \varphi W').$$

Now, being φ_p an almost complex structure on $\mathfrak{D}_p$ for any $p \in M$, from a well-known result analogous to Lemma 5.9 ([1]), in the case of a real vector space endowed with an almost complex structure, we deduce $T(X', Y', Z', W') = S(X', Y', Z', W')$. Then, in particular, we have

$$T(X', Y', X', Y') = S(X', Y', X', Y').$$

Now, if $X, Y \in \Gamma(TM)$ are linearly independent and non lightlike, we compute $T(X,Y,X,Y)$ and $S(X,Y,X,Y)$, writing $X = X' + \eta^\alpha(X)\xi_\alpha$ and $Y = Y' + \eta^\alpha(Y)\xi_\alpha$, and likewise to (9), by the $\mathfrak{F}(M)$-linearity of T and S, using *vi)*, we get $T(X,Y,X,Y) = S(X,Y,X,Y)$. □

Remark 5.11. Using Remark 5.7 and Proposition 5.1, the Riemannian (0,4)-type curvature tensor field R satisfies the properties listed in Proposition 5.10. Thus, it is uniquely determined by the φ-sectional curvature.

Theorem 5.12. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite $\mathcal{S}$-manifold. Then the φ-sectional curvature c is pointwise constant, $c \in \mathfrak{F}(M)$, if and only if the Riemannian $(0,4)$-type curvature tensor field R is given by*

$$\begin{aligned} R(X,Y,Z,W) = &-\frac{c+3\varepsilon}{4}\{g(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\} \quad (11)\\ &-\frac{c-\varepsilon}{4}\{\Phi(W,X)\Phi(Z,Y)\\ &-\Phi(Z,X)\Phi(W,Y) + 2\Phi(X,Y)\Phi(W,Z)\}\\ &-\{\bar\eta(W)\bar\eta(X)g(\varphi Z,\varphi Y) - \bar\eta(W)\bar\eta(Y)g(\varphi Z,\varphi X)\\ &+\bar\eta(Y)\bar\eta(Z)g(\varphi W,\varphi X) - \bar\eta(Z)\bar\eta(X)g(\varphi W,\varphi Y)\}. \end{aligned}$$

Proof: We suppose that the φ-sectional curvature c is pointwise constant and in order to prove (11), denote by $S(X,Y,Z,W)$ the right-hand side of (11). Obviously S is a tensor field of type (0,4) on M, and we shall prove that S coincides with R. To this end it is easy to check that for any $X,Y,Z,W \in \Gamma(TM)$ we have the properties of skew-symmetry $-S(X,Y,W,Z) = S(X,Y,Z,W) = -S(Y,X,Z,W)$ and the Bianchi identity $\mathfrak{S}_{Y,Z,W} S(X,Y,Z,W) = 0$, while the property *iii)* of Proposition 5.10, $S(X,Y,Z,W) = S(Z,W,X,Y)$, follows by the Bianchi identity and the skew-symmetries.

Now, for $X,Y,Z,W \in \Gamma(\mathfrak{D})$, computing $S(X,Y,Z,\varphi W) + S(X,Y,\varphi Z,W)$ we get

$$\begin{aligned}
&S(X,Y,Z,\varphi W) + S(X,Y,\varphi Z,W) = -\frac{c}{4}\{g(Y,Z)\Phi(X,W) - g(X,Z)\Phi(Y,W)\\
&\quad + \Phi(Y,Z)g(X,W) - \Phi(X,Z)g(Y,W) + g(W,X)\Phi(Z,Y)\\
&\quad - \Phi(Z,X)g(W,Y) + \Phi(W,X)g(Z,Y) - g(Z,X)\Phi(W,Y)\}\\
&\quad - \frac{\varepsilon}{4}\{3\Phi(X,W)g(Z,Y) - 3\Phi(Y,W)g(X,Z) + 3g(X,W)\Phi(Y,Z)\\
&\quad - 3g(Y,W)\Phi(X,Z) + \Phi(Y,Z)g(W,X) - \Phi(X,Z)g(W,Y)\\
&\quad + \Phi(X,W)g(Z,Y) - \Phi(Y,W)g(Z,X)\}\\
&= -\varepsilon\{\Phi(X,W)g(Z,Y) - \Phi(X,Z)g(Y,W) - \Phi(Y,W)g(X,Z) + g(X,W)\Phi(Y,Z)\}\\
&= \varepsilon P(X,Y;Z,W).
\end{aligned}$$

We continue verifying *vi)* of Proposition 5.10, and obtaining $S(X,\xi_\alpha,X,Y) = 0 = R(X,\xi_\alpha,X,Y)$, $S(\xi_\alpha,X,\xi_\beta,\xi_\gamma) = 0 = R(\xi_\delta,X,\xi_\beta,\xi_\gamma)$, $S(\xi_\alpha,\xi_\delta,\xi_\beta,\xi_\gamma) = 0 = R(\xi_\delta,\xi_\delta,\xi_\beta,\xi_\gamma)$ and

$$\begin{aligned}
S(\xi_\alpha,X,\xi_\beta,Y) = &-\frac{c+3\varepsilon}{4}\{g(\varphi X,\varphi\xi_\beta)g(\varphi\xi_\alpha,\varphi Y) - g(\varphi\xi_\alpha,\varphi\xi_\beta)g(\varphi X,\varphi Y)\}\\
&- \frac{c-\varepsilon}{4}\{\Phi(Y,\xi_\alpha)\Phi(\xi_\beta,X) - \Phi(\xi_\beta,\xi_\alpha)\Phi(Y,X)\\
&+ 2\Phi(\xi_\alpha,X)\Phi(Y,\xi_\beta)\} - \{\bar\eta(Y)\bar\eta(\xi_\alpha)g(\varphi\xi_\beta,\varphi X)\\
&- \bar\eta(Y)\bar\eta(X)g(\varphi\xi_\beta,\varphi\xi_\alpha) + \bar\eta(X)\bar\eta(\xi_\beta)g(\varphi Y,\varphi\xi_\alpha)\\
&- \bar\eta(\xi_\beta)\bar\eta(\xi_\alpha)g(\varphi Y,\varphi X)\} = \varepsilon_\alpha\varepsilon_\beta g(X,Y) = R(\xi_\alpha,X,\xi_\beta,Y).
\end{aligned}$$

For any $X \in \Gamma(\mathfrak{D})$ non lightlike vector field, we compute $S(X,\varphi X,X,\varphi X)$, obtaining:

$$\begin{aligned}
S(X,\varphi X,X,\varphi X) = &-\frac{c+3\varepsilon}{4}\{g(\varphi^2X,\varphi X)g(\varphi X,\varphi^2X) - g(\varphi X,\varphi X)g(\varphi^2X,\varphi^2X)\}\\
&- \frac{c-\varepsilon}{4}\{\Phi(\varphi X,X)\Phi(X,\varphi X) - \Phi(X,X)\Phi(\varphi X,\varphi X)\\
&+ 2\Phi(X,\varphi X)\Phi(\varphi X,X)\} \qquad (12)\\
&- \{\bar\eta(\varphi X)\bar\eta(X)g(\varphi X,\varphi^2X) - \bar\eta(\varphi X)\bar\eta(\varphi X)g(\varphi X,\varphi X)\\
&+ \bar\eta(\varphi X)\bar\eta(X)g(\varphi^2X,\varphi X) - \bar\eta(X)\bar\eta(X)g(\varphi^2X,\varphi^2X)\}\\
= &\frac{c+3\varepsilon}{4}g(X,X)^2 - \frac{c-\varepsilon}{4}\{-g(X,X)^2 - 2g(X,X)^2\}\\
= &\frac{c+3\varepsilon}{4}g(X,X)^2 + 3\frac{c-\varepsilon}{4}g(X,X)^2 = cg(X,X)^2.
\end{aligned}$$

Moreover, since by definition of φ-sectional curvature we have

$$R(X,\varphi X,X,\varphi X) = cg(X,X)^2. \qquad (13)$$

from (12) and (13) we get $R(X,\varphi X,X,\varphi X) = S(X,\varphi X,X,\varphi X)$, and, using Proposition 5.10, the previous Remark and the properties of the tensor field S, we obtain $R(X,Y,Z,W) = S(X,Y,Z,W)$, for any $X,Y,Z,W \in \Gamma(TM)$, that is the formula (11).

Conversely, if we assume (11), choosing a point $p \in M$ and a φ-plane $\pi = span\{X,\varphi X\}$, with $X \in \mathfrak{D}_p$ non lightlike vector, by direct computation, omitting the point p, we have

$$H(X) = \frac{c+3\varepsilon}{4g(X,X)^2}g(X,X)^2 + 3\frac{c-\varepsilon}{4g(X,X)^2}g(X,X)^2 = c.$$

□

6 Sectional Curvature in the case $\varepsilon = 0$, an example

In this section we consider the case $\varepsilon = 0$, as already pointed out, $r = 2p$ and $\xi_1,\dots,\xi_p$ are timelike vector field, $\xi_{p+1},\dots,\xi_{2p}$ are spacelike vector field. We call such a manifold a *special indefinite $\mathcal{S}$-manifold*. Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ be a special indefinite $\mathcal{S}$-manifold. The tensor Q is given by

$$\begin{aligned} Q(X,Y;Z,W) = &-g(W,\varphi Y)\bar\eta(Z)\bar\eta(X) + g(W,\varphi X)\bar\eta(Z)\bar\eta(Y) \\ &+ g(Z,\varphi Y)\bar\eta(X)\bar\eta(W) - g(Z,\varphi X)\bar\eta(Y)\bar\eta(W), \end{aligned}$$

and

$$g(R(X,Y,\varphi Z),W) + g(R(X,Y,Z),\varphi W) = -Q(X,Y;Z,W)$$

Moreover, being $Q(X,Y;Z,W) = 0$ for any $X,Y,Z,W \in \mathfrak{D}$, we have

a) $g(R(\varphi X,\varphi Y,\varphi Z),\varphi W) = g(R(X,Y,Z),W)$;

b) $g(R(X,\varphi X,Y),\varphi Y) = g(R(X,Y,X),Y) + g(R(X,\varphi Y,X),\varphi Y)$;

c) $g(R(\varphi X,Y,\varphi X),Y) = g(R(X,\varphi Y,X),\varphi Y)$.

Furthermore, for $X,Y \in \Gamma(\mathfrak{D})$

$$\begin{aligned} B(X,Y) = \frac{1}{32}\{&3D(X+\varphi Y) + 3D(X-\varphi Y) \\ &- D(X+Y) - D(X-Y) - 4D(X) - 4D(Y)\}, \end{aligned}$$

and for a non degenerate 2-plane $\pi = span\{X,Y\}$ of $\mathfrak{D}_p$, where X and Y are unit vectors of $\mathfrak{D}_p$,

$$\begin{aligned} K_p(X,Y) = \frac{1}{32(\varepsilon_X\varepsilon_Y - g(X,Y)^2)}\{&3(\varepsilon_X+\varepsilon_Y+2g(X,\varphi Y))^2H_p(X+\varphi Y) \\ &+ 3(\varepsilon_X+\varepsilon_Y-2g(X,\varphi Y))^2H_p(X-\varphi Y) \\ &- (\varepsilon_X+\varepsilon_Y+2g(X,Y))^2H_p(X+Y) \\ &- (\varepsilon_X+\varepsilon_Y-2g(X,Y))^2H_p(X-Y) - 4H_p(X) - 4H_p(Y)\}. \end{aligned}$$

Finally we have that the φ-sectional curvature c is pointwise constant, $c \in \mathfrak{F}(M)$, if and only if the Riemannian (0,4)-type curvature tensor field R is given by

$$\begin{aligned} R(X,Y,Z,W) = -\frac{c}{4}\{&g(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - g(\varphi X,\varphi Z)g(\varphi Y,\varphi W) \\ &+ \Phi(W,X)\Phi(Z,Y) - \Phi(Z,X)\Phi(W,Y) + 2\Phi(X,Y)\Phi(W,Z)\} \\ &- \{\bar{\eta}(W)\bar{\eta}(X)g(\varphi Z,\varphi Y) - \bar{\eta}(W)\bar{\eta}(Y)g(\varphi Z,\varphi X) \\ &+ \bar{\eta}(Y)\bar{\eta}(Z)g(\varphi W,\varphi X) - \bar{\eta}(Z)\bar{\eta}(X)g(\varphi W,\varphi Y)\}. \end{aligned} \tag{14}$$

An example of a special indefinite $\mathcal{S}$-manifold is $M = (\mathbb{R}^4_1, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$, which is described in Example 4.3. We observe that the metric is Lorentzian, ξ_1 is a spacelike vector field while ξ_2 is a timelike vector field, then, since $\varepsilon = 0$, the structure is a special indefinite $\mathcal{S}$-structure. Now, we compute the tensor field Q on some relevant set of vector fields, the sectional curvature and φ-sectional curvature. We know that $Q = 0$ on $\mathfrak{D}$, moreover we have

$$\begin{aligned} Q(\xi_1,Y;Z,W) &= -Q(\xi_2,Y;Z,W) = -g(W,\varphi Y)\bar{\eta}(Z) + g(Z,\varphi Y)\bar{\eta}(W) = 0, \\ Q(\xi_\alpha,Y;\xi_\beta,W) &= Q(Y,\xi_\alpha;W,\xi_\beta) = -\varepsilon_\alpha\varepsilon_\beta g(W,\varphi Y), \end{aligned} \tag{15}$$

for any $Y, Z, W \in \Gamma(\mathfrak{D})$ and for any $\alpha, \beta \in \{1,2\}$. Equation (15) shows that Q never vanishes. Now, computing the Christoffel's symbols we obtain:

$$\Gamma^3_{12} = \Gamma^4_{12} = \frac{1}{2}, \quad \Gamma^2_{13} = -\Gamma^2_{14} = -\Gamma^1_{23} = \Gamma^1_{24} = -1,$$

$$\Gamma^3_{23} = \Gamma^4_{23} = -\Gamma^3_{24} = -\Gamma^4_{24} = -y,$$

whereas the other Γ^k_{ij} vanish. To compute the φ-sectional curvature, being $\mathfrak{D}$ globally spanned by $X = \frac{\partial}{\partial x} - y\xi_1 - y\xi_2$ and $Y = \varphi X = \frac{\partial}{\partial y}$, we value $H(X)$. So, we have

$$\begin{aligned} R(X,\varphi X,X) &= \nabla_X\left(\Gamma^h_{21} - y(\Gamma^h_{23} + \Gamma^h_{24})\frac{\partial}{\partial x^h} - \xi_1 - \xi_2\right) - \nabla_{\xi_1}X - \nabla_{\xi_2}X \\ &= -\frac{1}{2}\nabla_X(\xi_1+\xi_2) - (\Gamma^h_{31} - y(\Gamma^h_{33} + \Gamma^h_{34}) + \Gamma^h_{41} - y(\Gamma^h_{43} + \Gamma^h_{44}))\frac{\partial}{\partial x^h} \\ &= [\Gamma^h_{11} - y(\Gamma^h_{31} + \Gamma^h_{41}) - y(\Gamma^h_{13} - y(\Gamma^h_{33} + \Gamma^h_{43}) + \\ &\quad + \Gamma^h_{14} - y(\Gamma^h_{34} + \Gamma^h_{44}))]\frac{\partial}{\partial x^h} = 0, \end{aligned}$$

$$\begin{aligned} g(X,X) = g(\frac{\partial}{\partial x},\frac{\partial}{\partial x}) &- 2y(g(\frac{\partial}{\partial x},\xi_1) + \\ &+ g(\frac{\partial}{\partial x},\xi_2)) + y^2(g(\xi_1,\xi_1) + g(\xi_1,\xi_2) + g(\xi_2,\xi_2)) = \frac{1}{2}. \end{aligned}$$

It follows that

$$H(X) = -\frac{1}{g(X,X)^2}g(R(X,\varphi X,X),\varphi X) = 0.$$

Then, M is an indefinite $\mathcal{S}$-space form with $c = 0 = \varepsilon$ and, from (14) for any $Y, Z, W \in \Gamma(TM)$, the Riemannian curvature tensor field R is given by:

$$R(\xi_\alpha, Y, Z, W) = -\varepsilon_\alpha\{\bar{\eta}(W)g(\varphi Z, \varphi Y) - \bar{\eta}(Z)g(\varphi W, \varphi Y)\},$$
$$R(\xi_\alpha, \xi_\beta, Z, W) = 0,$$
$$R(\xi_\alpha, Y, \xi_\beta, W) = \varepsilon_\alpha\varepsilon_\beta g(\varphi W, \varphi Y),$$

and R vanishes on $\mathfrak{D}$.

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