Curvature of a class of indefinite globally framed $f$-manifolds

by

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Abstract

We present a compared analysis of some properties of indefinite almost $S$-manifolds and indefinite $S$-manifolds. We give some characterizations in terms of the Levi-Civita connection and of the characteristic vector fields. We study the sectional and $\varphi$-sectional curvature of indefinite almost $S$-manifolds and state an expression of the curvature tensor field for the indefinite $S$-space forms. We analyse the sectional curvature of indefinite $S$-manifold in which the number of the spacelike characteristic vector fields is equal to that of the timelike characteristic vector fields. Some examples are also described.

Key Words: Semi-Riemannian manifolds, indefinite metrics, $f$-structures, sectional curvature, $\varphi$-sectional curvature.


1 Introduction

In the framework of Riemannian geometry, almost $S$-manifolds and $S$-manifolds represent a natural generalization of contact and Sasaki manifolds, respectively. Such manifolds have been extensively studied by several authors and from different points of view ([2, 3, 4, 7, 8, 12]). On the other hand, also Sasakian manifolds with semi-Riemannian metric have been considered ([10, 6, 17]), and in recent works many authors, (for example, in [13], K.L. Duggal and B. Sahin) study lightlike submanifolds of indefinite Sasakian manifolds. Indefinite $S$-manifolds are natural generalizations of indefinite Sasaki manifolds. Moreover many space-time manifolds can be endowed with $f$-structures ([9]).

After a first section on $f$-structures and indefinite metric $g.f.f$-structures, in section 3, we carry out an in-depth study of the indefinite (almost) $S$-manifolds. In section 4 we describe two examples of 6-dimensional indefinite $S$-manifolds.
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having two characteristic vector fields which are both spacelike or both timelike.

A third example is a Lorentzian indefinite $S$-manifold of dimension 4 with two characteristic vector fields of different causal type. In section 5, after some Lemmas, we prove that the \( \varphi \)-sectional curvatures completely determine the sectional curvatures. Then, we find an expression of the curvature tensor field \( R \) which characterizes the indefinite $S$-space forms, that is indefinite $S$-manifolds with constant \( \varphi \)-sectional curvature. Then, in section 6, we consider the curvature of special indefinite $S$-manifold in which the number of the characteristic vector fields is even with an equal number of spacelike and timelike characteristic vector fields; we prove that the special indefinite $S$-manifold described in the third example in section 4 turns out to be an indefinite $S$-space form whose \( \varphi \)-sectional curvature vanishes.

All manifolds and tensor fields are assumed to be smooth.

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2 Indefinite metric \( f \)-structure

We recall that an \( f \)-structure on a manifold \( M \) is a non null \((1,1)\)-tensor field \( \varphi \) on \( M \) of constant rank such that \( \varphi^3 + \varphi = 0 \). A manifold \( M \), provided with an \( f \)-structure, is said to be an \( f \)-manifold, and it is known that \( TM \) splits into two complementary subbundles \( \text{Im} \varphi \) and \( \ker \varphi \) and that the restriction of \( \varphi \) to \( \text{Im} \varphi \) determines a complex structure on it and the rank of \( \varphi \) is even. An interesting case of \( f \)-structure occurs when \( \ker \varphi \) is parallelizable for which there exist global vector fields \( \xi_\alpha \), \( \alpha \in \{1, \ldots, r\} \), with their dual 1-forms \( \eta_\alpha \), satisfying:

\[
\varphi^2 = -I + \sum_{\alpha=1}^{r} \eta_\alpha \otimes \xi_\alpha, \quad \text{and} \quad \eta_\alpha(\xi_\beta) = \delta_\alpha^\beta. 
\]

Such an \( f \)-structure is called an \( f \)-structure with parallelizable kernel or globally framed \( f \)-structure, briefly denoted \( g.f.f \)-structure ([14]). Moreover, a manifold \( M \) endowed with a \( g.f.f \)-structure is called a \( g.f.f \)-manifold, and it is denoted with \((M, \varphi, \xi_\alpha, \eta_\alpha)\); the vector fields \( \xi_\alpha \), \( \alpha = 1, \ldots, r \), are called characteristic vector fields.

It is also known that an \( f \)-structure, on a manifold \( M \), is called normal if the tensor field \( N = N_\varphi + 2 \sum_{\alpha=1}^{r} d\eta_\alpha \otimes \xi_\alpha \) vanishes, where \( N_\varphi \) is the Nijenhuis torsion of \( \varphi \).

Definition 2.1. Let \((M, \varphi)\) be a \((2n+r)\)-dimensional \( f \)-manifold and \( g \) a semi-Riemannian metric on \( M \) with index \( \nu \), \( 0 < \nu < 2n + r \). Then, the pair \((\varphi, g)\) is said to be an indefinite metric \( f \)-structure, and the triple \((M, \varphi, g)\) is called an indefinite metric \( f \)-manifold, if \( \varphi \) is skew-symmetric with respect to \( g \), that is, for any \( X, Y \in \Gamma(TM) \):

\[
g(\varphi X, Y) + g(X, \varphi Y) = 0.
\]

Definition 2.2. Let \((M^{2n+r}, \varphi, \xi_\alpha, \eta_\alpha)\) be a \( g.f.f \)-manifold, and \( g \) a semi-Riemannian metric on \( M \) with index \( \nu \), \( 0 < \nu < 2n + r \). Then, we say that the
two structures are compatible if for any $X, Y \in \Gamma(TM)$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{r} \varepsilon_{\alpha} \eta^{\alpha}(X)\eta^{\alpha}(Y), \quad \varepsilon_{\alpha}g(X, \xi_{\alpha}) = \eta^{\alpha}(X)$$

(1)

for any $\alpha \in \{1, \ldots, r\}$, where $\varepsilon_{\alpha} = \pm 1$ according to whether $\xi_{\alpha}$ is spacelike or timelike. Then $(M^{2n+r}, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$ is called an indefinite metric $g.f.f$. manifold.

We shall use the Einstein convention omitting the sum symbol for repeated indices above and below, writing, e.g., $\varepsilon_{\alpha} \eta^{\alpha}(X)\eta^{\alpha}(Y)$ to mean $\sum_{\alpha=1}^{r} \varepsilon_{\alpha} \eta^{\alpha}(X)\eta^{\alpha}(Y)$.

Observe that if $g$ is a semi-Riemannian metric on a $g.f.f$.-manifold $(M, \varphi, \xi_{\alpha}, \eta^{\alpha})$ compatible with the $f$-structure $\varphi$, then the pair $(\varphi, g)$ is necessarily an indefinite metric $f$-structure. The fundamental 2-form $\Phi$ is defined putting $\Phi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$. Let $(M, \varphi, \xi_{\alpha}, \eta^{\alpha})$, with $\alpha = 1, \ldots, r$, be a $g.f.f$.-manifold, and $g$ a compatible semi-Riemannian metric on $M$. We know that the orthogonal decomposition $TM = \text{Im} \varphi \oplus \ker \varphi$ holds, and that the induced structure $J$ on $\text{Im} \varphi$ is an almost complex structure; then $(\text{Im} \varphi, g = g|_{\text{Im} \varphi}, J)$ is an indefinite Hermitian distribution and the only possible signatures of $g$ are $(2p, 2q)$ with $p + q = n$; therefore $g$ cannot be a Lorentz metric, for $n > 1$. We shall denote $\text{Im} \varphi$ and $\ker \varphi$ with $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively and for a section of $\mathcal{D}$ ($\mathcal{D}^{\perp}$) we will write $X \in \mathcal{D}$ or $X \in \Gamma(\mathcal{D})$ ($X \in \mathcal{D}^{\perp}$ or $X \in \Gamma(\mathcal{D}^{\perp})$).

We recall the following result due to A. Bejancu and K.L. Duggal ([10]).

**Theorem 2.3.** Let $(M, \varphi, \xi_{\alpha}, \eta^{\alpha})$, $\alpha = 1, \ldots, r$, be a $g.f.f$.-manifold and $h_0$ a semi-Riemannian metric on $M$; we suppose that $\{\xi_{\alpha}\}_{1 \leq \alpha \leq r}$ are $h_0$-orthonormal and that $h_0(\xi_{\alpha}, \xi_{\alpha}) = -\varepsilon_{\alpha}$, for any $\alpha \in \{1, \ldots, r\}$. Then there exists a symmetric tensor field $g$ of type $(0, 2)$ on $M$ satisfying (1).

Now, with a standard computation as in the Riemannian setting ([2]), one can prove the following results.

**Proposition 2.4.** Let $(M, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$ be an indefinite metric $g.f.f$.-manifold. Then, the Levi-Civita connection satisfies the following equality, for any $X, Y, Z \in \Gamma(TM)$:

$$2g((\nabla_{X}\varphi)Y, Z) = 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + g(N(Y, Z), \varphi X) + \varepsilon_{\alpha}N^{(2)}_{\alpha}(X, Y)\eta^{\alpha}(X) + 2\varepsilon_{\alpha}d\eta^{\alpha}(\varphi Y, X)\eta^{\alpha}(Z) - 2\varepsilon_{\alpha}d\eta^{\alpha}(\varphi Z, X)\eta^{\alpha}(Y),$$

where $N^{(2)}_{\alpha}(X, Y) = (\mathcal{L}_{\varphi Y}\eta^{\alpha})(Y) - (\mathcal{L}_{\varphi X}\eta^{\alpha})(X) = 2\eta^{\alpha}(\varphi X, Y) - 2d\eta^{\alpha}(\varphi Y, X)$.

**Proposition 2.5.** Let $(M, \varphi, \xi_{\alpha}, \eta^{\alpha}, g)$ be an indefinite metric $g.f.f$.-manifold. Then the following statements hold:

a) $(\mathcal{L}_{\xi_{\alpha}}\Phi)(X, Y) = (\mathcal{L}_{\xi_{\alpha}}g)(X, \varphi Y) + g(X, (\mathcal{L}_{\xi_{\alpha}}\varphi)Y)$, for any $\alpha \in \{1, \ldots, r\}$.

b) $(\nabla_{X}\Phi)(Y, Z) = g(Y, (\nabla_{X}\varphi)Z)$, for any $X, Y, Z \in \Gamma(TM)$. 
c) If $\mathcal{L}_{\xi_\alpha} \varphi = 0$, then $\eta^\beta [\varphi Z, \xi_\alpha] = 0$, for any $\beta \in \{1, \ldots, r\}$.

d) $N = 0 \Rightarrow N^{(2)}_\alpha = 0$, for any $\alpha \in \{1, \ldots, r\}$.

Between the indefinite metric $g.f.f$-manifolds, we can define the following classes.

**Definition 2.6.** Let $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric $g.f.f$-manifold. $M$ is called **indefinite $K$-manifold** if it is normal and $d\Phi = 0$.

In this case $\mathcal{L}_{\xi_\alpha} \varphi = 0$ if and only if the characteristic vector fields $\xi_\alpha$ are Killing. Two subclasses of indefinite $K$-manifolds are those of indefinite $C$-manifolds and indefinite $S$-manifolds, that are defined as follows: an indefinite $K$-manifold is called **indefinite $C$-manifold** if $d\eta^\alpha = 0$ for any $\alpha \in \{1, \ldots, r\}$, while it is called **indefinite $S$-manifold** if $d\eta^\alpha = \Phi$ for any $\alpha \in \{1, \ldots, r\}$.

### 3 Indefinite $S$-manifolds

The properties of (almost) $S$-manifolds (with Riemannian metric) are studied in [12] and in [2]. Now, we discuss indefinite (almost) $S$-manifolds and their properties.

#### 3.1 Indefinite almost $S$-manifolds

**Definition 3.1.** Let $(M^{2n+r}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric $g.f.f$-manifold. $M$ is called **indefinite almost $S$-manifold** if $d\eta^\alpha = \Phi$ for any $\alpha \in \{1, \ldots, r\}$.

**Lemma 3.2.** Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $S$-manifold. Then the tensor fields $N^{(2)}_\alpha$ vanish and for any $X, Y \in \Gamma(D)$ and $\alpha \in \{1, \ldots, r\}$, we have

$$
\eta^\alpha [\varphi X, Y] = \eta^\alpha [\varphi Y, X].
$$

**Proof:** For $\alpha \in \{1, \ldots, r\}$, we have $N^{(2)}_\alpha(X, Y) = 2d\eta^\alpha(\varphi X, Y) - 2d\eta^\alpha(\varphi Y, X) = 2\Phi(\varphi X, Y) - 2\Phi(\varphi Y, X) = 0$. Then, for any $X, Y \in \Gamma(D)$, $2d\eta^\alpha(\varphi X, Y) = -\eta^\alpha([\varphi X, Y])$ implies $\eta^\alpha[\varphi X, Y] = \eta^\alpha[\varphi Y, X]$. \hfill \Box

**Proposition 3.3.** Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $S$-manifold and $\bar{\eta} := \sum_{\alpha=1}^r \varepsilon_\alpha \eta^\alpha$. Then, the following statements hold:

$$
2g((\nabla_X \varphi)Y, Z) = g(N(Y, Z), \varphi X) + 2g(\varphi Y, \varphi X)\bar{\eta}(Z) - 2g(\varphi Z, \varphi X)\bar{\eta}(Y),
$$

$$
\nabla_{\xi_\alpha} \varphi = 0, \quad \nabla_{\xi_\alpha} \xi_\beta = 0
$$

for all $\alpha, \beta \in \{1, \ldots, r\}$.
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**Proof:** Equation (3) follows from (2) using \( d\Phi = 0 \), \( N_\alpha^{(2)} = 0 \) and \( d\eta^\alpha = \Phi \), for \( \alpha \in \{1, \ldots, r\} \). Then, putting \( X = \xi_\alpha \), we obtain \( \nabla_{\xi_\alpha} \varphi = 0 \).

Hence, we have \( 0 = (\nabla_{\xi_\alpha} \varphi)(\xi_\beta) = -\varphi(\nabla_{\xi_\alpha} \xi_\beta) \), therefore \( \nabla_{\xi_\alpha} \xi_\beta \in \mathcal{D}^\perp \), which implies that \([\xi_\alpha, \xi_\beta] \in \mathcal{D}^\perp \). On the other hand, for any \( \gamma \in \{1, \ldots, r\} \)

\[
0 = \Phi(\xi_\alpha, \xi_\beta) = d\eta^\gamma(\xi_\alpha, \xi_\beta) = -\frac{1}{2} \eta^\gamma [\xi_\alpha, \xi_\beta] = -\frac{1}{2} \varepsilon_\gamma g([\xi_\alpha, \xi_\beta], \xi_\gamma).
\]

Therefore \([\xi_\alpha, \xi_\beta] \in \mathcal{D} \cap \mathcal{D}^\perp \) and we obtain \([\xi_\alpha, \xi_\beta] = 0 \) and \( \nabla_{\xi_\alpha} \xi_\beta = \nabla_{\xi_\beta} \xi_\alpha \).

Now we check that \( \nabla_{\xi_\alpha} \xi_\beta \in \mathcal{D} \), that is, for any \( \gamma \in \{1, \ldots, r\} \), \( g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) = 0 \).

Being \( g(\xi_\beta, \xi_\gamma) = \varepsilon_\beta \delta_{\gamma} \) and using the covariant derivative with respect to \( \xi_\alpha \), we find \( g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) + g(\xi_\beta, \nabla_{\xi_\alpha} \xi_\gamma) = 0 \), and, covariantly differentiating \( g(\xi_\alpha, \xi_\gamma) = \varepsilon_\alpha \delta_{\gamma \gamma} \) with respect to \( \xi_\delta \), we obtain \( g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) + g(\xi_\alpha, \nabla_{\xi_\beta} \xi_\gamma) = 0 \). From the last two equations, using \( \nabla_{\xi_\alpha} \xi_\beta = \nabla_{\xi_\beta} \xi_\alpha \), we have \( g(\xi_\beta, \nabla_{\xi_\alpha} \xi_\gamma) = g(\xi_\alpha, \nabla_{\xi_\beta} \xi_\gamma) \).

Therefore,

\[
g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) = g(\xi_\alpha, \nabla_{\xi_\beta} \xi_\gamma) = g(\xi_\alpha, \nabla_{\xi_\beta} \xi_\gamma) = -g(\nabla_{\xi_\beta} \xi_\alpha, \xi_\gamma) = -g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma),
\]

from which \( g(\nabla_{\xi_\alpha} \xi_\beta, \xi_\gamma) = 0 \) follows. This result and \( \nabla_{\xi_\alpha} \xi_\beta \in \mathcal{D} ^ \perp \) imply

\[
\nabla_{\xi_\alpha} \xi_\beta = 0.
\]

\[\square\]

**Proposition 3.4.** Let \((M, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an indefinite almost \( S \)-manifold. Then

a) for any \( \alpha \in \{1, \ldots, r\} \) the operator \( h_\alpha = \frac{1}{2} \mathcal{L}_{\xi_\alpha} \varphi \) is self-adjoint,

b) for any \( \alpha, \beta \in \{1, \ldots, r\} \), \( h_\alpha(\xi_\beta) = 0 \),

c) for any \( \alpha \in \{1, \ldots, r\} \), \( h_\alpha \circ \varphi + \varphi \circ h_\alpha = 0 \).

**Proof:** As first step, using (4), for any \( X, Y \in \Gamma(TM) \) and any \( \alpha \in \{1, \ldots, r\} \), we easily obtain,

\[
g(\mathcal{L}_{\xi_\alpha} \varphi X, Y) = \varepsilon_\alpha (-(\varphi X)(\eta^\alpha(Y)) + \eta^\alpha(\nabla_{\varphi X} Y + \nabla_X (\varphi Y))).
\]

It follows that

\[
2g(h_\alpha(X), Y) - 2g(h_\alpha(Y), X) = -\varepsilon_\alpha (\varphi X)(\eta^\alpha(Y)) + \varepsilon_\alpha \eta^\alpha [\varphi X, Y] + \varepsilon_\alpha (\varphi Y)(\eta^\alpha(X)) - \varepsilon_\alpha \eta^\alpha [\varphi Y, X] = -\varepsilon_\alpha (\mathcal{L}_{\varphi X} \eta^\alpha)(Y) + \varepsilon_\alpha (\mathcal{L}_{\varphi Y} \eta^\alpha)(X) = 0.
\]

Obviously, for any \( \alpha, \beta \in \{1, \ldots, r\} \) we have \( h_\alpha(\xi_\beta) = 0 \) and finally

\[
2(h_\alpha \circ \varphi + \varphi \circ h_\alpha)(X) = \mathcal{L}_{\xi_\alpha}(\varphi^2 X) - \varphi(\mathcal{L}_{\xi_\alpha}(\varphi X)) + \varphi(\mathcal{L}_{\xi_\alpha}(\varphi X)) - \varphi(\mathcal{L}_{\xi_\alpha} X)
= \xi_\alpha(\eta^\beta(X))\xi_\beta - \eta^\beta[\xi_\alpha, X]\xi_\beta = 0
\]

for any \( \alpha \in \{1, \ldots, r\} \) and any \( X \in \Gamma(TM) \).

\[\square\]
Proposition 3.5. Let \((M, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an indefinite almost \(S\)-manifold. Then, for any \(X, Y \in \Gamma(TM)\), the following properties hold:

a) \(\varphi(N(X, Y)) + N(\varphi X, Y) = 2\eta^\alpha(X)h_\alpha(Y)\),

b) \(N(X, Y) \in \mathcal{D}\).

Proof: Using Lemma 3.2, we obtain

\[
\varphi(N(X, Y)) + N(\varphi X, Y) = -(L_{\varphi X}\eta^\alpha)(X)\xi_\alpha + (L_{\varphi X}\eta^\alpha)(Y)\xi_\alpha + \eta^\alpha(X)(L_{\xi_\alpha}\varphi)(Y) = 2\eta^\alpha(X)h_\alpha(Y).
\]

Now, we observe that for any \(\alpha \in \{1, \ldots, r\}\) we have \([\xi_\alpha, \mathcal{D}] \subset \mathcal{D}\), in fact, if \(\beta \in \{1, \ldots, r\}\) and \(X \in \Gamma(TM)\), we have \(\eta^\beta [\xi_\alpha, \varphi X] = -2d\eta^\beta(\xi_\alpha, \varphi X) = 0\) and in particular, if \(X \in \mathcal{D}\) and \(\alpha = \beta\), we get \(\eta^\alpha[\xi_\alpha, X] = 0\). So, if \(Z \in \mathcal{D}\) then \(N(\xi_\alpha, Z) = -[\xi_\alpha, Z] - \varphi[\xi_\alpha, \varphi Z] \in \mathcal{D}\). It is easy to check that \(N(\xi_\alpha, \xi_\beta) = 0\) for any \(\alpha, \beta \in \{1, \ldots, r\}\); therefore, we have that \(N(\xi_\alpha, X) \in \mathcal{D}\) for any \(X \in \Gamma(TM)\). Finally, applying a), we have \(g(N(\varphi X, Y), \xi_\alpha) = 2\eta^\beta(X)g(h_\beta(Y), \xi_\alpha) = 0\). Hence, if \(X, Y \in \Gamma(TM)\), we get \(N(X, Y) = -N(\varphi^2 X, Y) + \eta^\alpha(X)N(\xi_\alpha, Y)\), and being \(N(\varphi^2 X, Y) \in \mathcal{D}\) and \(N(\xi_\alpha, Y) \in \mathcal{D}\), we conclude that \(N(X, Y) \in \mathcal{D}\). \(\square\)

Proposition 3.6. Let \((M, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an indefinite almost \(S\)-manifold. For any \(X \in \Gamma(TM)\) and for any \(\alpha \in \{1, \ldots, r\}\),

\[\nabla_X\xi_\alpha = -\varepsilon_\alpha \varphi(X) - \varphi(h_\alpha X).\]

Proof: Putting \(X = \xi_\alpha\) in a) of Proposition 3.5, we have that for any \(Z, Y \in \Gamma(TM)\)

\[g(N(\xi_\alpha, Y), \varphi Z) = -g(\varphi(N(\xi_\alpha, Y)), Z) = -2\eta^\beta(\xi_\alpha) g(h_\beta(Y), Z) = -2g(h_\alpha(Y), Z).\]

Moreover, applying (3) of Proposition 3.3, for any \(\alpha \in \{1, \ldots, r\}\) we find:

\[
g(-\varphi(\nabla_X\xi_\alpha), Z) = \frac{1}{2}g(N(\xi_\alpha, Z), \varphi X) - g(\varphi Z, \varphi X)\eta(\xi_\alpha)
= -g(h_\alpha(Z), X) - \varepsilon_\alpha g(Z, X) + \varepsilon_\alpha \varepsilon_\beta \eta^\beta(X)\eta^\alpha(Z)
= g(-h_\alpha(X) - \varepsilon_\alpha X + \varepsilon_\alpha \eta^\beta(X)\xi_\beta, Z),
\]

then \(\varphi(\nabla_X\xi_\alpha) = h_\alpha(X) + \varepsilon_\alpha X - \varepsilon_\alpha \eta^\beta(X)\xi_\beta\), and, applying \(\varphi\), we complete the proof. Note that \(\nabla_X\xi_\alpha \in \mathcal{D}\). \(\square\)

Proposition 3.7. Let \((M, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an indefinite almost \(S\)-manifold. For \(X, Y \in \Gamma(TM)\), we have

\[
(\nabla_X\varphi)(Y) + (\nabla_{\varphi X}\varphi)(\varphi Y) = 2g(\varphi X, \varphi Y)\xi + \eta(Y)\varphi^2(X) - \eta^\alpha(Y)h_\alpha(X).
\]

where \(\xi := \sum_{\alpha=1}^r \xi_\alpha\) and \(\eta(X) = g(X, \xi)\), for any \(X \in \Gamma(TM)\).
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**Proof:** Using (3), Proposition 3.5 and Proposition 3.6, for any $X, Y, Z \in \Gamma(TM)$ we have

\[
2g((\nabla_X \varphi)(Y), Z) + 2g((\nabla_{\varphi X} \varphi)(\varphi Y), Z) = -g(\varphi(N(Y, Z)) + N(\varphi Y, Z), X) \\
+ 4g(\varphi Y, \varphi X)\bar{\eta}(Z) - 2g(\varphi Z, \varphi X)\bar{\eta}(Y) \\
= -2g(Z, \eta^\alpha(Y)h_{\alpha}(X)) + \\
+ 4g(\varphi Y, \varphi X)g(Z, \bar{\xi}) + 2g(Z, \bar{\eta}(Y)\varphi^2 X).
\]

Then, we deduce

\[
(\nabla_X \varphi)(Y) + (\nabla_{\varphi X} \varphi)(\varphi Y) = 2g(X, Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X) - \eta^\alpha(Y)h_{\alpha}(X).
\]

Obviously, $\bar{\eta}(X) = \sum_{\alpha=1}^r \varepsilon_\alpha \eta^\alpha(X) = \sum_{\alpha=1}^r g(X, \xi_{\alpha}) = g(X, \bar{\xi}).$ \hfill $\square$

**Corollary 3.8.** Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $S$-manifold. Then, for any $X, Y \in \mathcal{D}$:

a) $(\nabla_X \varphi)(Y) + (\nabla_{\varphi X} \varphi)(\varphi Y) = 2g(X, Y)\bar{\xi},$

b) $(\nabla_X \varphi)(\varphi X) = (\nabla_{\varphi X} \varphi)(X)$.

**Proof:** The first statement follows from the above proposition. Putting $Y := \varphi X$ in a), we have $(\nabla_X \varphi)(\varphi X) + (\nabla_{\varphi X} \varphi)(\varphi^2 X) = 2g(X, \varphi X)\bar{\xi} = 0$, therefore, being $\varphi^2 X = -X$, we obtain $(\nabla_X \varphi)(\varphi X) = (\nabla_{\varphi X} \varphi)(X)$. \hfill $\square$

**Remark 3.9.** The statement b) can be written as $\nabla_X(\varphi^2 X) - \varphi(\nabla_X \varphi) = \nabla_{\varphi X}(\varphi X) - \varphi(\nabla_{\varphi X} \varphi)$, i.e. as $\nabla_X X + \nabla_{\varphi X}(\varphi X) = \varphi[\varphi X, X]$.

### 3.2 Indefinite $S$-manifolds

**Definition 3.10.** Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric $g,f,f$-manifold. $M$ is said an **indefinite $S$-manifold** if it is a normal indefinite almost $S$-manifold.

**Proposition 3.11.** Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $S$-manifold. Then $M$ is an indefinite $S$-manifold if and only if, for any $X, Y \in \Gamma(TM)$, the Levi-Civita connection satisfies:

\[
(\nabla_X \varphi)Y = g(X, Y)\bar{\xi} - \bar{\eta}(Y)X - \varepsilon_\alpha \eta^\alpha(Y)\eta^\alpha(X)\bar{\xi} + \bar{\eta}(Y)\eta^\alpha(X)\xi_{\alpha},
\]

or equivalently

\[
(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X).
\]
\textbf{Proof:} Assuming that $M$ is an indefinite $S$-manifold, (3) becomes
\[ g((\nabla_X \varphi) Y, Z) = g(\varphi Y, \varphi X) \tilde{\eta}(Z) - g(\varphi Z, \varphi X) \tilde{\eta}(Y) = g(Z, g(\varphi Y, \varphi X) \xi + \tilde{\eta}(Y) \varphi^2 X), \]
from which
\[
(\nabla_X \varphi) Y = g(\varphi X, \varphi Y) \xi + \tilde{\eta}(Y) \varphi^2 X = g(X, Y) \xi - \varepsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y) \xi - \tilde{\eta}(Y) X + \eta(Y) \eta^\alpha(X) \xi.
\]
Vice versa, we suppose that $\nabla$ satisfies (5). Then we obtain $g((\nabla_X \varphi) Y, Z) = g(\varphi Y, \varphi X) \tilde{\eta}(Z) - g(\varphi Z, \varphi X) \tilde{\eta}(Y)$, and comparing with (3), we deduce for any $X, Y, \in \Gamma(TM)$, $g(N(Y, Z), \varphi X) = 0$. From Proposition 3.5, we obtain that $N(Y, Z) = 0$ for any $Y, Z \in \Gamma(TM)$, that is $M$ is normal.

\textbf{Remark 3.12.} In an indefinite $S$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, the operators $L_{\xi_\alpha} \varphi$, and then $h_\alpha$, vanish. In fact, by direct computation for any $X \in \Gamma(TM)$ and for any $\alpha \in \{1, \ldots, r\}$ we get $N(\varphi X, \xi_\alpha) = (L_{\xi_\alpha} \varphi) X = 2h_\alpha(X)$, and the normality condition implies $h_\alpha = 0$. Using Proposition 3.6, we obtain, for any $\alpha \in \{1, \ldots, r\}$, $\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X$.

Now, we give the condition of indefinite $S$-manifold in terms of the fundamental 2-form:

\textbf{Proposition 3.13.} Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite almost $S$-manifold. Then $M$ is an indefinite $S$-manifold if and only if for any $X, Y, Z \in \Gamma(TM)$:
\[ (\nabla_X \Phi)(Y, Z) = \tilde{\eta}(Y)g(\varphi X, \varphi Z) - \tilde{\eta}(Z)g(\varphi X, \varphi Y). \]

\textbf{Proof:} One simply uses $(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi) Z)$ in (5).

\textbf{Proposition 3.14.} Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite metric g.f.f-manifold.

If the vector fields $\xi_\alpha$ are Killing, $L_{\xi_\alpha} \eta^\beta = 0$ for any $\alpha, \beta \in \{1, \ldots, r\}$ and $M$ satisfies (5) or equivalently (6), then $M$ is an indefinite $S$-manifold.

\textbf{Proof:} Being $3d \Phi(X, Y, Z) = \delta(X, Y, Z)(\nabla_X \Phi)(Y, Z)$, from (6) we get $d \Phi = 0$ and $(L_{\xi_\alpha} \Phi)(X, Y) = 0$, since $L_{\xi_\alpha} \Phi = i_{\xi_\alpha} d \Phi + di_{\xi_\alpha} \Phi$. Proposition 2.5 implies $(L_{\xi_\alpha} g)(X, \varphi Y) + g(X, (L_{\xi_\alpha} \varphi) Y) = 0$, for any $\alpha \in \{1, \ldots, r\}$ and $X, Y \in \Gamma(TM)$.

Hence, being $\xi_\alpha$ a Killing vector field, we find $L_{\xi_\alpha} \varphi = 0$ and then $\eta^\beta ([\xi_\alpha, \varphi Y]) = 0$, for any $\alpha, \beta \in \{1, \ldots, r\}$. In these hypotheses, (2) becomes
\[ 2g((\nabla_X \varphi) Y, Z) = g(N(Y, Z), \varphi X) + 2\varepsilon_\alpha [d\eta^\alpha(\varphi Y, Z) \eta^\alpha(X) - d\eta^\alpha(\varphi Z, Y) \eta^\alpha(X) + d\eta^\alpha(\varphi Y, X) \eta^\alpha(Z) - d\eta^\alpha(\varphi Z, X) \eta^\alpha(Y)]. \]

On the other hand, (6) implies $g(Y, (\nabla_X \varphi) Z) = \tilde{\eta}(Y)g(\varphi X, \varphi Z) - \tilde{\eta}(Z)g(\varphi X, \varphi Y)$, therefore we deduce
\[ g(N(Y, Z), \varphi X) = -2\varepsilon_\alpha [(d\eta^\alpha(\varphi Y, Z) - d\eta^\alpha(\varphi Z, Y)) \eta^\alpha(X) + (d\eta^\alpha(\varphi Y, X) - g(\varphi X, \varphi Y)) \eta^\alpha(Z) - (d\eta^\alpha(\varphi Z, X) - g(\varphi X, \varphi Z)) \eta^\alpha(Y)]. \]
Putting $Y = \xi_\beta$ in the above equation, we get
\[ g(N(\xi_\beta, Z), \varphi X) = 2\xi_\beta(d\eta^\beta(\varphi Z, X) - g(\varphi X, \varphi Z)). \] (7)

Since $N(\xi_\beta, Z) = -[\xi_\beta, Z] - \varphi[\xi_\beta, \varphi Z] + \xi_\beta(\eta^\alpha(Z))\xi_\alpha$, then $\varphi N(\xi_\beta, Z) = (L_{\xi_\alpha}\varphi)Z - \eta^\alpha[\xi_\beta, \varphi Z]\xi_\alpha = 0$ and (7) gives $d\eta^\beta(\varphi Z, X) = g(\varphi X, \varphi Z) = \Phi(\varphi Z, X)$. Finally, $L_{\xi_\alpha}\eta^\beta = 0$ implying $i_{\xi_\alpha}d\eta^\beta = 0$ and being $Y = -\phi^2Y + \eta^\alpha(Y)\xi_\alpha$, for any $Y \in \Gamma(TM)$, we obtain $d\eta^\beta(Y, X) = -d\eta^\beta(\phi^2Y, X) + \eta^\alpha(Y)d\eta^\beta(\xi_\alpha, X) = -\Phi(\phi^2Y, X) = \Phi(Y, X)$. Then $M$ is an indefinite almost $S$-manifold and we apply Proposition 3.11.

\[ \square \]

4 Examples of indefinite $S$-manifolds

We describe some examples of indefinite $S$-manifolds, where the characteristic vector fields are either timelike or spacelike or of both types.

**Example 4.1.** We consider $\mathbb{R}^6$ with its standard coordinates $\{x^1, x^2, y^1, y^2, z^1, z^2\}$. We introduce on $\mathbb{R}^6$ an indefinite $g.f.f.-structure (\varphi, \xi_1, \xi_2, \eta_1, \eta_2, g)$ by setting
\[ \xi_\alpha = \frac{\partial}{\partial z^\alpha}, \quad \eta^\alpha = dz^\alpha - 2y^i dx^i, \quad \alpha \in \{1, 2\}, \]
\[ g = -2\sum_{\alpha=1}^2 \eta^\alpha \otimes \eta^\alpha + \frac{1}{2} \sum_{i=1}^2 ((dx^i)^2 + (dy^i)^2), \]
and $\varphi$ given, with respect to the frame $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \xi_1, \xi_2\}$, by the matrix
\[ F = \begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}, \quad \text{where} \quad Y = \begin{pmatrix} y^1 & y^2 \\ y^1 & y^2 \end{pmatrix}. \]

We put $M = (\mathbb{R}^6, \varphi, \xi_1, \xi_2, \eta_1, \eta_2, g)$. A straightforward computation shows that $g$ is a metric tensor field. Firstly we check that $g$ is non-degenerate and then we compute its index. The matrix $G$ of $g$ is given by
\[ G = \begin{pmatrix} \frac{1}{2} - 2(y^1)^2 & -2y^1 y^2 & 0 & 0 & y^1 & y^1 \\ -2y^1 y^2 & \frac{1}{2} - 2(y^2)^2 & 0 & 0 & y^2 & y^2 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ y^1 & y^2 & 0 & 0 & -1 & 0 \\ y^1 & y^2 & 0 & 0 & 0 & -1 \end{pmatrix}, \]
and \( \det G = \frac{1}{16} \neq 0 \). Now, to determine the index of \( g \), we look for the eigenvalues of \( G \). Since
\[
\det(G - \lambda I) = -(\frac{1}{2} - \lambda)^4 \left( (1 + \lambda)(\lambda^2 + (2y^1)^2 + 2(y^2)^2 + \frac{1}{2})\lambda - \frac{1}{2} \right),
\]
we find that the index of \( g \) is two; therefore \( g \) is a semi-Riemannian metric of the index 2 on \( \mathbb{R}^6 \). We remark that \( \xi_1 \) and \( \xi_2 \) are timelike vector fields. It is easy to prove that \( M \) is an indefinite \( S \)-manifold.

**Example 4.2.** The second example of an indefinite \( S \)-manifold is \( M = (\mathbb{R}^6_2, \varphi, \xi_\alpha, \eta^\alpha, g) \), where, for any \( \alpha \in \{1, 2\} \), we put
\[
\xi_\alpha := \frac{\partial}{\partial z^\alpha}, \quad \eta^\alpha := dz^\alpha - \sum_{i=1}^{2} \tau_i y^i dx^i,
\]

\( \varphi, g \) are given by
\[
F = \begin{pmatrix}
0 & I_2 & 0 \\
-I_2 & 0 & 0 \\
0 & Y & 0
\end{pmatrix}, \quad \text{where} \quad Y = \begin{pmatrix} -y^1 & y^2 \\ -y^1 & y^2 \end{pmatrix},
\]

and
\[
g = \sum_{\alpha=1}^{2} \eta^\alpha \otimes \eta^\alpha + \frac{1}{2} \sum_{i=1}^{2} \tau_i ((dx^i)^2 + (dy^i)^2),
\]

respectively, where \( \tau_i = \pm 1 \) according to whether \( i = 1 \) or \( i = 2 \). Moreover, the symmetric \((0, 2)\)-type tensor field \( g \) is a semi-Riemannian metric because \( \det G = \frac{1}{16} \neq 0 \). Therefore \( g \) is non degenerate, and
\[
\det(G - \lambda I) = -(\frac{1}{2} + \lambda)^2 (\frac{1}{2} - \lambda)(\lambda - 1)(\lambda^2 - (\frac{3}{2} + 2(y^1)^2 + 2(y^2)^2)\lambda + \frac{1}{2}),
\]

so, since the signs of eigenvalues are independent from the coordinates, the index of \( g \) is constant. We note that in this example \( \xi_1 \) and \( \xi_2 \) are spacelike. One proves that \( M \) is an indefinite \( S \)-manifold.

**Example 4.3.** The third example is \( M = (\mathbb{R}^4_2, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g) \) constructed as follows. Denoting the standard coordinates with \( \{x, y, z^1, z^2\} \), we endow \( \mathbb{R}^4 \) with the structure \((\varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)\) where
\[
\xi_\alpha = \frac{\partial}{\partial z^\alpha}, \quad \eta^\alpha = dz^\alpha + ydx,
\]

for any \( \alpha \in \{1, 2\} \) and where the tensor fields \( \varphi \) and \( g \) are given by
\[
F := \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & y & 0 & 0 \\
0 & y & 0 & 0
\end{pmatrix}, \quad G := \begin{pmatrix}
\frac{1}{2} & 0 & y & -y \\
0 & \frac{1}{2} & 0 & 0 \\
y & 0 & 1 & 0 \\
-y & 0 & 0 & -1
\end{pmatrix}.
\]
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respectively. An immediate computation shows that g is non-degenerate and its index is constant. In fact, we have \( \det G = -\frac{1}{4} \), and

\[
\det(G - \lambda I) = (\frac{1}{2} - \lambda)(\lambda^3 - \frac{1}{2} \lambda^2 - (2y^2 + 1)\lambda + \frac{1}{2}),
\]

hence \( \det G \neq 0 \) and, using Cartesius’s rule, we deduce that the index is 1. Therefore, the tensor field \( g \) is a Lorentzian metric. Now, we observe that \( \xi_1 \) is a spacelike vector field while \( \xi_2 \) is a timelike vector field. One can check that \( M \) is an indefinite \( S \)-manifold.

5 Sectional curvature and \( \varphi \)-sectional curvature

In this section, we look for some results about the sectional curvature of indefinite \( S \)-manifolds. Following the notations in ([15]), for the curvature tensor \( R \) we have \( R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \), and \( R(X,Y,Z,W) = g(R(Z,W,Y),X) \), for any \( X, Y, Z, W, \in \Gamma(TM) \).

A two-dimensional subspace \( \pi \) of the tangent space \( T_p M \) is called non-degenerate if and only if we have \( \Delta(\pi) = g_p(X,X)g_p(Y,Y) - g_p(X,Y)^2 \neq 0 \) for any basis \( \{X,Y\} \) of \( \pi \). We know that if \( \pi \) is a non-degenerate 2-plane of \( T_p M \) then we can define the sectional curvature \( K_p(\pi) \) at \( p \) with respect to the 2-plane \( \pi \), putting

\[
K_p(\pi) = \frac{R_p(X,Y,X,Y)}{\Delta(\pi)} = \frac{g_p(R_p(X,Y,Y),X)}{\Delta(\pi)},
\]

where \( \pi = \text{span}\{X,Y\} \). In the following we denote \( K_p(\pi) = K_p(X,Y) \).

**Proposition 5.1.** In an indefinite \( S \)-manifold \( (M, \varphi, \xi_\alpha, \eta^\alpha, g) \) one has:

a) the distribution \( \ker \varphi \) is integrable and flat;

b) the sectional curvatures \( K(X,\xi_\alpha) = \varepsilon_\alpha \), for any \( \alpha \in \{1, \ldots, r\} \), and non lightlike \( X \in \text{Im} \varphi \).

**Proof:** For \( X, Y \in \ker \varphi \) we have \( X = f^\alpha \xi_\alpha, Y = t^\beta \xi_\beta \) then \( [X,Y] = [f^\alpha \xi_\alpha, t^\beta \xi_\beta] = f^\alpha \xi_\alpha (t^\beta \xi_\beta) - t^\beta \xi_\beta (f^\alpha \xi_\alpha) \in \ker \varphi \) and \( \ker \varphi \) is integrable. Furthermore, since \( \nabla_X \xi_\beta = 0 \) and \( [\xi_\alpha, \xi_\beta] = 0 \), we have \( R(\xi_\alpha, \xi_\beta, \xi_\gamma) = 0 \) and \( \ker \varphi \) is flat. Note that a) holds also for indefinite almost \( \mathit{S} \)-manifolds. Now, being \( M \) an indefinite \( \mathit{S} \)-manifold, we know that \( \nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X, \mathcal{L}_{\xi_\alpha} \varphi = 0 \) and we have

\[
R(\xi_\alpha, X, \xi_\beta) = -\varepsilon_\beta \nabla_{\xi_\alpha}(\varphi X) + \varepsilon_\beta \varphi [\xi_\alpha, X] \\
= \varepsilon_\beta (\varphi [\xi_\alpha, X] - [\xi_\alpha, \varphi X] - \varphi X \xi_\alpha) = \varepsilon_\beta \varepsilon_\alpha \varphi^2 X.
\]

So, for \( X \in \text{Im} \varphi, X \) non lightlike, we have \( K(X,\xi_\alpha) = -\frac{\varepsilon_\alpha g(\varphi^2 X, X)}{g(X,X)} = \varepsilon_\alpha \). □
As usual, we say that a 2-plane $\pi$ in $T_pM$, $p \in M$, is a $\varphi$-plane if $\pi = \text{span}\{X, \varphi X\}$ with $X \in \mathcal{D}_p$, and the sectional curvature at $p$ of such a plane, with $X$ a non lightlike vector, is said the $\varphi$-sectional curvature at $p$ and is denoted by $H_p(X)$.

We shall prove that on an indefinite $\mathcal{S}$-manifold, as in the Sasakian case, the $\varphi$-sectional curvatures determine the sectional curvatures.

As in [3], we define a tensor field of type (0,4) given for any $X, Y, Z, W$ in $\Gamma(TM)$ by

$$P(X, Y; Z, W) = \Phi(X, Z)g(Y, W) - \Phi(X, W)g(Y, Z)$$

$$- \Phi(Y, Z)g(X, W) + \Phi(Y, W)g(X, Z).$$

The following lemmas can be easily proved.

**Lemma 5.2.** Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite $\mathcal{S}$-manifold. Then:

a) $P(X, Y; Z, W) = -P(Z, W; X, Y)$, for any $X, Y, Z, W \in \Gamma(TM)$,

b) $P(X, Y; X, \varphi Y) = g(X, \varphi Y)^2 + g(X, Y)^2 - \varepsilon_X \varepsilon_Y$, where $X, Y$ are unit vector fields in $\mathcal{D}$ and $\varepsilon_X = g(X, X)$ and $\varepsilon_Y = g(Y, Y)$.

**Proposition 5.3.** Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite $\mathcal{S}$-manifold. Then, putting

$$\varepsilon = \sum_{\alpha=1}^r \varepsilon_\alpha,$$

for any $X, Y, Z, W \in \Gamma(TM)$

$$g(R(X, Y, \varphi Z), W) + g(R(X, Y, Z), \varphi W) = -\varepsilon P(X, Y; Z, W) - Q(X, Y; Z, W)$$

where

$$Q(X, Y; Z, W) = g(W, \varphi Y)(\varepsilon(g(X, Z) - g(\varphi X, \varphi Z)) - \bar{\eta}(Z)\bar{\eta}(X))$$

$$- g(W, \varphi X)(\varepsilon(g(Y, Z) - g(\varphi Y, \varphi Z)) - \bar{\eta}(Z)\bar{\eta}(Y))$$

$$- g(Z, \varphi Y)(\varepsilon(g(X, W) - g(\varphi X, \varphi W)) - \bar{\eta}(X)\bar{\eta}(W))$$

$$+ g(Z, \varphi X)(\varepsilon(g(Y, W) - g(\varphi Y, \varphi W)) - \bar{\eta}(Y)\bar{\eta}(W)).$$

Moreover if $X, Y, Z, W \in \mathcal{D}$ then obviously $Q(X, Y; Z, W) = 0$ and the following statements hold:

a) $g(R(\varphi X, \varphi Y, \varphi Z), \varphi W) = g(R(X, Y, Z), W)$;

b) $g(R(X, \varphi X, Y), \varphi Y) = g(R(X, Y, X), Y) + g(R(X, \varphi Y, X), \varphi Y) - 2\varepsilon P(X, Y, X, \varphi Y)$;

c) $g(R(\varphi X, \varphi Y, \varphi X), Y) = g(R(X, \varphi Y, X), \varphi Y)$.

**Remark 5.4.** We remark that $\varepsilon$ can vanish only if $r$ is an even number and the number of timelike characteristic vector fields is equal to the number of spacelike characteristic vector fields. Moreover, $\varepsilon = 0$ means that $g(\xi, \bar{\xi}) = 0$, i.e. $\xi = \sum_{\alpha=1}^r \xi_\alpha$ is a lightlike vector field.
We put
\[ B(X, Y) = g(R(X, Y, X), Y), \quad X, Y \in \Gamma(TM) \]
and
\[ D(X) = B(X, \varphi X), \quad X \in \Gamma(\mathfrak{D}). \]

The following Lemma, of which we omit the long proof, gives the useful expression of \( B(X, Y) \), for any \( X, Y \in \Gamma(\mathfrak{D}) \).

Lemma 5.5. Let \((M, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an indefinite S-manifold. Then, for any \(X, Y \in \Gamma(\mathfrak{D})\),
\[ B(X, Y) = \frac{1}{32} \{ 3D(X + \varphi Y) + 3D(X - \varphi Y) - D(X + Y) - D(X - Y) - 4D(X) - 4D(Y) + 24\varepsilon P(X, Y; X, \varphi Y) \}. \]

Using the previous Lemmas it is possible to compute the sectional curvature of a non degenerate 2-plane \( \pi = \text{span}\{X, Y\} \) of \( \mathfrak{D}_p \), as follows.

Proposition 5.6. Let \((M, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an indefinite S-manifold and \( p \) in \( M \). We consider a non degenerate 2-plane \( \pi = \text{span}\{X, Y\} \) of \( \mathfrak{D}_p \), where \( X \) and \( Y \) are unit vectors of \( \mathfrak{D}_p \). Then the sectional curvature \( K_p(X, Y) \) is given by
\[ K_p(X, Y) = \frac{1}{32(\varepsilon_X \varepsilon_Y - g(X, Y))^2} \{ 3(\varepsilon_X + \varepsilon_Y + 2g(X, \varphi Y))^2 H_p(X + \varphi Y) + 3(\varepsilon_X + \varepsilon_Y - 2g(X, \varphi Y))^2 H_p(X - \varphi Y) - (\varepsilon_X + \varepsilon_Y + 2g(X, Y))^2 H_p(X + Y) - (\varepsilon_X + \varepsilon_Y - 2g(X, Y))^2 H_p(X - Y) - 4H_p(X) - 4H_p(Y) + 24\varepsilon (g(X, \varphi Y)^2 + g(X, Y)^2 - \varepsilon_X \varepsilon_Y) \}. \]

Proof: We note that if \( X \in \mathfrak{D}_p \), we have
\[ D_p(X) = B_p(X, \varphi X) = \varphi_p(R_p(X, \varphi X, X), \varphi X) = -g_p(X, X)^2 H_p(X) \]
and if \( X \) and \( Y \) are unit vectors of \( \mathfrak{D}_p \), we find
\[ g(X + \varphi Y, X + \varphi Y) = \varepsilon_X + \varepsilon_Y + 2g(X, \varphi Y), \quad g(X + Y, X + Y) = \varepsilon_X + \varepsilon_Y + 2g(X, Y). \]

Being \( \Delta(\pi) = \varepsilon_X \varepsilon_Y - g_p(X, Y)^2 \), we get \( K_p(\pi) = -g_p(R_p(X, Y, X), Y)/\Delta(\pi) = -B_p(X, Y)/\Delta(\pi) \). Then, using (8) and Lemma 5.2, we get the required formula. \( \square \)

Remark 5.7. We note that if \( X \in \Gamma(\mathfrak{D}) \) is a unit vector field we have
\[ R(\xi_\alpha, X, \xi_\beta) = -\varepsilon_\beta \varepsilon_\alpha X, \quad R(X, \xi_\alpha, X) = -\varepsilon_X \varepsilon_\alpha \xi. \]
In fact, if \(Y \in \Gamma(TM)\), for any \(\alpha \in \{1, \ldots, r\}\), we have

\[
g(R(X, \xi_\alpha, X), Y) = -g(R(X, Y, \xi_\alpha), X) = \varepsilon_\alpha g(\nabla_X (\varphi Y) - \nabla_Y (\varphi X) - \varphi [X, Y], X)
\]

\[
= \varepsilon_\alpha g(\nabla_X \varphi) Y - 2 \varepsilon_\alpha \tilde{\eta}(Y) X - \tilde{\eta}(X) \varphi^2 Y, X
\]

\[
-\varepsilon_\alpha \varepsilon_\alpha \tilde{\eta}(Y) - \varepsilon_\alpha \varepsilon_\alpha g(\tilde{\xi}, Y).
\]

Finally, if \(X, Y \in \Gamma(D)\) and \(Z \in \Gamma(TM)\) then we get

\[
g(R(X, \xi_\alpha, Y), Z) = -\varepsilon_\alpha g(Y, X) \tilde{\eta}(Z) = -\varepsilon_\alpha g(Y, X) g(\tilde{\xi}, Z).
\]

**Theorem 5.8.** The \(\varphi\)-sectional curvatures completely determine the sectional curvatures of an indefinite \(S\)-manifold.

**Proof:** We show that for any \(p \in M\) and for any non degenerate 2-plane \(\pi = \text{span}\{X, Y\}\) in \(T_p(M)\) the sectional curvature \(K_p(X, Y)\) is uniquely determined by the \(\varphi\)-sectional curvature. In the sequel of the proof we suppose that \(p \in M\) is fixed. If \(X, Y \in D_p\), then we apply the previous Proposition and if \(X\) or \(Y\) is \(\xi_\alpha\), for any \(\alpha \in \{1, \ldots, r\}\), we have already seen that \(K_p(X, Y) = \varepsilon_\alpha\). If \(X, Y \in T_pM\), they can be written in the following way:

\[
X = aZ + \eta^\alpha(X)\xi_\alpha, \quad Y = bW + \eta^\alpha(Y)\xi_\alpha,
\]

where \(Z, W \in D, g_p(Z, Z) = \varepsilon_Z, g_p(W, W) = \varepsilon_W\), and \(a\) and \(b\) must satisfy:

\[
a^2 \varepsilon_Z = \varepsilon_X - \varepsilon_\alpha (\eta^\alpha(X))^2, \quad b^2 \varepsilon_W = \varepsilon_Y - \varepsilon_\alpha (\eta^\alpha(Y))^2.
\]

Therefore, we compute

\[
g_p(R_p(X, Y, X), Y) = a^2b^2 g_p(R_p(Z, W, Z), W) + 2a^2b \eta^\beta(Y) g_p(R_p(Z, W, Z), \xi_\beta)
\]

\[
+ 2ab^2 \eta^\alpha(X) g_p(R_p(Z, W, \xi_\alpha), W)
\]

\[
+ 2ab \eta^\alpha(X) \eta^\beta(Y) g_p(R_p(Z, \xi_\beta, \xi_\alpha), W) + 2ab \eta^\beta(Y) \eta^\alpha(X) g_p(R_p(Z, \xi_\beta, \xi_\alpha), \xi_\beta)
\]

\[
+ 2ab \eta^\beta(Y) \eta^\alpha(X) \eta^\beta(Y) g_p(R_p(Z, \xi_\beta, \xi_\alpha), \xi_\beta)
\]

\[
+ b^2 \eta^\alpha(X) \eta^\gamma(X) g_p(R_p(\xi_\alpha, W, \xi_\gamma), W)
\]

\[
+ 2b \eta^\alpha(X) \eta^\beta(Y) \eta^\gamma(X) g_p(R_p(\xi_\alpha, Z, \xi_\gamma), \xi_\beta)
\]

\[
+ \eta^\alpha(X) \eta^\beta(Y) \eta^\gamma(X) \eta^\delta(Y) g_p(R_p(\xi_\alpha, \xi_\beta, \xi_\gamma), \xi_\delta).
\]

Now, separately we take the terms of previous expression into account, using
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Remark 5.7 and the Bianchi identity, as follows:

\[ g_p(R_p(Z, W, Z), \xi, \eta) = g_p(R_p(Z, \xi, Z), W) = -\varepsilon_Z g_p(\xi, W) = 0, \]
\[ g_p(R_p(Z, W, \xi), W) = g_p(R_p(\xi, W, Z), W) = g_p(R_p(W, \xi, Z), Z) \]
\[ = -\varepsilon_W g_p(\xi, Z) = 0, \]
\[ g_p(R_p(Z, W, \xi), \xi, \eta) = -g_p(R_p(Z, \xi, \xi), W) - g_p(R_p(Z, \xi, \eta), W) \]
\[ = g_p(R_p(Z, \xi, Z), W) + \varepsilon_W g_p(Z, W) \]
\[ = -\varepsilon_W g_p(Z, W) + \varepsilon_W g_p(Z, W) = 0, \]
\[ g_p(R_p(Z, \xi, \xi), W) = -g_p(R_p(Z, \xi, W)g_p(\xi, W) \]
\[ = \varepsilon_W g_p(Z, W), \]
\[ g_p(R_p(Z, \xi, \xi), \xi) = -g_p(R_p(Z, \xi, \xi), W) = \varepsilon_W g_p(Z, W) = 0, \]
\[ g_p(R_p(Z, \xi, \xi), \xi) = \varepsilon_W g_p(Z, \xi) = 0. \]

Therefore, replacing the previous expressions in (9), we have:

\[ g_p(R_p(X, Y, X), Y) = a^2b^2 g_p(R_p(Z, W, Z), W) - a^2\varepsilon_Z \tilde{\eta}(Y) \tilde{\eta}(Y) \]
\[ + 2ab\tilde{\eta}(Y) \tilde{\eta}(X) g_p(Z, W) - b^2\varepsilon_W \tilde{\eta}(X) \tilde{\eta}(X). \]

Hence, being $K_p(X, Y) = -\varepsilon_X \varepsilon_Y g_p(R_p(X, Y, X), Y)$, we deduce

\[ K_p(X, Y) = \varepsilon_X \varepsilon_Y \left\{ a^2b^2 g_p(R_p(Z, W, W), Z) - 2ab\tilde{\eta}(Y) \tilde{\eta}(X) g_p(Z, W) \right\} \]
\[ + b^2\varepsilon_W \tilde{\eta}(X)^2 + a^2\varepsilon_Z \tilde{\eta}(Y)^2. \]

Now, we note that

\[ g_p(Z, W) = \frac{1}{ab} g_p(X - \eta^\alpha(X) \xi, Y - \eta^\beta(X) \xi) + \eta^\alpha(X) \eta^\beta(Y) g_p(\xi, \eta) \]
\[ = -\frac{1}{ab} \varepsilon^\alpha \eta^\alpha(X) \eta^\alpha(Y), \]
\[ g_p(R_p(Z, W, W), Z) = [\varepsilon_Z g_p(Z, W)^2] K_p(Z, W) \]
\[ = \frac{1}{ab^2} \left\{ a^2\varepsilon_Z b^2 \varepsilon_W - (\varepsilon^\alpha \eta^\alpha(X) \eta^\alpha(Y))^2 \right\} K_p(Z, W) \]
\[ = \frac{1}{ab^2} \left\{ (\varepsilon_X - \varepsilon^\alpha \eta^\alpha(X))^2 (\varepsilon_Y - \varepsilon^\alpha \eta^\alpha(Y))^2 \right\} \]
\[ - (\varepsilon^\alpha \eta^\alpha(X) \eta^\alpha(Y))^2 K_p(Z, W). \]

Thus, (10) becomes

\[ K_p(X, Y) = \varepsilon_X \varepsilon_Y \left\{ (\varepsilon_X - \varepsilon^\alpha \eta^\alpha(X))^2 (\varepsilon_Y - \varepsilon^\beta \eta^\beta(Y))^2 \right\} \]
\[ - (\varepsilon^\alpha \eta^\alpha(X) \eta^\alpha(Y))^2 K_p(Z, W) + 2\tilde{\eta}(Y) \tilde{\eta}(X) \varepsilon^\alpha \eta^\alpha(X) \eta^\alpha(Y) \]
\[ + (\varepsilon_Y - \varepsilon^\beta \eta^\beta(Y))^2 \tilde{\eta}(X)^2 + (\varepsilon_X - \varepsilon^\alpha \eta^\alpha(X))^2 \tilde{\eta}(Y)^2, \]

and this completes the proof, since $K_p(Z, W)$ is given as in Proposition 5.6. □
We recall the following result.

**Lemma 5.9** ([16]). Let \((V, g)\) be a semi-Euclidean vector space and \(R\) a \((0, 4)\)-type tensor on \(V\) such that for any \(X, Y, Z, W \in V\) the following conditions hold:

a) \(R(X, Y, Z, W) = -R(Y, X, Z, W),\)

b) \(R(X, Y, Z, W) = -R(X, Y, W, Z),\)

c) \(R(X, Y, Z, W) = R(Z, W, X, Y),\)

d) \(\mathcal{S}_{Y, Z, W} R(X, Y, Z, W) = 0.\)

If \(R(X, Y, X, Y) = 0\) for any linearly independent and non lightlike vectors \(X, Y \in V\), then \(R = 0\). Moreover, if \(R\) and \(S\) are \((0, 4)\)-type tensors on \(V\) such that the conditions (a-d) are satisfied and \(R(X, Y, X, Y) = S(X, Y, X, Y)\) for any \(X, Y \in V\) linearly independent non lightlike vectors, then \(R = S\).

**Proposition 5.10.** Let \((M, \varphi, \xi_\alpha, \eta^\alpha, g)\) be an indefinite \(\mathcal{S}\)-manifold, \(T\) and \(S\) be \((0, 4)\)-type tensor fields on \(M\) such that the following conditions hold:

i) \(T(X, Y, Z, W) = -T(Y, X, Z, W), \quad S(X, Y, Z, W) = -S(Y, X, Z, W), \quad X, Y, Z, W \in \Gamma(TM)\)

ii) \(T(X, Y, Z, W) = -T(X, Y, W, Z), \quad S(X, Y, Z, W) = -S(X, Y, W, Z), \quad X, Y, Z, W \in \Gamma(TM)\)

iii) \(T(X, Y, Z, W) = T(Z, W, X, Y), \quad S(X, Y, Z, W) = S(Z, W, X, Y), \quad X, Y, Z, W \in \Gamma(TM)\)

iv) \(\mathcal{S}_{Y, Z, W} T(X, Y, Z, W) = 0, \quad \mathcal{S}_{Y, Z, W} S(X, Y, Z, W) = 0, \quad X, Y, Z, W \in \Gamma(TM)\)

v) for any \(X, Y, Z, W \in \Gamma(\mathfrak{D})\)

\[
T(X, Y, \varphi Z, W) + T(X, Y, Z, \varphi W) = \varepsilon P(X, Y; Z, W)
\]

\[
S(X, Y, \varphi Z, W) + S(X, Y, Z, \varphi W) = \varepsilon P(X, Y; Z, W)
\]

vi) for any \(X, Y \in \Gamma(\mathfrak{D})\) and for any \(\alpha, \beta, \gamma, \delta \in \{1, \ldots, r\}\)

(a) \(T(X, \xi_\alpha, X, Y) = S(X, \xi_\alpha, X, Y),\)

(b) \(T(\xi_\alpha, X, \xi_\beta, Y) = S(\xi_\alpha, X, \xi_\beta, Y),\)

(c) \(T(\xi_\alpha, X, \xi_\beta, \xi_\gamma) = S(\xi_\alpha, X, \xi_\beta, \xi_\gamma),\)

(d) \(T(\xi_\alpha, \xi_\beta, \xi_\gamma, \xi_\delta) = S(\xi_\alpha, \xi_\beta, \xi_\gamma, \xi_\delta).\)

Then, if \(T(X, \varphi X, X, \varphi X) = S(X, \varphi X, X, \varphi X)\) for any \(X \in \Gamma(\mathfrak{D})\) non light-like vector field, one has \(T = S\).
Proof: It is easy to verify that $vi)$ implies that for any $X', Y', Z', W' \in \Gamma(\mathcal{D})$

$$T(\varphi X', \varphi Y', \varphi Z', \varphi W') = T(X', Y', Z', W'),$$

and, using the above formula, we obtain

$$T(\varphi X', \varphi Y', Z', W') = T(X', Y', \varphi Z', \varphi W').$$

Analogously, for the tensor field $S$ we have

$$S(\varphi X', \varphi Y', Z', W') = S(X', Y', \varphi Z', \varphi W').$$

Now, being $\varphi_p$ an almost complex structure on $\mathcal{D}_p$ for any $p \in M$, from a well-known result analogous to Lemma 5.9 ([1]), in the case of a real vector space endowed with an almost complex structure, we deduce $T(X', Y', Z', W') = S(X', Y', Z', W')$. Then, in particular, we have

$$T(X', Y', X', Y') = S(X', Y', X', Y').$$

Now, if $X, Y \in \Gamma(TM)$ are linearly independent and non lightlike, we compute $T(X, Y, X, Y)$ and $S(X, Y, X, Y)$, writing $X = X' + \eta^\alpha(X)\xi^\alpha$ and $Y = Y' + \eta^\alpha(Y)\xi^\alpha$, and likewise to (9), by the $\mathcal{F}(M)$-linearity of $T$ and $S$, using $vi)$, we get $T(X, Y, X, Y) = S(X, Y, X, Y)$.

Remark 5.11. Using Remark 5.7 and Proposition 5.1, the Riemannian $(0,4)$-type curvature tensor field $R$ satisfies the properties listed in Proposition 5.10. Thus, it is uniquely determined by the $\varphi$-sectional curvature.

Theorem 5.12. Let $(M, \varphi, \xi^\alpha, \eta^\alpha, g)$ be an indefinite $S$-manifold. Then the $\varphi$-sectional curvature $c$ is pointwise constant, $c \in \mathcal{F}(M)$, if and only if the Riemannian $(0,4)$-type curvature tensor field $R$ is given by

$$R(X, Y, Z, W) = -\frac{c + 3\varepsilon}{4}\{g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\} \quad (11)$$

$$- \frac{c - \varepsilon}{4}\{\Phi(W, X)\Phi(Z, Y)$$

$$- \Phi(Z, X)\Phi(W, Y) + 2\Phi(X, Y)\Phi(W, Z)\}$$

$$- \{\bar{\eta}(W)\bar{\eta}(X)g(\varphi Z, \varphi Y) - \bar{\eta}(W)\bar{\eta}(Y)g(\varphi Z, \varphi X)$$

$$+ \bar{\eta}(Y)\bar{\eta}(Z)g(\varphi W, \varphi X) - \bar{\eta}(Z)\bar{\eta}(X)g(\varphi W, \varphi Y)\}.$$

Proof: We suppose that the $\varphi$-sectional curvature $c$ is pointwise constant and in order to prove (11), denote by $S(X, Y, Z, W)$ the right-hand side of (11). Obviously $S$ is a tensor field of type $(0,4)$ on $M$, and we shall prove that $S$ coincides with $R$. To this end it is easy to check that for any $X, Y, Z, W \in \Gamma(TM)$ we have the properties of skew-symmetry $-S(X, Y, Z, W) = S(X, Y, Z, W) = -S(Y, X, Z, W)$ and the Bianchi identity $\mathcal{G}_{Y, Z, W}S(X, Y, Z, W) = 0$, while the
property \( iii \) of Proposition 5.10, \( S(X, Y, Z, W) = S(Z, W, X, Y) \), follows by the Bianchi identity and the skew-symmetries.

Moreover, since by definition of \( W \) we continue verifying property 200

\[\begin{align*}
S &= \frac{e}{4} \{ g(Y, Z)\Phi(X, W) - g(X, Z)\Phi(Y, W) \\
+ \Phi(Y, Z)g(X, W) - \Phi(X, Z)g(Y, W) + g(W, X)\Phi(Z, Y) \\
- \Phi(Z, X)g(W, Y) + \Phi(W, X)g(Z, Y) - g(Z, X)\Phi(W, Y) \} \\
- \frac{\varepsilon}{4} \{ 3\Phi(X, W)g(Z, Y) - 3\Phi(Y, W)g(X, Z) + 3g(X, W)\Phi(Y, Z) \\
- 3g(Y, W)\Phi(X, Z) + \Phi(Y, Z)g(W, X) - \Phi(X, Z)g(W, Y) \\
+ \Phi(X, W)g(Z, Y) - \Phi(Y, W)g(Z, X) \} \\
= -\varepsilon \{ \Phi(X, W)g(Z, Y) - \Phi(X, Z)g(Y, W) - \Phi(Y, W)g(X, Z) + g(W, X)\Phi(Y, Z) \} \\
= \varepsilon P(X, Y; Z, W).
\end{align*}\]

We continue verifying \( vi \) of Proposition 5.10, and obtaining \( S(X, \xi_\alpha, X, Y) = 0 = R(X, \xi_\alpha, X, Y), S(\xi_\alpha, X, \xi_\beta, \xi_\gamma) = 0 = R(\xi_\beta, X, \xi_\beta, \xi_\gamma), S(\xi_\alpha, \xi_\beta, \xi_\beta, \xi_\gamma) = 0 = R(\xi_\alpha, \xi_\beta, \xi_\beta, \xi_\gamma) \)

\[S(\xi_\alpha, X, \xi_\beta, Y) = -\frac{c + 3\varepsilon}{4} \{ g(\varphi X, \varphi \xi_\beta)g(\varphi \xi_\alpha, \varphi Y) - g(\varphi \xi_\alpha, \varphi \xi_\beta)g(\varphi X, \varphi Y) \} \\
- \frac{c}{4} \{ \Phi(Y, \xi_\alpha)\Phi(\xi_\beta, X) - \Phi(\xi_\beta, \xi_\alpha)\Phi(Y, X) \\
+ 2\Phi(\xi_\alpha, X)\Phi(Y, \xi_\beta) \} - \{ \eta(Y)\eta(\xi_\alpha)g(\varphi \xi_\beta, \varphi X) \\
- \eta(\xi_\alpha)\eta(Y)g(\varphi \xi_\beta, \varphi \xi_\beta) + \eta(Y)\eta(\xi_\beta)g(\varphi Y, \varphi \xi_\alpha) \\
- \eta(\xi_\beta)\eta(\xi_\alpha)g(\varphi Y, \varphi X) \} = \varepsilon \varepsilon g(Y, X) = R(\xi_\alpha, X, \xi_\beta, Y).\]

For any \( X \in \Gamma(\mathcal{D}) \) non lightlike vector field, we compute \( S(X, \varphi X, X, \varphi X) \), obtaining:

\[\begin{align*}
S(X, \varphi X, X, \varphi X) &= -\frac{c + 3\varepsilon}{4} \{ g(\varphi^2 X, \varphi X)g(\varphi X, \varphi^2 X) - g(\varphi X, \varphi X)g(\varphi^2 X, \varphi^2 X) \} \\
- \frac{c}{4} \{ \Phi(\varphi X, X)\Phi(X, \varphi X) - \Phi(X, \varphi X)\Phi(\varphi X, X) \\
+ 2\Phi(\varphi X, X)\Phi(\varphi X, X) \} \\
- \{ \eta(\varphi X)\eta(X)g(\varphi X, \varphi^2 X) - \eta(\varphi X)\eta(\varphi X)g(\varphi X, \varphi X) \\
+ \eta(\varphi X)\eta(X)g(\varphi^2 X, \varphi X) - \eta(\varphi X)\eta(\varphi X)g(\varphi^2 X, \varphi^2 X) \} \\
= \frac{c + 3\varepsilon}{4} g(X, X)^2 - \frac{c - \varepsilon}{4} \{ -g(X, X)^2 - 2g(X, X)^2 \} \\
= \frac{c + 3\varepsilon}{4} g(X, X)^2 + 3\frac{c - \varepsilon}{4} g(X, X)^2 = cg(X, X)^2.
\end{align*}\]

Moreover, since by definition of \( \varphi \)-sectional curvature we have

\[R(X, \varphi X, X, \varphi X) = cg(X, X)^2. \] (13)
from (12) and (13) we get \( R(X, \varphi X, X, \varphi X) = S(X, \varphi X, X, \varphi X) \), and, using Proposition 5.10, the previous Remark and the properties of the tensor field \( S \), we obtain \( R(X, Y, Z, W) = S(X, Y, Z, W) \), for any \( X, Y, Z, W \in \Gamma(TM) \), that is the formula (11).

Conversely, if we assume (11), choosing a point \( p \in M \) and a \( \varphi \)-plane \( \pi = \text{span}\{X, \varphi X\} \), with \( X \in \mathcal{D}_p \) non lightlike vector, by direct computation, omitting the point \( p \), we have

\[
H(X) = \frac{c + 3\varepsilon}{4g(X, X)^2} g(X, X)^2 + 3\frac{c - \varepsilon}{4g(X, X)^2} g(X, X)^2 = c.
\]

\[\square\]

6 Sectional Curvature in the case \( \varepsilon = 0 \), an example

In this section we consider the case \( \varepsilon = 0 \), as already pointed out, \( r = 2p \) and \( \xi_1, \ldots, \xi_p \) are timelike vector field, \( \xi_{p+1}, \ldots, \xi_{2p} \) are spacelike vector field. We call such a manifold a special indefinite \( S \)-manifold. Let \((M, \varphi, \xi_\alpha, \eta^\alpha, g)\) be a special indefinite \( S \)-manifold. The tensor \( Q \) is given by

\[
Q(X, Y; Z, W) = -g(W, \varphi Y)\tilde{\eta}(Z)\tilde{\eta}(X) + g(W, \varphi X)\tilde{\eta}(Z)\tilde{\eta}(Y) + g(Z, \varphi Y)\tilde{\eta}(X)\tilde{\eta}(W) - g(Z, \varphi X)\tilde{\eta}(Y)\tilde{\eta}(W),
\]

and

\[
g(R(X, Y, \varphi Z), W) + g(R(X, Y, Z), \varphi W) = -Q(X, Y; Z, W)
\]

Moreover, being \( Q(X, Y; Z, W) = 0 \) for any \( X, Y, Z, W \in \mathcal{D} \), we have

a) \( g(R(\varphi X, \varphi Y, \varphi Z), \varphi W) = g(R(X, Y, Z), W) \);

b) \( g(R(X, \varphi X, Y), \varphi Y) = g(R(X, Y, X), Y) + g(R(X, \varphi Y, X), \varphi Y) \);

c) \( g(R(\varphi X, \varphi Y, X), Y) = g(R(X, \varphi Y, X), \varphi Y) \).

Furthermore, for \( X, Y \in \Gamma(\mathcal{D}) \)

\[
B(X, Y) = \frac{1}{32}\{3D(X + \varphi Y) + 3D(X - \varphi Y) - D(X + Y) - D(X - Y) - 4D(X) - 4D(Y)\},
\]

and for a non degenerate 2-plane \( \pi = \text{span}\{X, Y\} \) of \( \mathcal{D}_p \), where \( X \) and \( Y \) are unit vectors of \( \mathcal{D}_p \),

\[
K_p(X, Y) = \frac{1}{32(\varepsilon_X \varepsilon_Y - g(X, Y)^2)} \{3(\varepsilon_X + \varepsilon_Y + 2g(X, \varphi Y))^2 H_p(X + \varphi Y) + 3(\varepsilon_X + \varepsilon_Y - 2g(X, \varphi Y))^2 H_p(X - \varphi Y) - (\varepsilon_X + \varepsilon_Y + 2g(X, Y))^2 H_p(X + Y) - (\varepsilon_X + \varepsilon_Y - 2g(X, Y))^2 H_p(X - Y) - 4H_p(X) - 4H_p(Y)\}.
\]
Finally we have that the \( \varphi \)-sectional curvature \( c \) is pointwise constant, \( c \in \mathcal{F}(M) \), if and only if the Riemannian \((0,4)\)-type curvature tensor field \( R \) is given by

\[
R(X, Y, Z, W) = -\frac{c}{4} \left\{ \varphi(Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W) \right\} \\
+ \Phi(W, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(W, Y) + 2\Phi(X, Y)\Phi(W, Z) \\
- \left\{ \eta(W)\eta(Y)g(\varphi Z, \varphi Y) - \eta(W)\eta(Y)g(\varphi Z, \varphi X) \\
+ \eta(Y)\eta(Z)g(\varphi W, \varphi X) - \eta(Z)\eta(X)g(\varphi W, \varphi Y) \right\}.
\]

An example of a special indefinite \( S \)-manifold is \( M = (\mathbb{R}^4_{\varepsilon}, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g) \), which is described in Example 4.3. We observe that the metric is Lorentzian, \( \xi_1 \) is a spacelike vector field while \( \xi_2 \) is a timelike vector field, then, since \( \varepsilon = 0 \), the structure is a special indefinite \( S \)-structure. Now, we compute the tensor field \( Q \) on some relevant set of vector fields, the sectional curvature and \( \varphi \)-sectional curvature. We know that \( Q = 0 \) on \( \mathcal{D} \), moreover we have

\[
Q(\xi_1, Y; Z, W) = -Q(\xi_2, Y; Z, W) = -g(W, \varphi Y)\eta(Z) + g(Z, \varphi Y)\eta(W) = 0, \\
Q(\xi_\alpha, Y; \xi_\beta, W) = Q(Y; \xi_\alpha; W, \xi_\beta) = -\varepsilon_\alpha\varepsilon_\beta g(W, \varphi Y),
\]

for any \( Y, Z, W \in \Gamma(\mathcal{D}) \) and for any \( \alpha, \beta \in \{1, 2\} \). Equation (15) shows that \( Q \) never vanishes. Now, computing the Christoffel’s symbols we obtain:

\[
\Gamma^3_{12} = \Gamma^4_{12} = \frac{1}{2}, \quad \Gamma^2_{13} = -\Gamma^2_{23} = -\Gamma^1_{23} = \Gamma^1_{24} = -1, \\
\Gamma^3_{23} = \Gamma^3_{24} = -\Gamma^4_{24} = -y,
\]

whereas the other \( \Gamma^k_{ij} \) vanish. To compute the \( \varphi \)-sectional curvature, being \( \mathcal{D} \) globally spanned by \( X = \frac{\partial}{\partial x} - y\xi_1 - y\xi_2 \) and \( Y = \varphi X = \frac{\partial}{\partial y} \), we value \( H(X) \). So, we have

\[
R(X, \varphi X, X, X) = \nabla_X \left( \Gamma^h_{21} - y(\Gamma^h_{23} + \Gamma^h_{24}) \frac{\partial}{\partial x^h} - \xi_1 - \xi_2 \right) - \nabla_{\xi_1} X - \nabla_{\xi_2} X \\
= -\frac{1}{2} \nabla_X (\xi_1 + \xi_2) - (\Gamma^h_{31} - y(\Gamma^h_{33} + \Gamma^h_{34}) + \Gamma^h_{41} - y(\Gamma^h_{43} + \Gamma^h_{44})) \frac{\partial}{\partial x^h} \\
= [\Gamma^h_{11} - y(\Gamma^h_{31} + \Gamma^h_{41}) - y(\Gamma^h_{13} + \Gamma^h_{14}) + \\
\Gamma^h_{14} - y(\Gamma^h_{34} + \Gamma^h_{44})] \frac{\partial}{\partial x^h} = 0, \\
\]

\[
g(X, X) = g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) - 2g(\frac{\partial}{\partial x}, \xi_1) + \\
+ g(\frac{\partial}{\partial x}, \xi_2) + y^2(g(\xi_1, \xi_1) + g(\xi_1, \xi_2) + g(\xi_2, \xi_2)) = \frac{1}{2}.
\]

It follows that

\[
H(X) = -\frac{1}{g(X, X)^2}g(R(X, \varphi X, X, \varphi X, \varphi X) = 0.
\]
Then, $M$ is an indefinite $S$-space form with $c = 0 = \varepsilon$ and, from (14) for any $Y, Z, W \in \Gamma(TM)$, the Riemannian curvature tensor field $R$ is given by:

$$R(\xi_\alpha, Y, Z, W) = -\varepsilon_\alpha \{\bar{\eta}(W)g(\varphi Z, \varphi Y) - \bar{\eta}(Z)g(\varphi W, \varphi Y)\},$$

$$R(\xi_\alpha, \xi_\beta, Z, W) = 0,$$

$$R(\xi_\alpha, Y, \xi_\beta, W) = \varepsilon_\alpha \varepsilon_\beta g(\varphi W, \varphi Y),$$

and $R$ vanishes on $D$.

References


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