Asymptotic behavior of discrete and continuous semigroups on Hilbert spaces

by

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Abstract

Let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $\phi(t) > 0$ for all $t > 0$, $H$ be a complex Hilbert space and let $T$ be a bounded linear operator acting on $H$. Among our results is the fact that $T$ is power stable (i.e. its spectral radius is less than 1) if

$$\sum_{n=0}^{\infty} \phi(|\langle T^n x, x \rangle|) < \infty$$

for all $x \in H$ with $\|x\| \leq 1$.

In the continuous case we prove that a strongly continuous uniformly bounded semigroup of operators acting on a Hilbert space $H$ is spectrally stable (i.e. the spectrum of its infinitesimal generator lies in the open left half plane) if and only if for each $x \in H$ and each $\mu \in \mathbb{R}$ one has:

$$\sup_{s \geq 0} \left| \int_{0}^{s} e^{-i\mu t} \langle T(t)x, x \rangle \, dt \right| < \infty.$$

Key Words: Spectral radius, discrete semigroups, strongly continuous semigroups, uniform exponential stability, Orlicz space.

2000 Mathematics Subject Classification: Primary 47D03, Secondary 11M35.

1 Introduction

Let $X$ be a complex Banach space and $X^*$ be its dual. The resolvent set and the spectrum of a linear operator $T$ (acting on $X$) will be denoted respectively by $\rho(T)$ and $\sigma(T)$. When $T$ is bounded, the spectrum radius of $T$ is given by the formula

$$r(T) = \sup\{ |\lambda| : \lambda \in \sigma(T) \} = \lim_{n \to \infty} \|T^n\|^{1/n}.$$
Jan van Neerven ([13]) has shown that $r(T) < 1$ if there exists a Banach function space $E$ over $\mathbb{N}$ with
\[ \lim_{n \to \infty} \| \chi_{\{0,1,\ldots,n-1\}} \|_E = \infty \] (1.1)
such that for each $x \in X$ and $x^* \in X^*$ the map $n \mapsto |\langle T^n x, x^* \rangle|$ belongs to $E$. By applying this result to an appropriate Orlicz space $E$, he also able to show that $r(T) < 1$ if there exists a non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) > 0$ for all $t > 0$ and
\[ \sum_{n=0}^{\infty} \phi(|\langle T^n x, y \rangle|) < \infty. \] (1.2)
for all $x \in X$ and $y \in X^*$. The particular case where $\phi(t) = t^p$ for some $(1 \leq p < \infty)$ has been noticed by G. Weiss ([21]). It is worth to mention the contribution of K. L. Przyulski ([18]) who considered the case where $\phi(t) = t$, in the context of weakly sequentially complete Banach spaces. The similar topics in the continuous case started with a question raised by A. J. Pritchard and J. Zabczyk [20]. Precisely, they asked whether any weakly-$L^p$-stable semigroup is necessarily uniformly exponentially stable. See the next section for relevant definitions. An account on the research related to this question can be found in [7], [22], [13].

The aim of this paper is to prove the following.

**Theorem 1.** Let $T$ be a bounded linear operator acting on a complex Hilbert space $H$ and let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\phi(t) > 0$ for all $t > 0$ and
\[ \sum_{n=0}^{\infty} \phi(|\langle T^n x, x \rangle|) < \infty. \] (1.3)
for every $x \in H$. Then $r(T) < 1$.

**Theorem 2.** Let $T = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on a complex Hilbert space $H$ and let $\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing function with $\phi(t) > 0$ for all $t > 0$. If $T$ is uniformly bounded (that is, $\sup_{t \geq 0} \|T(t)\| < \infty$) and
\[ \int_0^\infty \phi(|\langle T(t)x, x \rangle|)dt < \infty, \] (1.4)
then the semigroup $T$ is uniformly exponentially stable, that is, its uniform growth bound $\omega_0(T) := \inf_{t>0} \frac{\ln \|T(t)\|}{t}$ is negative.

**Theorem 3.** Let $T = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on a complex Hilbert space $H$. If for each $x \in H$ one has
\[ \sup_{\mu \in \mathbb{R}} \sup_{s \geq 0} \left| \int_0^s e^{-i\mu t} \langle T(t)x, x \rangle dt \right| = M(x) < \infty \] (1.5)
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then the semigroup \( T \) is uniformly exponentially stable.

**Theorem 4.** Let \( T = \{ T(t) \}_{t \geq 0} \) be a strongly continuous and uniformly bounded bounded semigroup on a complex Hilbert space \( H \). The following five statements are equivalent:

1. The semigroup \( T \) is spectrally stable (that is, the spectrum of its infinitesimal generator \( \sigma(A) \) lies in the open left half plane \( \mathbb{C}_- := \{ z \in \mathbb{C} : \Re(z) < 0 \} \))

2. For each \( \mu \in \mathbb{R} \) and each \( x \in H \), we have that

\[
\sup_{s \geq 0} \left\| \int_0^s e^{-i\mu t} T(t)x dt \right\| = L(\mu, x) < \infty.
\]

3. For each \( \mu \in \mathbb{R} \) and each \( x \in H \), one has:

\[
\sup_{s \geq 0} \left\| \int_0^s \langle e^{-i\mu t} T(t)x, x \rangle dt \right\| = M(\mu, x) < \infty.
\]

4. \( \sigma(A) \cap i\mathbb{R} = \emptyset \).

5. For each \( \mu \in \mathbb{R} \) and each \( x \in X \) the solution of the inhomogeneous Cauchy Problem

\[
\dot{u}(t) = Au(t) + e^{i\mu t} x, \quad u(0) = 0,
\]

\((A, \mu, x)\)

is bounded.

To the best of our knowledge these results are new even if the proofs are not very difficult. The proof of the Theorem 1 is based on a technical lemma stated in the third section of our paper. In fact, if combining this Lemma with the proof of Theorem 3.3 from [13], originally given by Jan van Neerven, we obtain that the our condition (1.3) and condition (1.2) are equivalent.

If in addition to (1.3) assume that the discrete semigroup \( (T^n) \) is strongly asymptotically stable (i.e. \( T^n x \) tends to 0 when \( n \to \infty \) for any \( x \in H \)) then we can give a completely different proof for Theorem 1 using a very recent result from the operator theory, originally given by V. Müller, see [12].

If \( \phi \) is the identity map and (1.3) is fulfilled then the formal series \( \sum T^n x \) is convergent in the norm of \( H \), so in particular \( (T^n) \) is strongly asymptotically stable. This result remain true even if put \( T_n \) instead of \( T^n \), where \( T_n \) is an arbitrary bounded linear operator acting on \( H \), such as is stated in Proposition 1 from the third section of this paper.

The paper is organized as follows. Section 2 contains the results in the case of self-adjoint operators. In the Section 3 we prove Theorem 1 and consider some natural consequences, while the last section is devoted to the proof of the Theorems 2, 3 and 4.

2 The case of self-adjoint operators

In this section consider the case when the semigroups are self-adjoint and \( \phi \) is the identity map. We begin with the following well-known lemma. See for example [2] for a proof.
Lemma 1. Let \( S \) be a bounded linear operator acting on a Banach space \( X \) and \( \mu \) be a real number such that
\[
\sup_{N \in \mathbb{N}} \left\| \sum_{n=0}^{N} e^{-i\mu n} S^n \right\| = M_\mu < \infty.
\]
Then \( S \) is power bounded and \( e^{i\mu} \in \rho(T) \).

Theorem 5. Let \( T \) be a self-adjoint operator acting on a complex Hilbert space \( H \). If
\[
\sup_{||x||\leq 1} \sum_{n=0}^{\infty} ||T^n x, x|| := K < \infty
\]
then the spectral radius of \( T \) is less then 1.

Proof: For every integer number \( k \) and any positive integer number \( n \) the operator \( e^{-ikn\pi}T^n \) is self-adjoint. Then using a well-known result ([23], Theorem 3, page 201) we get:

\[
\left\| \sum_{n=0}^{N} e^{-ikn\pi}T^n \right\| = \sup_{||x||\leq 1} \left| \sum_{n=0}^{N} e^{-ikn\pi} \langle T^n x, x \rangle \right| \\
\leq \sup_{||x||\leq 1} \sum_{n=0}^{\infty} ||\langle T^n x, x || = K < \infty.
\]

Then the operator \( T \) is power bounded and its spectral radius is less than or equal 1. On the other hand the spectrum of \( T \) is real. The spectral radius of \( T \) is less than 1 because in according with Lemma 1 above, the set \( \{-1, 1\} \) belongs to \( \rho(T) \).

Let \( 1 \leq p < \infty \). In order to introduce similar results in the continuous case we recall that a strongly continuous semigroup \( T = \{ T(t) \}_{t \geq 0} \) on a Hilbert space \( H \) is called weakly-\( L^p \)-stable if for each \( x \in H \) and \( y \in H \) one has
\[
\int_{0}^{\infty} ||(T(t)x, y)||^p dt < \infty.
\]
It is known ([7], [22]) that every weakly-\( L^p \)-stable semigroup \( T \) is uniformly exponentially stable, that is, its uniform growth bound
\[
\omega_0(T) := \inf_{t>0} \frac{\ln ||T(t)||}{t}
\]
is negative. A possible new proof of this result in the case \( p = 1 \) can be stated as follows.
Lemma 2. Let $T = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on a Banach space $X$. If for each $x \in X$ one has
\[
\sup_{t \geq 0} \left\| \int_0^t T(s)x ds \right\| := M(x) < \infty \tag{2.1}
\]
then the half-plane $\{\Re(\lambda) > 0\}$ lies in $\rho(A)$. Moreover $0 \in \rho(A)$ and if
\[
M := \sup_{t \geq 0} \left\| \int_0^t T(s) ds \right\|_{L(X)}
\]
then $||R(0,A)|| \leq M$.

Proof: See [16].

We remark that if (2.1) holds with $e^{-i\mu t}T(t)$ instead of $T(t)$ for all $\mu \in \mathbb{R}$, then there exists a positive constant $L$ such that
\[
\sup_{\Re(\lambda) \geq 0} \|R(\lambda,A)\| = L < \infty.
\]
Combining this with the Gerhart-Prüss theorem (see e.g. [17], [7]) we get the following well known result ([16]):

Corollary 1. Let $T = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on a Hilbert space $H$. If for each $x \in H$ one has
\[
\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \left\| \int_0^t e^{-i\mu s}T(s)x ds \right\| := N(x) < \infty
\]
then the semigroup $T$ is uniformly exponentially stable.

Suppose now that the semigroup $T$ acts on the Hilbert space $H$ and it is weakly-$L^1$-stable. Then for all $x, y \in H$ with $\|x\| \leq 1$ and $\|y\| \leq 1$ one has:
\[
\left\| \int_0^t e^{-i\mu s}T(s)ds \right\| \leq \int_0^\infty |\langle T(s)x, y \rangle| ds \leq \text{Constant} < \infty.
\]
From the Corollary 1 above follows that the semigroup $T$ is uniformly exponentially stable.

Theorem 6. Let $T = \{T(t)\}_{t \geq 0}$ be a strongly continuous and self-adjoint operator semigroup on a Hilbert space $H$. If
\[
\sup_{\|x\| \leq 1} \int_0^\infty |\langle T(t)x, x \rangle| dt = K < \infty,
\]
then the semigroup $T$ is uniformly exponentially stable.
**Proof:** First remark that the above inequality (2.1) holds. Indeed:

\[
\sup_{t \geq 0} \left\| \int_0^t T(s) ds \right\|_{L(H)} = \sup_{t \geq 0} \sup_{|x| \leq 1} |\int_0^t T(s)x ds, x| \leq \sup_{|x| \leq 1} \int_0^\infty |\langle T(s)x, x \rangle| ds \leq K.
\]

From the above Lemma 2 follows that the spectrum of \( A \) lies in the interval \((-\infty, 0)\). On the other hand

\[
\sup_{\Re(\lambda) > 0} \| R(\lambda, A) \| \leq \sup_{\Re(\lambda) > 0} \sup_{|x| \leq 1} \int_0^\infty e^{-\Re(\lambda)t} |\langle T(t)x, x \rangle| dt \leq K.
\]

Now we can apply the Gerhart-Prüss theorem.

\[ \square \]

### 3 Proof of Theorem 1 and some natural consequences

We recall some well-known facts about Orlicz spaces. For further details we refer to [9], [10], [11], [4] and references therein. Let \( \Phi : [0, \infty) \rightarrow [0, \infty] \) be a convex, non-decreasing function such that \( \Phi(0) = \Phi(0+) = 0 \), and \( \Phi \) is not identically with 0 or with \( \infty \) on \( (0, \infty) \). Let \( \mathbb{Z}_+ \) be the set of all non-negative integers. For each scalar-valued sequence \( a = (a_\nu)_{\nu \in \mathbb{Z}_+} \) let us consider \( M^\Phi(a) := \sum_{n \in \mathbb{Z}_+} \Phi(|a_n|) \) and the set \( L^\Phi \) of all sequences \( (a_\nu) \) for which there exists a positive real number \( \lambda \) such that \( I_\Phi(\lambda a) < \infty \). The space \( L^\Phi \) can be endowed with the Luxemburg norm, given by:

\[
|\!
\!|a|_{L^\Phi} := \inf \{ \lambda > 0 : M^\Phi(\lambda^{-1}a) \leq 1 \}.
\]

The Orlicz spaces over \( \mathbb{R}_+ \) can be defined by a similar manner. Precisely in this case \( L^\Phi \) is the set of all complex valued measurable functions \( f \) defined on \( \mathbb{R}_+ \) for which there exists a positive \( \lambda \) such that \( \int_0^\infty \Phi(\lambda|f(t)|) dt < \infty \).

The Luxemburg norm of a function \( f \in L^\Phi \) is defined by

\[
\rho^\Phi(f) := \inf\{ k > 0 : \int_0^\infty \Phi(k^{-1}|f(t)|) dt \leq 1 \}.
\]

Some useful identities are collected in the next Lemma.

**Lemma 3.** Let \( H \) be a complex Hilbert space, \( x \) and \( y \) in \( H \), \( \mu \) be a real number, \( T \) be a bounded linear operator acting on \( H \) and \( \mathbf{T} = \{ T(t) \}_{t \geq 0} \) be a strongly continuous semigroup of bounded linear operators on \( H \). For each \( t \geq 0 \) let

\[
\rho_{\mu}(t) := e^{-i\mu t} T(t).
\]

The following identities are fulfilled:

\[
\langle Tx, y \rangle = \frac{1}{2i} [(1 - i)\langle (T x, x) + (T y, y) \rangle + i\langle (T(x + y), x + y) - \langle T(x + iy), x + iy) \rangle].
\]

(3.1)
and
\[
\int_0^s \langle \rho_\mu(t)x, y \rangle \, dt = \frac{1}{2i}(1 - i) \left( \int_0^s \langle \rho_\mu(t)x, x \rangle \, dt + \int_0^s \langle \rho_\mu(t)y, y \rangle \, dt \right) + \\
\frac{1}{2} \int_0^s \langle \rho_\mu(t)(x + y), x + y \rangle \, dt + \frac{1}{2i} \int_0^s \langle \rho_\mu(t)(x + iy), x + iy \rangle \, dt.
\] (3.2)

**Proof of the Theorem 1**

Using the identity (3.1) it is easily to establish the following inequality:
\[
|\langle T^nx, y \rangle| \leq \frac{\sqrt{2}}{2}(|\langle T^nx, x \rangle| + |\langle T^ny, y \rangle|) + \\
\frac{1}{2}(|\langle T^n(x + y), x + y \rangle| + |\langle T^n(x + iy), x + iy \rangle|).
\] (3.3)

In view of (1.3) follows that for each \( x \in H, \phi(\|T^n x\|) \) tends to 0 when \( n \) tends to \( \infty \) and then \( \|T^n x\| \) tends to 0 as well. As a consequence, the maps \( n \mapsto |\langle T^n z, z \rangle| \) with \( z \in \{x, y, x + y, x + iy\} \) are bounded and moreover them belong to a same Orlicz space \( E \) satisfying the condition (1.1), see the proof of Theorem 3.3 by [13]. In view of (3.3), and using the fact the every Orlicz space has the ideal property, follows that for each \( x \) and \( y \) in \( H \) the map \( n \mapsto |\langle T^n x, y \rangle| \) belongs to \( E \). Now we can apply van Neerven’s theorem, which was reminded in the beginning of our paper, in order to obtain that \( r(T) < 1 \).

In particular, from (1.3) follows that \( \|T^n x\| \) decays to 0 for any \( x \in H \). In the case when \( \phi \) is the identity map, we generalize the latter result, as follows:

**Proposition 1.** Let \( (T_n) \) be a sequence of bounded linear operators acting on a complex Hilbert space \( H \). The following two statements are equivalent:

(i) For each \( x \in H \) the series \( \sum_{n \geq 0} |\langle T_n x, x \rangle| \) is convergent.

(ii) For each \( x \in H \) and each \( y \in H \) the series \( \sum_{n \geq 0} |\langle T_n x, y \rangle| \) is convergent.

Moreover, these statements imply the fact that the (formal) series \( \sum_{n \geq 0} T_n x \) is convergent in the norm of \( H \).

**Proof:**
\[
|\langle T_n x, y \rangle| \leq \frac{\sqrt{2}}{2}(|\langle T_n x, x \rangle| + |\langle T_n y, y \rangle|) + \\
\frac{1}{2}(|\langle T_n(x + y), x + y \rangle| + |\langle T_n(x + iy), x + iy \rangle|).
\]

Now, it is clear that the statement (ii) is a consequence of (i). On the other hand by the inequality
\[
|\sum_{n \geq 0} \langle T_n x, y \rangle| \leq \sum_{n \geq 0} |\langle T_n x, y \rangle|
\]
and the statement \((\text{ii})\) follows that any subseries of the series
\[
\left(\sum_{n \geq 0} \langle T_n x, y \rangle\right)
\]
is convergent. Then the latter assertion follows by the Orlicz-Pettis theorem, see [5], page 22.

We are very grateful to Nigel Kalton for pointing out of the above result [8].

With the supplementary hypothesis that \(\|T^nx\|\) decays to 0 for any \(x \in H\) we can give another proof of Theorem 1. We use a recent result of V. Müller, [12], which reads as follows: Let \(T\) be a linear and bounded operator acting on a complex Hilbert space \(H\) such that \(1 \in \sigma(T)\) and \(\|T^nx\|\) decays to 0 for all \(x \in H\). Let \((a_n)_{n=1}^{\infty}\) be a non-increasing sequence with \(\lim_{n \to \infty} a_n = 0\) and \(\sup a_n < 1\).

Then there exists \(x \in H\) of norm one such that \(\text{Re} \langle T^n x, x \rangle > a_n\) for all \(n \geq 1\). In such circumstances the operator \(T\) is power bounded hence its spectral radius \(r(T)\) is less than or equal one. Suppose for the contrary that \(r(T) = 1\). Then there exists \(\mu \in \mathbb{R}\) such that \(e^{i\mu} \in \sigma(T)\). In order to apply the above result we may suppose that \(1 \in \sigma(T)\). Indeed, if it is not true, put \(S := e^{-i\mu}T\) instead of \(T\).

It is clear that \(\phi(0) = 0\). We may suppose that \(\phi(1) = 1\) and that \(\phi\) is a strictly increasing and continuous function on \(\mathbb{R}_+\). If not, put a multiple of \(\tilde{\phi}\) instead of \(\phi\), where

\[
\tilde{\phi}(t) := \int_0^t \phi(s) ds \quad \text{if } 0 \leq t \leq 1 \quad \text{and} \quad \tilde{\phi}(t) := \frac{at}{at + 1 - a} \quad \text{if } t > 1.
\]

Here \(a := \int_0^1 \phi(s) ds\). Let us consider \(a_n := \phi^{-1}\left(\frac{1}{n+1}\right)\). Then as stated in the above Müller result, there exists a \(x_0 \in H\), of norm one, such that

\[
\sum_{n=1}^{\infty} \phi(|\langle T^n x_0, x_0 \rangle|) \geq \sum_{n=1}^{\infty} \phi(a_n) = \infty
\]

which is a contradiction.

We can complete the result from Theorem 1 in the following way:

**Proposition 2.** Let \(T\) be a bounded linear operator acting on a complex Hilbert space \(H\). The following two statements are equivalent:

\((i)\) There exists \(\varepsilon > 0\) such that for all positive integer \(n\) there is a norm one vector \(x \in X\) such that

\[
\text{card} \ \{k = 0, 1, \ldots : |\langle T^k x, x \rangle| \geq \varepsilon \} \geq n.
\]
(ii) For every non-decreasing function $\phi$ on $\mathbb{R}_+$, with $\phi(t) > 0$ for all $t > 0$, there exists a norm one vector $x \in X$ such that

$$\sum_{k=1}^{\infty} \phi(|\langle T^k x, x \rangle|) = \infty.$$ \hfill ($\ast$)

**Proof:** ($i$) $\Rightarrow$ ($ii$) Follows using the inclusion:

$$\{ k \in \mathbb{N} : |\langle T^k x, x \rangle| \geq \varepsilon \} \subset \{ j \in \mathbb{N} : \phi(|\langle T^j x, x \rangle|) \geq \phi(\varepsilon) \}.$$ 

($ii$) $\Rightarrow$ ($i$) Assume the contrary. For each $n = 1, 2, \cdots$ let us consider

$$w_n := \sup_{||x||=1} \text{card} \{ k \in \mathbb{N} : |\langle T^k x, x \rangle| \geq \frac{1}{2^n} \}.$$ 

The function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\phi(0) = 0$ and

$$\phi(t) = \frac{1}{w_1} 1_{[\frac{1}{2}, \infty)}(t) + \sum_{n=1}^{\infty} \frac{1}{2^{n+1} w_n} 1_{[\frac{1}{2^n}, \frac{1}{2^{n+1}})}(t)$$

for $t > 0$ is non-decreasing, $\phi(t) > 0$ for each $t > 0$, and for each norm one vector $x \in H$, one has:

$$\sum_{n=1}^{\infty} \phi(|\langle T^n x, x \rangle|) \leq 1.$$ 

This is a contradiction. \hfill $\Box$

It is clear that if $r(T) \geq 1$ then the above statements ($i$) and ($ii$) are fulfilled.

In particular, follows that if $T$ is an isometry on a complex Hilbert space $H$ and $\phi : [0, \infty) \to [0, \infty)$ is a non-decreasing function with $\phi(t) > 0$ for all $t > 0$ then there exists a norm one $x \in H$ such that:

$$\sum_{n=0}^{\infty} \phi(|\langle T^n x, x \rangle|) = \infty.$$ 

We can compare this result with a similar one given by Jan van Neerven in [14] which states that if $T$ is a non-unitary isometry on a real or complex Hilbert space $H$ then for all $\varepsilon > 0$ and all $\alpha \in c_0$ of norm one, there exists a norm one vector $x$ such that

$$|\langle T^n x, x \rangle| \geq (1 - \varepsilon)|\alpha_n| \quad \forall n \in \mathbb{N}.$$ 

This result do not holds for unitary isometry, (cf. [14], Example 2.4).
4 Proofs of the Theorems 2, 3 and 4

**Proof of the Theorem 2.** Applying (3.1) with $T(t)$ instead of $T$ we get the following inequality.

$$|\langle T(t)x, y \rangle| \leq \frac{\sqrt{2}}{2} (|\langle T(t)x, x \rangle| + |\langle T(t)y, y \rangle|) +$$

$$+ \frac{1}{2} (|\langle T(t)(x + y), x + y \rangle| + |\langle T(t)(x + iy), x + iy \rangle|).$$

In view of (1.3), and using the Lemma 3.2 from [13], it follows that the maps $t \mapsto |\langle T(t)z, z \rangle|$ with $z \in \{x, y, x + y, x + iy\}$ belong to the same Orlicz space $E$ over $\mathbb{R}^+$ which satisfies the condition $\lim_{t \to \infty} \|1_{[0,t]}\|_E = \infty$. Then the map $t \mapsto |\langle T(t)x, y \rangle|$ belongs to $E$ and the desired assertion follows immediately. See also [15], Theorem 4.6.3 (ii).

**Remarks:**

1. We leave open the question whether the uniform boundedness condition on the semigroup can be dropped.
2. Under the different assumptions on the function $\phi$ the uniform boundedness condition on the semigroup can be dropped, see [3].
3. If the semigroup $T$ acts on the finite dimensional space $\mathbb{C}^n$ then the uniform boundedness condition can be dropped.

**Proof of the Theorem 3.**

In view of (3.2) and (1.4) we get:

$$\left| \int_0^s \langle \rho_\mu(t)x, y \rangle dt \right| \leq \frac{\sqrt{2}}{2} \left[ \int_0^s \langle \rho_\mu(t)x, x \rangle dt \right] + \left[ \int_0^s \langle \rho_\mu(t)y, y \rangle dt \right] +$$

$$+ \frac{1}{2} \left[ \int_0^s \langle \rho_\mu(t)(x + y), x + y \rangle dt \right] + \left[ \int_0^s \langle \rho_\mu(t)(x + iy), x + iy \rangle dt \right]$$

$$\leq M||x|| ||y|| < \infty.$$

Now we obtain

$$\sup_{\mu \in \mathbb{R}} \sup_{s \geq 0} \left\| \int_0^s e^{-i\mu t}T(s)x ds \right\| \leq \sup_{||y|| \leq 1} \sup_{s \geq 0} \left| \int_0^s \langle e^{-i\mu t}T(t)x, y \rangle dt \right|$$

$$\leq M||x|| < \infty$$

and we can apply Corollary 1 to end the proof.

**Proof of Theorem 4.** The implication 1. $\Rightarrow$ 2. was stated in [19] without the assumption of uniform boundedness on $T$. A proof of 2. $\Rightarrow$ 1. can be found in [1]. In fact by the identity

$$\int_0^s e^{-i\mu t}T(t)(A - i\mu I)(A - i\mu I)^{-1}x = T(s)(A - i\mu I)^{-1}x - (A - i\mu I)^{-1}x$$
follows that
\[
\sup_{s \geq 0} \left\| \int_0^s e^{-\mu t} T(t)x \, dt \right\| \leq \|(A - i\mu I)^{-1}\|(1 + \sup_{s \geq 0} \|T(s)\|)\|x\|.
\]

It is clear that the second statement implies the third one. On the other hand from (3.2) follows that for each \( \mu \in \mathbb{R} \) and each \( x, y \in H \) have that
\[
\sup_{s \geq 0} \left| \int_0^s e^{-i\mu t} T(t)x \, dt, y \right| = N(\mu, x, y) < \infty,
\]
and then (1.6) is fulfilled. The equivalences between 1. and 4. and between 2. and 5. are obvious.

The semigroup \( T = \{T(t)\}_{t \geq 0} \), or its infinitesimal generator \( A \), is called \textit{strongly stable} if \( \lim_{t \to \infty} T(t)x = 0 \) for every \( x \in X \). The \textit{point spectrum} of \( A \), denoted by \( \sigma_p(A) \), is the set of all complex scalars \( \lambda \) for which there exists a non-zero vector \( x \) such that \( Ax = \lambda x \), while that the \textit{residual spectrum} of \( A \), denoted by \( \sigma_r(A) \), is the set of all scalar \( \lambda \in \sigma(A) \) such that the range of \( (\lambda I - A) \) is not dense in \( X \). As consequence of the Hahn-Banach theorem \( \sigma_p(A^*) = \sigma_r(A) \). In particular, the punctual spectrum of \( A^* \) is a subset of \( \sigma(A) \). We recall here the very famous and well-known stability theorem of Arendt-Batty-Lyubich-Vu.

**Theorem 7.** Let \( T = \{T(t)\}_{t \geq 0} \) be a strongly continuous and uniformly bounded semigroup on a Banach space \( X \) and let \( A \) its infinitesimal generator. If

(i) \( \sigma(A) \cap (i\mathbb{R}) \) is a countable set and

(ii) \( \sigma_p(A^*) \cap (i\mathbb{R}) = \emptyset \),

then the semigroup \( T \) is strongly stable.

Combining this theorem with the Theorem 4 above, shall obtain:

**Corollary 2.** Let \( T = \{T(t)\}_{t \geq 0} \) be a strongly continuous uniformly bounded semigroup on a complex space \( H \) and let \( A \) its infinitesimal generator. If \( \sigma(A) \cap (i\mathbb{R}) = \emptyset \) or if for each \( x \in H \) and each \( \mu \in \mathbb{R} \) one has:
\[
\sup_{s \geq 0} \left| \int_0^s \langle e^{-i\mu t} T(t)x, x \rangle \, dt \right| = M(\mu, x) < \infty,
\]
then the semigroup \( T \) is strongly stable.

**Acknowledgement**

The authors are very grateful to Nigel Kalton (University of Missouri) and Constantin P. Niculescu (University of Craiova) for stimulating discussions on the topic of this article.
References


Asymptotic behavior of semigroups on Hilbert spaces


Received: 28.02.2007.

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