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A class of transversal polymatroids with Gorenstein base ring

by
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Abstract

In this paper, the principal tool to describe transversal polymatroids with Gorenstein base ring is polyhedral geometry, especially the Danilov-Stanley theorem for the characterization of canonical module. Also, we compute the a-invariant and the Hilbert series of base ring associated to this class of transversal polymatroids.

Key Words: Base ring, transversal polymatroid, equations of a cone, a-invariant, canonical module, Hilbert series.
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1 Introduction

In this paper we determine the facets of the polyhedral cone generated by the exponent set of the monomials defining the base ring associated to a transversal polymatroid. The importance of knowing those facets comes from the fact that the canonical module of the base ring can be expressed in terms of the relative interior of the cone. This would allow one to compute the a-invariant of those base rings. The results presented were discovered by extensive computer algebra experiments performed with *Normaliz* [4].

2 Preliminaries

Let $n \in \mathbb{N}$, $n \geq 3$, $\sigma \in S_n$, $\sigma = (1, 2, \ldots, n)$ the cycle of length n, $[n] := \{1, 2, \ldots, n\}$ and $\{e_i\}_{1 \leq i \leq n}$ be the canonical base of $\mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, we will denote $\mid x \mid := x_1 + \ldots + x_n$. If x^a is a monomial in $K[x_1, \ldots, x_n]$ we set $\log(x^a) = a$. Given a set A of monomials, the *log set of* A, denoted $\log(A)$, consists of all $\log(x^a)$ with $x^a \in A$.

We consider the following set of integer vectors of $\mathbb{N}^n$:

$$\downarrow i^{th} column$$

$$\nu_{\sigma^0[i]} := \Big(\ -(n-i-1), -(n-i-1), \ldots, -(n-i-1), (i+1), \ldots, (i+1)\ \Big),$$

$$\downarrow (2)^{nd} column \qquad \qquad \downarrow (i+1)^{st} column$$

$$\nu_{\sigma^1[i]} := \Big(\ (i+1), -(n-i-1), \ldots, -(n-i-1), (i+1), \ldots, (i+1)\ \Big),$$

$$\downarrow (3)^{rd} column \qquad \qquad \downarrow (i+2)^{nd} column$$

$$\nu_{\sigma^2[i]} := \Big(\ (i+1), (i+1), -(n-i-1), \ldots, -(n-i-1), (i+1), \ldots, (i+1)\ \Big),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\downarrow (i-2)^{nd} column \quad \downarrow (n-2)^{nd} column$$

$$\nu_{\sigma^{n-2}[i]} := \Big(\ \ldots, -(n-i-1), (i+1), \ldots, (i+1), -(n-i-1), -(n-i-1)\ \Big),$$

$$\downarrow (i-1)^{st} column \qquad \downarrow (n-1)^{st} column$$

$$\nu_{\sigma^{n-1}[i]} := \Big(\ -(n-i-1), \ldots, -(n-i-1), (i+1), \ldots, (i+1), -(n-i-1)\ \Big),$$

where $\sigma^k[i] := \{\sigma^k(1), \ldots, \sigma^k(i)\}$ for all $1 \leq i \leq n-1$ and $0 \leq k \leq n-1$.

Remark: $\nu_{\sigma^k[n-1]} = n\ e_{[n]\setminus\sigma^k[n-1]}$ for all $0 \leq k \leq n$.
For example, if $n = 4$, $\sigma = (1,2,3,4) \in S_4$ then we have the following set of integer vectors:

$$\begin{aligned}
\nu_{\sigma^0[1]} &= \nu_{\{1\}} = (-2,2,2,2), \quad & \nu_{\sigma^0[2]} &= \nu_{\{1,2\}} = (-1,-1,3,3),\\
\nu_{\sigma^1[1]} &= \nu_{\{2\}} = (2,-2,2,2), \quad & \nu_{\sigma^1[2]} &= \nu_{\{2,3\}} = (3,-1,-1,3),\\
\nu_{\sigma^2[1]} &= \nu_{\{3\}} = (2,2,-2,2), \quad & \nu_{\sigma^2[2]} &= \nu_{\{3,4\}} = (3,3,-1,-1),\\
\nu_{\sigma^3[1]} &= \nu_{\{4\}} = (2,2,2,-2), \quad & \nu_{\sigma^3[2]} &= \nu_{\{1,4\}} = (-1,3,3,-1).
\end{aligned}$$

$$\begin{aligned}
\nu_{\sigma^0[3]} &= \nu_{\{1,2,3\}} = (0,0,0,4),\\
\nu_{\sigma^1[3]} &= \nu_{\{2,3,4\}} = (4,0,0,0),\\
\nu_{\sigma^2[3]} &= \nu_{\{1,3,4\}} = (0,4,0,0),\\
\nu_{\sigma^3[3]} &= \nu_{\{1,2,4\}} = (0,0,4,0).
\end{aligned}$$

If $0 \neq a \in \mathbb{R}^n$, then H_a will denote the hyperplane of $\mathbb{R}^n$ through the origin with normal vector a, that is,

$$H_a = \{x \in \mathbb{R}^n \mid \langle x, a\rangle = 0\},$$

where $\langle , \rangle$ is the usual inner product in $\mathbb{R}^n$. The two closed halfspaces bounded by H_a are:

$$H_a^+ = \{x \in \mathbb{R}^n \mid \langle x, a\rangle \geq 0\} \text{ and } H_a^- = \{x \in \mathbb{R}^n \mid \langle x, a\rangle \leq 0\}.$$

We will denote by $H_{\sigma^k[i]}$ the hyperplane of $\mathbb{R}^n$ through the origin with normal vector $\nu_{\sigma^k[i]}$, that is,

$$H_{\nu_{\sigma^k[i]}} = \{x \in \mathbb{R}^n \mid \langle x, \nu_{\sigma^k[i]}\rangle = 0\},$$

for all $1 \leq i \leq n-1$ and $0 \leq k \leq n-1$.

An *affine space* in $\mathbb{R}^n$ is the translate of a linear subspace of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$, we denote by *aff*(A) the affine space generated by A. There is a unique linear subspace V of $\mathbb{R}^n$ such that *aff*$(A) = x_0 + V$, for some $x_0 \in \mathbb{R}^n$. The dimension of *aff*(A) is $dim(aff(A)) = dim_{\mathbb{R}}(V)$.

Recall that a *polyhedral cone* $Q \subset \mathbb{R}^n$ is the intersection of a finite number of closed subspaces of the form H_a^+. If $Q = H_{a_1}^+ \cap \ldots \cap H_{a_m}^+$ is a polyhedral cone, then *aff*(Q) is the intersection of those hyperplanes H_{a_i} , $i = 1, \ldots, m$, that contain Q.(see [3, Proposition 1.2.]) The *dimension of* Q is the dimension of *aff*(Q), $dim(Q) = dim(aff(Q))$.
If $A = \{\gamma_1, \ldots, \gamma_r\}$ is a finite set of points in $\mathbb{R}^n$ the *cone* generated by A, denoted by $\mathbb{R}_+A$, is defined as

$$\mathbb{R}_+A = \{\sum_{i=1}^{r} a_i\gamma_i \mid a_i \in \mathbb{R}_+ \text{ for all } 1 \leq i \leq n\}.$$

An important fact is that Q is a polyhedral cone in $\mathbb{R}^n$ if and only if there exists a finite set $A \subset \mathbb{R}^n$ such that $Q = \mathbb{R}_+A$ (see [3] or [10, Theorem 4.1.1.]).

Next we give some important definitions and results (see [1], [2], [3], [8], [9]).

Definition 2.1. *A* proper face *of a polyhedral cone is a subset* $F \subset Q$ *such that there is a supporting hyperplane* H_a *satisfying:*

1) $F = Q \cap H_a \neq \emptyset$,
2) $Q \nsubseteq H_a$ *and* $Q \subset H_a^+$.

The *dimension* of a proper face F of a polyhedral cone Q is: $dim(F) = dim(aff(F))$.

Definition 2.2. *A cone* C *is* pointed *if* 0 *is a face of* C. *Equivalently we can require that* $x \in C$ *and* $-x \in C \Rightarrow x = 0$.

Definition 2.3. *The 1-dimensional faces of a pointed cone are called extremal rays.*

Definition 2.4. *A proper face F of a polyhedral cone $Q \subset \mathbb{R}^n$ is called a facet of Q if* $\dim(F) = \dim(Q) - 1$.

Definition 2.5. *If a polyhedral cone Q is written as*

$$Q = H^+_{a_1} \cap \ldots \cap H^+_{a_r}$$

such that no $H^+_{a_i}$ can be omitted, then we say that this is an irreducible representation *of Q.*

Theorem 2.6. *Let $Q \subset \mathbb{R}^n$, $Q \neq \mathbb{R}^n$, be a polyhedral cone with $dim(Q) = n$. Then the halfspaces $H^+_{a_1}, \ldots, H^+_{a_m}$ in an irreducible representation $Q = H^+_{a_1} \cap \ldots \cap H^+_{a_m}$ are uniquely determined. In fact, the sets $F_i = Q \cap H_{a_i}$, $i = 1, \ldots, n$, are the facets of Q.*

Proof: See [3, Theorem 1.6.] □

Definition 2.7. *Let Q be a polyhedral cone in $\mathbb{R}^n$ with* $\dim\ Q = n$ *and such that $Q \neq \mathbb{R}^n$. Let*

$$Q = H^+_{a_1} \cap \ldots \cap H^+_{a_r}$$

be the irreducible representation of Q. If $a_i = (a_{i1}, \ldots, a_{in})$, then we call

$$H_{a_i}(x) := a_{i1}x_1 + \ldots + a_{in}x_n = 0, \quad i \in [r],$$

the equations of the cone Q.

Definition 2.8. *The* relative interior *$ri(Q)$ of a polyhedral cone is the interior of Q with respect to the embedding of Q into its affine space* $\mathrm{aff}(Q)$, *in which Q is full-dimensional.*

The following result gives us the description of the relative interior of a polyhedral cone when we know its irreducible representation.

Theorem 2.9. *Let $Q \subset \mathbb{R}^n$, $Q \neq \mathbb{R}^n$, be a polyhedral cone with* $\dim(Q) = n$ *and let*

$$(*)\ Q = H^+_{a_1} \cap \ldots \cap H^+_{a_m}$$

be an irreducible representation of Q with $H^+_{a_1}, \ldots, H^+_{a_n}$ pairwise distinct, where $a_i \in \mathbb{R}^n \setminus \{0\}$ for all i. Set $F_i = Q \cap H_{a_i}$ for $i \in [r]$. Then:
a) $ri(Q) = \{x \in \mathbb{R}^n \mid \langle x, a_1 \rangle > 0, \ldots, \langle x, a_r \rangle > 0\}$, where $ri(Q)$ is the relative interior of Q, which in this case is just the interior.
b) Each facet F of Q is of the form $F = F_i$ for some i.
c) Each F_i is a facet of Q.

Proof: See [1, Theorems 8.2.15 and 3.2.1]. □

Theorem 2.10. (Danilov, Stanley) *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and F a finite set of monomials in R. If $K[F]$ is normal, then the canonical module $\omega_{K[F]}$ of $K[F]$, with respect to standard grading, can be expressed as an ideal of $K[F]$ generated by monomials*

$$\omega_{K[F]} = (\{x^a \mid a \in \mathbb{N}A \cap ri(\mathbb{R}_+A)\}),$$

where $A = \log(F)$ and $ri(\mathbb{R}_+A)$ denotes the relative interior of $\mathbb{R}_+A$.

The formula above represents the canonical module of $K[F]$ as an ideal of $K[F]$ generated by monomials. For a comprehensive treatment of the Danilov-Stanley formula see [2], [8] or [9] .

3 Polymatroids

Let K be an infinite field, n and m be positive integers, $[n] = \{1, 2, \ldots, n\}$. A nonempty finite set B of $\mathbb{N}^n$ is the *base set of a discrete polymatroid* $\mathcal{P}$ if for every $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n) \in B$ one has $u_1 + u_2 + \ldots + u_n = v_1 + v_2 + \ldots + v_n$ and for all i such that $u_i > v_i$ there exists j such that $u_j < v_j$ and $u + e_j - e_i \in B$, where e_k denotes the k^{th} vector of the standard basis of $\mathbb{N}^n$. The notion of discrete polymatroid is a generalization of the classical notion of matroid, see [5], [6], [7], [11]. Associated with the base B of a discrete polymatroid $\mathcal{P}$ one has a $K-$algebra $K[B]$, called the *base ring* of $\mathcal{P}$, defined to be the $K-$subalgebra of the polynomial ring in n indeterminates $K[x_1, x_2, \ldots, x_n]$ generated by the monomials x^u with $u \in B$. From [7] the algebra $K[B]$ is known to be normal and hence Cohen-Macaulay.

If A_i are some nonempty subsets of $[n]$ for $1 \leq i \leq m$, $\mathcal{A} = \{A_1, \ldots, A_m\}$, then the set of the vectors $\sum_{k=1}^{m} e_{i_k}$ with $i_k \in A_k$ is the base of a polymatroid, called the *transversal polymatroid presented* by $\mathcal{A}$. The *base ring* of a transversal polymatroid presented by $\mathcal{A}$ is the ring

$$K[\mathcal{A}] := K[x_{i_1} \cdots x_{i_m} \mid i_j \in A_j, 1 \leq j \leq m].$$

4 Cones of dimension n with $n+1$ facets

Lemma 4.1. *Let $1 \leq i \leq n-2$, $A := \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_k,\ for\ all\ 1 \leq k \leq n\} \subset \mathbb{N}^n$ the exponent set of generators of the $K-$algebra $K[\mathcal{A}]$, where $\mathcal{A} = \{A_1 = [n], \ldots, A_i = [n], A_{i+1} = [n] \setminus [i], \ldots, A_{n-1} = [n] \setminus [i], A_n = [n]\}$. Then the cone generated by A has the irreducible representation*

$$\mathbb{R}_+A = \bigcap_{a \in N} H_a^+,$$

where $N = \{\nu_{\sigma^0[i]}, \nu_{\sigma^k[n-1]} \mid 0 \leq k \leq n-1\}$.

Proof: We denote $J_k = \begin{cases} (i+1)\ e_k + (n-i-1)\ e_{i+1}\ , & \text{if } 1 \leq k \leq i \\ (i+1)\ e_1 + (n-i-1)\ e_k\ , & \text{if } i+2 \leq k \leq n \end{cases}$ and $J = n\ e_n$. Since $A_t = [n]$ for any $t \in \{1, \ldots, i\} \cup \{n\}$ and $A_r = [n] \setminus [i]$ for any $r \in \{i+1, \ldots, n-1\}$, it is easy to see that for any $k \in \{1, \ldots, i\}$ and $r \in \{i+2, \ldots, n\}$ the set of monomials $x_k^{i+1}\ x_{i+1}^{n-i-1},\ x_1^{i+1}\ x_r^{n-i-1},\ x_n^n$ is a subset of the generators of the $K-$algebra $K[\mathcal{A}]$. Thus one has

$$\{J_1, \ldots, J_i, J_{i+2}, \ldots, J_n, J\} \subset A.$$

If we denote by C the $n \times n-$matrix whose rows are the entries of the vectors $J_1, \ldots, J_i, J_{i+2}, \ldots, J_n, J$, then by a simple computation we get $|\det(C)| = n\ (i+1)^i\ (n-i-1)^{n-i-1}$. Therefore the set

$$\{J_1, \ldots, J_i, J_{i+2}, \ldots, J_n, J\}$$

is linearly independent and it follows that $\dim \mathbb{R}_+ A = n$.
Since $\{J_1, \ldots, J_i, J_{i+2}, \ldots, J_n\}$ is linearly independent and lies on the hyperplane $H_{\sigma^0[i]}$, we have that $\dim(H_{\sigma^0[i]} \cap \mathbb{R}_+ A) = n-1$.

Now we will prove that $\mathbb{R}_+ A \subset H_a^+$ for all $a \in N$. It is enough to show that for all vectors $P \in A$, $\langle P, a \rangle \geq 0$ for all $a \in N$. Since $\nu_{\sigma^k[n-1]} = n\ e_{[n]\setminus\sigma^k[n-1]}$, where $\{e_i\}_{1 \leq i \leq n}$ is the canonical base of $\mathbb{R}^n$, we get that $\langle P, \nu_{\sigma^k[n-1]} \rangle \geq 0$. Let $P \in A$, $P = \log(x_{j_1} \cdots x_{j_i} x_{j_{i+1}} \cdots x_{j_{n-1}} x_{j_n})$ and let t be the number of j_{k_s} such that $1 \leq k_s \leq i$ and $j_{k_s} \in [i]$. Thus $1 \leq t \leq i$. Now we have only two cases to consider:
1) If $j_n \in [i]$, then

$$\langle P, \nu_{\sigma^0[i]} \rangle = -t(n-i-1)+(i-t)(i+1)+(n-i-1)(i+1)-(n-i-1) = n(i-t) \geq 0.$$

2) If $j_n \in [n] \setminus [i]$, then

$$\langle P, \nu_{\sigma^0[i]} \rangle = -t(n-i-1)+(i-t)(i+1)+(n-i-1)(i+1)+(i+1) = n(i-t+1) > 0.$$

Thus

$$\mathbb{R}_+ A \subseteq \bigcap_{a \in N} H_a^+.$$

To prove the converse inclusion, it is clearly enough to prove that all the extremal rays of the cone $\bigcap_{a \in N} H_a^+$ are contained in $\mathbb{R}_+ A$. Any extremal ray of the cone $\bigcap_{a \in N} H_a^+$ can be written as the intersection of $n-1$ hyperplanes H_a, with $a \in N$. There are two possibilities to obtain extremal rays by intersection of $n-1$ hyperplanes.
First case.
Let $1 \leq i_1 < \ldots < i_{n-1} \leq n$ be a sequence and $\{t\} = [n] \setminus \{i_1, \ldots, i_{n-1}\}$. The system of equations $(*)$ $\begin{cases} z_{i_1} = 0, \\ \vdots \\ z_{i_{n-1}} = 0 \end{cases}$ admits the solution $x \in \mathbb{Z}_+^n,\ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

with $\mid x \mid = n$ and $x_k = n \cdot \delta_{kt}$ for all $1 \leq k \leq n$, where δ_{kt} is the Kronecker symbol.
There are two possibilities:
1) If $1 \leq t \leq i$, then $H_{\sigma^0[i]}(x) < 0$ and therefore $x \notin \bigcap_{a \in N} H_a^+$.
2) If $i+1 \leq t \leq n$, then $H_{\sigma^0[i]}(x) > 0$ and thus $x \in \bigcap_{a \in N} H_a^+$ and is an extremal ray.
Hence, there exist $n-i$ sequences $1 \leq i_1 < \ldots < i_{n-1} \leq n$ such that the system of equations $(*)$ has a solution $x \in \mathbb{Z}_+^n$ with $\mid x \mid = n$ and $H_{\sigma^0[i]}(x) > 0$. The extremal rays are: $\{ne_k \mid i+1 \leq k \leq n\}$.
Second case.
Let $1 \leq i_1 < \ldots < i_{n-2} \leq n$ be a sequence and $\{j, \ k\} = [n] \setminus \{i_1, \ldots, i_{n-2}\}$, with $j < k$, and

$$(**) \quad \begin{cases} z_{i_1} = 0, \\ \vdots \\ z_{i_{n-2}} = 0, \\ -(n-i-1)z_1 - \ldots - (n-i-1)z_i + (i+1)z_{i+1} + \ldots + (i+1)z_n = 0 \end{cases}$$

be the system of linear equations associated to this sequence.
There are two possibilities:
1) If $1 \leq j \leq i$ and $i+1 \leq k \leq n$, then the system of equations $(**)$ admits the solution $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{Z}_+^n$, with $\mid x \mid = n$ and $x_t = (i+1)\delta_{jt} + (n-i-1)\delta_{kt}$ for all $1 \leq t \leq n$.
2) If $1 \leq j, k \leq i$ or $i+1 \leq j, k \leq n$, then there exist no solutions $x \in \mathbb{Z}_+^n$ with $\mid x \mid = n$ for the system of equations $(**)$ because otherwise $H_{\sigma^0[i]}(x) > 0$ or $H_{\sigma^0[i]}(x) < 0$.
Thus, there exist $i(n-i)$ sequences $1 \leq i_1 < \ldots < i_{n-2} \leq n$ such that the system of equations $(**)$ has a solution $x \in \mathbb{Z}_+^n$ with $\mid x \mid = n$ and the extremal rays are: $\{(i+1)e_j + (n-i-1)e_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\}$.

In conclusion, there exist $(i+1)(n-i)$ extremal rays of the cone $\bigcap_{a \in N} H_a^+$:

$$R := \{ne_k \mid i+1 \leq k \leq n\} \cup \{(i+1)e_j + (n-i-1)e_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\}.$$

Since $R \subset A$ we have $\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+$.

It is easy to see that the representation is irreducible because if we delete, for some k, the hyperplane with the normal $\nu_{\sigma^k[n-1]}$, then a coordinate of a $\log(x_{j_1} \cdots x_{j_i} x_{j_{i+1}} \cdots x_{j_{n-1}} x_{j_n})$ would be negative, which is impossible; and if we delete the hyperplane with the normal $\nu_{\sigma^0[i]}$, then the cone $\mathbb{R}_+ A$ would be generated by $A = \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in [n], \text{ for all } 1 \leq k \leq n\}$, which again is impossible. Thus the representation $\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+$ is irreducible. □

Lemma 4.2. *Let $1 \leq i \leq n-2$, $1 \leq t \leq n-1$, $A := \{\log(x_{j_1} \cdots x_{j_n}) \mid j_{\sigma^t(k)} \in A_{\sigma^t(k)}, 1 \leq k \leq n\} \subset \mathbb{N}^n$ the exponent set of generators of $K-$algebra $K[\mathcal{A}]$, where*

$\mathcal{A} = \{A_{\sigma^t(k)} \mid A_{\sigma^t(k)} = [n],\ for\ 1 \leq k \leq i\ and\ A_{\sigma^t(k)} = [n] \setminus \sigma^t[i],\ for\ i+1 \leq k \leq n-1,\ A_{\sigma^t(n)} = [n]\}$. Then the cone generated by A has the irreducible representation

$$\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+,$$

where $N = \{\nu_{\sigma^t[i]},\ \nu_{\sigma^k[n-1]} \mid 0 \leq k \leq n-1\}$.

Proof: The proof goes as that for Lemma 4.1. □

5 The a-invariant and the canonical module

Lemma 5.1. *The $K-$algebra $K[\mathcal{A}]$, where $\mathcal{A} = \{A_{\sigma^t(k)} \mid A_{\sigma^t(k)} = [n],\ for\ 1 \leq k \leq i,\ and\ A_{\sigma^t(k)} = [n] \setminus \sigma^t[i],\ for\ i+1 \leq k \leq n-1,\ A_{\sigma^t(n)} = [n]\}$, is a Gorenstein ring for all $0 \leq t \leq n-1$ and $1 \leq i \leq n-2$.*

Proof: Since the algebras from Lemmas 4.1 and 4.2 are isomorphic, it is enough to prove the case $t = 0$.

We will show that the canonical module $\omega_{K[\mathcal{A}]}$ is generated by $(x_1 \cdots x_n)K[\mathcal{A}]$. Since $K-$ algebra $K[\mathcal{A}]$ is normal, using the Danilov-Stanley theorem we get that the canonical module $\omega_{K[\mathcal{A}]}$ is

$$\omega_{K[\mathcal{A}]} = \{x^\alpha \mid \alpha \in \mathbb{N}A \cap ri(\mathbb{R}_+ A)\}.$$

Let $d = \gcd(n, i+1)$ be the greatest common divisor of n and $i+1$, then the equation of the facet $H_{\nu_{\sigma^0[i]}}$ is

$$H_{\nu_{\sigma^0[i]}}:\ -\frac{(n-i-1)}{d} \sum_{k=1}^{i} x_k + \frac{(i+1)}{d} \sum_{k=i+1}^{n} x_k = 0.$$

The relative interior of the cone $\mathbb{R}_+ A$ is

$$ri(\mathbb{R}_+ A) = \{x \in \mathbb{R}^n \mid x_k > 0,\ \forall\, k \in [n],\ -\frac{(n-i-1)}{d} \sum_{k=1}^{i} x_k + \frac{(i+1)}{d} \sum_{k=i+1}^{n} x_k > 0\}.$$

We will show that $\mathbb{N}A \cap ri(\mathbb{R}_+ A) = (1, \ldots, 1) + (\mathbb{N}A \cap \mathbb{R}_+ A)$.

It is clear that $ri(\mathbb{R}_+ A) \supset (1, \ldots, 1) + \mathbb{R}_+ A$. If $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}A \cap ri(\mathbb{R}_+ A)$, then $\alpha_k \geq 1$ for all $1 \leq k \leq n$ and

$$-\frac{(n-i-1)}{d} \sum_{k=1}^{i} \alpha_k + \frac{(i+1)}{d} \sum_{k=i+1}^{n} \alpha_k \geq 1 \text{ and } \sum_{k=1}^{n} \alpha_k = t\ n \text{ for some } t \geq 1.$$

We claim that there exists $(\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}A \cap \mathbb{R}_+ A$ such that $(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1 + 1, \beta_2 + 1, \ldots, \beta_n + 1)$. Let $\beta_k = \alpha_k - 1$ for all $1 \leq k \leq n$. It is clear that $\beta_k \geq 0$ and

$$-\frac{(n-i-1)}{d}\sum_{k=1}^{i}\beta_k+\frac{(i+1)}{d}\sum_{k=i+1}^{n}\beta_k = -\frac{(n-i-1)}{d}\sum_{k=1}^{i}\alpha_k+\frac{(i+1)}{d}\sum_{k=i+1}^{n}\alpha_k-\frac{n}{d}.$$

If

$$-\frac{(n-i-1)}{d}\sum_{k=1}^{i}\alpha_k + \frac{(i+1)}{d}\sum_{k=i+1}^{n}\alpha_k = j \text{ with } 1 \leq j \leq \frac{n}{d} - 1,$$

then we will get a contradiction. Indeed, since n divides $\sum_{k=1}^{n} \alpha_k$, it follows that $\frac{n}{d}$ divides j, which is false. So we have

$$-\frac{(n-i-1)}{d}\sum_{k=1}^{i}\beta_k + \frac{(i+1)}{d}\sum_{k=i+1}^{n}\beta_k =$$

$$-\frac{(n-i-1)}{d}\sum_{k=1}^{i}\alpha_k + \frac{(i+1)}{d}\sum_{k=i+1}^{n}\alpha_k - \frac{n}{d} \geq 0.$$

Thus $(\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}A \cap \mathbb{R}_+ A$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}A \cap ri(\mathbb{R}_+ A)$.
Since $\mathbb{N}A \cap ri(\mathbb{R}_+ A) = (1, \ldots, 1) + (\mathbb{N}A \cap \mathbb{R}_+ A)$, we get that $\omega_{K[\mathcal{A}]} = (x_1 \cdots x_n)K[\mathcal{A}]$. ∎

Let S be a standard graded $K-$algebra over a field K. Recall that the *$a-$invariant* of S, denoted $a(S)$, is the degree as a rational function of the Hilbert series of S, see for instance ([9, p. 99]). If S is Cohen-Macaulay and ω_S is the canonical module of S, then

$$a(S) = -\min\{i \mid (\omega_S)_i \neq 0\},$$

see [2, p. 141] and [9, Proposition 4.2.3]. In our situation $S = K[\mathcal{A}]$ is normal [7] and consequently Cohen-Macaulay, thus this formula applies. We have the following consequence of Lemma 5.1.

Corollary 5.2. *The $a-$invariant of $K[\mathcal{A}]$ is $a(K[\mathcal{A}]) = -1$.*

Proof: Let $\{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be generators of the $K-$algebra $K[\mathcal{A}]$. Then $K[\mathcal{A}]$ is a standard graded algebra with the grading

$$K[\mathcal{A}]_i = \sum_{|c|=i} K(x^{\alpha_1})^{c_1} \cdots (x^{\alpha_q})^{c_q}, \text{ where } |c| = c_1 + \ldots + c_q.$$

Since $\omega_{K[\mathcal{A}]} = (x_1 \cdots x_n)K[\mathcal{A}]$, it follows that $\min\{i \mid (\omega_{K[\mathcal{A}]})_i \neq 0\} = 1$, thus $a(K[\mathcal{A}]) = -1$. ∎

6 Ehrhart function

We consider a fixed set of distinct monomials $F = \{x^{\alpha_1}, \ldots, x^{\alpha_r}\}$ in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. Let

$$\mathcal{P} = conv(\log(F))$$

be the convex hull of the set $\log(F) = \{\alpha_1, \ldots, \alpha_r\}$. The *normalized Ehrhart ring* of $\mathcal{P}$ is the graded algebra

$$A_{\mathcal{P}} = \bigoplus_{j=0}^{\infty} (A_{\mathcal{P}})_j \subset R[T],$$

where the j^{th} component is given by

$$(A_{\mathcal{P}})_j = \sum_{\alpha \in \mathbb{Z}\log(F) \cap \ j\mathcal{P}} K\ x^{\alpha}\ T^j.$$

The *normalized Ehrhart function* of $\mathcal{P}$ is defined as

$$E_{\mathcal{P}}(j) = \dim_K (A_{\mathcal{P}})_j = \mid \mathbb{Z}\ \log(F) \cap j\mathcal{P} \mid.$$

From [9, Proposition 7.2.39 and Corollary 7.2.45] we have the following important result.

Theorem 6.1. *If $K[F]$ is a standard graded subalgebra of R and h is the Hilbert function of $K[F]$, then:*
a) $h(j) \leq E_{\mathcal{P}}(j)$ for all $j \geq 0$, and
b) $h(j) = E_{\mathcal{P}}(j)$ for all $j \geq 0$ if and only if $K[F]$ is normal.

In this section we will compute the Hilbert function and the Hilbert series for the $K-$algebra $K[\mathcal{A}]$, where $\mathcal{A}$ satisfies the hypothesis of Lemma 4.1.

Proposition 6.2. *In the hypothesis of Lemma 4.1, the Hilbert function of the $K-$algebra $K[\mathcal{A}]$ is*

$$h(t) = \sum_{k=0}^{(i+1)t} \binom{k+i-1}{k} \binom{nt-k+n-i-1}{nt-k}.$$

Proof: From [7] we know that the $K-$algebra $K[\mathcal{A}]$ is normal. Therefore, to compute the Hilbert function of $K[\mathcal{A}]$ is equivalent to compute the Ehrhart function of $\mathcal{P}$, where $\mathcal{P} = conv(A)$.
It is clearly enough to show that $\mathcal{P}$ is the intersection of the cone $\mathbb{R}_+ A$ with the hyperplane $x_1 + \ldots + x_n = n$, that is,

$$\mathcal{P} = \{\alpha \in \mathbb{R}^n \mid \alpha_k \geq 0 \text{ for any } k \in [n],\ 0 \leq \alpha_1 + \ldots + \alpha_i \leq i+1 \text{ and } \alpha_1 + \ldots + \alpha_n = n\},$$

whence it follows that

$t\,\mathcal{P} = \{\alpha \in \mathbb{R}^n \mid \alpha_k \geq 0,\ \forall\, k \in [n],\ 0 \leq \alpha_1+\ldots+\alpha_i \leq (i+1)\,t \text{ and } \alpha_1+\ldots+\alpha_n = n\,t\}.$

Since for any $0 \leq k \leq (i+1)\ t$ the equation $\alpha_1 + \ldots + \alpha_i = k$ has $\binom{k+i-1}{k}$ nonnegative integer solutions and the equation $\alpha_{i+1} + \ldots + \alpha_n = n\ t - k$ has $\binom{nt-k+n-i-1}{nt-k}$ nonnegative integer solutions, we get that

$$E_{\mathcal{P}}(t) = |\ \mathbb{Z}\ A\ \cap\ t\ \mathcal{P}\ | = \sum_{k=0}^{(i+1)t} \binom{k+i-1}{k}\binom{nt-k+n-i-1}{nt-k}.$$

□

Corollary 6.3. *The Hilbert series of the $K-$algebra $K[\mathcal{A}]$, where $\mathcal{A}$ satisfies the hypothesis of Lemma 4.1, is*

$$H_{K[\mathcal{A}]}(t) \ = \ \frac{1 + h_1\ t + \ldots + h_{n-1}\ t^{n-1}}{(1-t)^n},$$

where

$$h_j = \sum_{s=0}^{j} (-1)^s\ h(j-s)\ \binom{n}{s}$$

and $h(s)$ is the Hilbert function of $K[\mathcal{A}]$

Proof: Since the $a-$invariant of $K[\mathcal{A}]$ is $a(K[\mathcal{A}]) = -1$, it follows that to compute the Hilbert series of $K[\mathcal{A}]$ it is necessary to know the first n values of the Hilbert function of $K[\mathcal{A}]$, $h(i)$ for $0 \leq i \leq n-1$. Since $\dim(K[\mathcal{A}]) = n$, applying n times the difference operator Δ (see [2]) on the Hilbert function of $K[\mathcal{A}]$ we get the conclusion.

Let $\Delta^0(h)_j := h(j)$ for any $0 \leq j \leq n-1$. For $k \geq 1$ let $\Delta^k(h)_0 := 1$ and $\Delta^k(h)_j := \Delta^{k-1}(h)_j - \Delta^{k-1}(h)_{j-1}$ for any $1 \leq j \leq n-1$. We claim that

$$\Delta^k(h)_j = \sum_{s=0}^{k} (-1)^s h(j-s) \binom{k}{s}$$

for any $k \geq 1$ and $0 \leq j \leq n-1$. We proceed by induction on k.

If $k = 1$, then

$$\Delta^1(h)_j = \Delta^0(h)_j - \Delta^0(h)_{j-1} = h(j) - h(j-1) = \sum_{s=0}^{1} (-1)^s h(j-s) \binom{1}{s}$$

for any $1 \leq j \leq n-1$.

If $k > 1$, then

$$\Delta^k(h)_j = \Delta^{k-1}(h)_j - \Delta^{k-1}(h)_{j-1} =$$

$$\sum_{s=0}^{k-1}(-1)^s h(j-s)\binom{k-1}{s}-\sum_{s=0}^{k-1}(-1)^s h(j-1-s)\binom{k-1}{s}=h(j)\binom{k-1}{0}+$$

$$\sum_{s=1}^{k-1}(-1)^s h(j-s)\binom{k-1}{s}-\sum_{s=0}^{k-2}(-1)^s h(j-1-s)\binom{k-1}{s}+(-1)^k h(j-k)\binom{k-1}{k-1}$$

$$=h(j)+\sum_{s=1}^{k-1}(-1)^s h(j-s)\left[\binom{k-1}{s}+\binom{k-1}{s-1}\right]+(-1)^k h(j-k)\binom{k-1}{k-1}$$

$$=h(j)+\sum_{s=1}^{k-1}(-1)^s h(j-s)\binom{k}{s}+(-1)^k h(j-k)\binom{k-1}{k-1}=\sum_{s=0}^{k}(-1)^s h(j-s)\binom{k}{s}.$$

Thus, if $k=n$ it follows that

$$h_j=\Delta^n(h)_j=\sum_{s=0}^{n}(-1)^s h(j-s)\binom{n}{s}=\sum_{s=0}^{j}(-1)^s h(j-s)\binom{n}{s}$$

for any $1\leq j\leq n-1$.

$\square$

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