Disjoint paths spanning simple polytopal graphs

by

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Abstract

In this paper, the definition of traceability is extended to n-path-traceability, i.e. the existence of a spanning set of n disjoint paths. Subsequently, an algorithm is provided to show that for each natural number \( n > 1 \), there is a simple 3-polytopal graph with at most \( 44n + 46 \) vertices which is not n-path-traceable.

Key Words: Spanning subgraphs, spanning set of paths, simple 3-polytopes.

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1 Introduction

The question of the existence of hamiltonian cycles in graphs, a historically old question, was in modern times mainly motivated by the well-known 4-colour conjecture. For cubic 3-connected planar graphs, the question was answered through a counterexample by Tutte [6] in 1946. From then on, the question of smallest counterexamples arose.

The corresponding question about smallest non-traceable graphs, i.e. without hamiltonian paths, is mentioned by V. Klee [3]. T. Zamfirescu found in 1968 an example with 88 vertices, which remained the smallest known until these days (see [1], [8]). Moreover, he generalized the last question replacing the hamiltonian paths by n-paths, which are disjoint unions of at most n paths. He also separately considered the question when the n paths are not necessarily pairwise disjoint. A related problem was treated in [7].

By a graph we shall always understand here a cubic 3-connected planar graph, which means by Steinitz' Theorem the 1-skeleton of a simple polytope. A graph is n-path-traceable if it has a spanning n-path.

By using the well-known Lederberg-Bosák-Barnette graph \( T \), which is a smallest non-hamiltonian graph [2], one easily gets a non-n-path-traceable graph. It suffices to insert \( 2n+1 \) copies of \( T' \), the graph \( T \) minus one vertex, into equally
many vertices of an arbitrary graph $H$. To keep the example small, we take $H$ to have $2n + 2$ vertices. Since $T'$ contains 37 vertices, the constructed graph has $74n + 38$ vertices. (An example for $n = 2$ is shown in Figure 1. Each pointed triangle stands for a copy of $T'$.)

Figure 1: A non-2-path-traceable graph, 186 vertices

This paper describes the construction of non-$n$-path-traceable simple 3-polytopal graphs consisting of at most $44n + 46$ vertices. This result has already been announced in [4].

2 The construction

The first graph needed for our construction is shown in Figure 2. It consists of three Tutte triangles (see [6]), contains 43 vertices and will be called $T_3$.

Lemma 1. $T_3$ is not traceable.

Proof: Assume $T_3$ contains a spanning path $P$. Then, $P$ cannot enter and leave the triangle $aa'b'$, since this would lead to a hamilton path from $a'$ to $b'$. (The same holds true for $a''b''b$.)

Thus, each of the small triangles must contain an endpoint of $P$. If these triangles are contracted to $a$ respectively $b$, $P$ would become a hamiltonian path from $a$ to $b$ in the big Tutte triangle, which is impossible.

Lemma 2. If a graph $G$ contains a copy $I$ of $T_3$ and a spanning $n$-path $J$, then $I \cap J$ is not connected.
The proof is easy, since no subpath of $J$ can span $I$.

**Theorem 1.** Let $G$ be the graph of Figure 3, the black-white partition of the vertex set being $\{V_1, V_2\}$, with

$$\text{card } V_1 = \text{card } V_2 = \begin{cases} n+1 & \text{for } n \text{ odd} \\ n+2 & \text{for } n \text{ even}. \end{cases}$$

If $n+1$ vertices in $V_2$ are replaced by copies of $T_3$, then the resulting graph is not $n$-path-traceable and has $44n + 44$ vertices if $n$ is odd and $44n + 46$ if $n$ is even.

**Proof:** Let $H$ be the resulting graph. Assume $H$ contains a spanning $n$-path $J$. Then, for each copy $I$ of $T_3$ in $H$, $I \cap J$ must consist of more than one component. If each of these components is contracted to a single vertex and all edges of $H \setminus J$ are deleted, then we obtain a union $K$ of $n$ disjoint paths (see Figure 4).

During the process of transformation from $G$ over $H$ to $K$, the number of black vertices does not change. If all other vertices in $K$ are considered white, then no two white vertices can be adjacent to each other. But since $I \cap J$ consists of more than one component for each copy $I$ of $T_3$ and $H$ contains $n+1$ such copies, $K$ must contain at least $n+1$ more white than black vertices. Thus, the number of paths must be larger than $n$, and a contradiction is obtained. $\square$
Figure 3: Graph $G$ for $n = 6$ or $n = 7$

Figure 4: A spanning 3-path is transformed
For $n = 2$, the graph described in the Theorem is depicted in Figure 5. It was found by Strauch [5] in 2002.

References


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