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Uniqueness Of Meromorphic Functions Concerning Differential Monomials Sharing The Same Value

by
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Abstract

Using the notion of weakly weighted sharing and truncated sharing of values we prove some uniqueness theorems concerning differential monomials which will improve respectively a result of Lahiri [5] and a recent result of Banerjee [2]. Also we introduce a new sharing definition namely relaxed weighted sharing which is weaker than weakly weighted sharing as well as weighted sharing and with the aid of which we supplement a recent result of Banerjee.[2]

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1 Introduction Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{k)}(a;f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a;f)$=$E_{\infty)}(a;g)$ we say that f, g share the value a CM.

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure. We use I to denote any set of infinite linear measure of $0 < r < \infty$.

Let $N_E(r,a;f,g)$ $(\overline{N}_E(r,a;f,g))$ be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and

$N_0(r,a;f,g)$ ($\overline{N}_0(r,a;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities.
If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_E(r,a;f,g)=S(r,f)+S(r,g)$$

then we say that f and g share a "CM".

On the other hand if

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a;f,g)=S(r,f)+S(r,g)$$

then we say that f and g share a "IM".

Yang and Hua [10] studied the problem of non linear differential polynomials when they share the same value a CM.

They proved the following result.

Theorem A. *[10] Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbb{C}-\{0\}$. If $f^n f^{'}$ and $g^n g^{'}$ share the value a CM, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.*

To state the next result we require the following definition.

Definition 1.1. *[5] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Since $E_k(a;f) = E_k(a;g)$ implies $E_p(a;f) = E_p(a;g)$ for any integer p $(0 \leq p < k)$, clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

With the notion of weighted sharing of values improving *Theorem A* the following result is proved in [5].

Theorem B. *[5] Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbb{C}-\{0\}$. If $f^n f^{'}$ and $g^n g^{'}$ share $(a,2)$, then then conclusion of the* Theorem A *holds.*

In 2005 the first author obtained the following result.

Theorem C. *[1]Let f and g be two nonconstant meromorphic functions such that $n > 22-[5\Theta(\infty;f)+5\Theta(\infty;g)+\min\{\Theta(\infty;f),\Theta(\infty;g)\}]$, where n is an integer. If for $a \in \mathbb{C}-\{0\}$, $f^n f^{'}$ and $g^n g^{'}$ share $(a,0)$, then conclusion of the* Theorem A *holds.*

Recently in [2] Banerjee improved *Theorem A* with the idea of truncated sharing of values. In the same paper as a supplementary result Banerjee [2] obtained the following.

Theorem D. *[2] Let f and g be two nonconstant meromorphic functions and $n > max\{8, 14 - 3\Theta(\infty; f) - 3\Theta(\infty; g) - min\{\Theta(\infty, f), \Theta(\infty; g)\}\}$ an integer. be an integer. If $E_{2)}(a; f^n f^{'}) = E_{2)}(a; g^n g^{'})$ then conclusion of the* Theorem A *holds.*

We now require the following definition.

Definition 1.2. *[4] For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity.*

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ *are defined similarly, where in counting the a-points of f we ignore the multiplicities.*

Also $N(r, a; f \mid< m), N(r, a; f \mid> m), \overline{N}(r, a; f \mid< m)$and $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Recently Lin and Lin [7] introduced the notion of weakly weighted sharing which we shall define next.

Definition 1.3. *[7] Let f g share a "IM" and k be a positive integer or ∞.*

(i) $\overline{N}^E(r, a; f, g \mid\leq k)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k.

(ii) $\overline{N}^0(r, a; f, g \mid> k)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k.

Definition 1.4. *[7] For $a \in \mathbb{C} \cup \{\infty\}$, if k is a positive integer or ∞ and*

$$\overline{N}(r, a; f \mid\leq k) - \overline{N}^E(r, a; f, g \mid\leq k) = S(r, f),$$

$$\overline{N}(r, a; g \mid\leq k) - \overline{N}^E(r, a; f, g \mid\leq k) = S(r, g)$$

$$\overline{N}(r, a; f \mid\geq k+1) - \overline{N}^0(r, a; f, g \mid\geq k+1) = S(r, f),$$

$$\overline{N}(r, a; g \mid\geq k+1) - \overline{N}^0(r, a; f, g \mid\geq k+1) = S(r, g)$$

or if $k = 0$ and

$$\overline{N}(r, a; f) - \overline{N}_0(r, a; f, g) = S(r, f), \quad \overline{N}(r, a; g) - \overline{N}_0(r, a; f, g) = S(r, g),$$

then we say f, g weakly share a with weight k. Here we write f, g share "(a, k)" to mean that f, g weakly share a with weight k

Obviously if f, g share "(a,k)", then f, g share "(a,p)" for any integer p, $0 \le p < k$. Also we note that f, g share a value a "IM" or "CM" if and only if f, g share "$(a,0)$" or "(a,∞)" respectively.

Now it is clear from *Definition 1.1* and *Definition 1.4* that weighted sharing and weakly weighted sharing are respectively scalings between IM, CM and "IM", "CM". Also weakly weighted sharing includes the definition of weighted sharing.

In the paper we introduce another sharing notion which is also a scaling between "IM" and "CM" but weaker than weakly weighted sharing and hence include the same definition. To this end we first require the following notation.

Definition 1.5. *We denote by $\overline{N}(r,a;f \mid= p; g \mid= q)$ the reduced counting function of common a-points of f and g with multiplicities p and q respectively.*

We are now at a stage to introduce the definition of relaxed weighted sharing.

Definition 1.6. *Let f, g share a "IM". Also let k be a positive integer or ∞ and $a \in \mathbb{C} \cup \{\infty\}$. If*

$$\sum_{\substack{p,q \le k \\ (p \ne q)}} \overline{N}(r,a;f \mid= p; g \mid= q) = S(r)$$

then we say f, g share a with weight k in a relaxed manner. Here we write f, g share $(a,k)^*$ to mean that f, g share a with weight k in a relaxed manner.

Obviously if f, g share $(a,k)^*$, then f, g share $(a,p)^*$ for any integer p, $1 \le p < k$. Also we note that f, g share "$(a,0)$" or "(a,∞)" if and only if f, g share $(a,1)^*$ or $(a,\infty)^*$ respectively.

We note that f, g share "(a,k)" means they share $(a,k)^*$ for $k \ge 1$ but not conversely. Also from the definition of relaxed weighted sharing it is clear that for finite k f, g share $(a,k)^*$ actually means they share a "IM" with some restrictions imposed on the common zeros of $f-a$ and $g-a$ up to multiplicity k. In particular if $k=2$ the restrictions are minimum. In the paper we will show that if in *Theorem C* f and g share $(a,2)^*$ instead of $(a,0)$ the lower bound of n can be significantly reduced.

With the notion of weakly weighted sharing and truncated sharing of values we improve respectively *Theorem B* and *Theorem D* and show that the lower bound of n can be reduced by 1 under certain restriction on f and g in *Theorem B.*

Definition 1.7. *[14] For $a \in \mathbb{C} \cup \{\infty\}$ we put*

$$\delta_{1)}(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r,a;f \mid= 1)}{T(r,f)}$$

Clearly $0 \le \delta(a;f) \le \Theta(a;f) \le \delta_{1)}(a;f)$

Following theorems are the main results of the paper.

Theorem 1.1. *Let* f *and* g *be two nonconstant meromorphic functions,* $n > max\{8, 10-\min\{\delta_{1)}(\infty, f), \delta_{1)}(\infty; g)\}-3\min\{\Theta(\infty, f), \Theta(\infty; g)\}\}$ *an integer and* $a \in \mathbb{C} - \{0\}$. *If* $f^n f^{'}$ *and* $g^n g^{'}$ *share "*$(a, 2)$*", then conclusion of the* Theorem A *holds..*

Theorem 1.2. *Let* f *and* g *be two nonconstant meromorphic functions and* $n > max\{8, 12-5\min\{\Theta(\infty, f), \Theta(\infty; g)\}\}$ *an integer. If* $E_{2)}(a; f^n f^{'}) = E_{2)}(a; g^n g^{'})$ *then conclusion of the* Theorem A *holds.*

Theorem 1.3. *Let* f *and* g *be two nonconstant meromorphic functions,* $n > 14 - 3\Theta(\infty, f) - 3\Theta(\infty; g)$ *an integer and* $a \in \mathbb{C} - \{0\}$. *If* $f^n f^{'}$ *and* $g^n g^{'}$ *share* $(a, 2)^*$, *then conclusion of the* Theorem A *holds.*

Remark 1.1. *In* Theorem 1.1 *if we take* $\min\{\Theta(\infty, f), \Theta(\infty; g)\} > 0$ *then the theorem is true for an integer* $n \geq 10$. *Also in the same theorem if we take* f *and* g *be two nonconstant entire functions then the theorem is true for an integer* $n \geq 7$.

Remark 1.2. *In* Theorem 1.2 *if we take* f *and* g *be two nonconstant entire functions then the theorem is true for an integer* $n \geq 7$.

Though the standard definitions and notations of the value distribution theory are available in [3], we explain some definitions and notations which are used in the paper.

Definition 1.8. $\{5$, *cf.[12]*$\}$ *For* $a \in \mathbb{C} \cup \{\infty\}$ *and a positive integer* p, *we denote by* $N_p(r, a; f)$ *the sum* $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots \overline{N}(r, a; f \mid\geq p)$. *Clearly* $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.9. *Let* k *be a positive integer and for* $a \in \mathbb{C} - \{0\}$, $E_{k)}(a; f) = E_{k)}(a; g)$. *Let* z_0 *be a zero of* $f(z) - a$ *of multiplicity* p *and a zero of* $g(z) - a$ *of multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ *the reduced counting function of those* a*-points of* f *and* g *for which* $p > q \geq k+1 (q > p \geq k+1)$, *by* $\overline{N}_E^{(k+1}(r, a; f)$ *the reduced counting function of those* a*-points of* f *and* g *for which* $p = q \geq k+1$, *by* $\overline{N}_{f\geq k+1}(r, a; f| \, g \neq a)$ *the reduced counting functions of those* a*-points of* f *and* g *for which* $p \geq k+1$ *and* $q = 0$.

Definition 1.10. *Let* k *be a positive integer and for* $a \in \mathbb{C} - \{0\}$, *let* f, g *share* a *"IM". Let* z_0 *be a zero of* $f(z) - a$ *of multiplicity* p *and a zero of* $g(z) - a$ *of multiplicity* q. *We denote by* $\overline{N}_{f\geq k+1}(r, a; f \mid g = m)$ *the reduced counting functions of those* a*-points of* f *and* g *for which* $p \geq k+1$ *and* $q = m$. *We can define* $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ *and* $\overline{N}_E^{(k+1}(r, a; f)$ *in a similar manner as defined in the previous definition.*

Definition 1.11. *[6] Let* $a, b \in \mathbb{C} \cup \{\infty\}$. *We denote by* $N(r, a; f \mid g = b)$ *the counting function of those* a*-points of* f, *counted according to multiplicity, which are* b*-points of* g.

Definition 1.12. *[6] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.*

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f, g, F, G be four nonconstant meromorphic functions. Henceforth we shall denote by h and H the following two functions.

$$h = (\frac{f^{''}}{f^{'}} - \frac{2f^{'}}{f-1}) - (\frac{g^{''}}{g^{'}} - \frac{2g^{'}}{g-1})$$

and

$$H = (\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1}) - (\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1}).$$

Lemma 2.1. *If f, g be share "$(1,1)$" and $h \not\equiv 0$. Then*

$$\begin{aligned} N(r,1; f \mid\leq 1) &\leq N(r,0;h) + S(r,f) \\ &\leq N(r,\infty;h) + S(r,f) + S(r,g). \end{aligned}$$

Proof: Since f, g share "$(1,1)$" it follows that if z_0 be a common simple 1-point of f and g, then in some neighborhoods of z_0 we have $h = (z-z_0)\alpha(z)$, where $\alpha(z)$ is analytic at z_0. Hence by the first fundamental theorem and Milloux theorem {p.55 [3]} we get

$$\begin{aligned} N(r,1; f \mid\leq 1) &= N^E(r,1; f, g \mid\leq 1) + S(r,f) \\ &\leq N(r,0;h) + S(r,f) \\ &\leq N(r,\infty;h) + S(r,f) + S(r,g) \end{aligned}$$

□

Lemma 2.2. *If f, g be two nonconstant meromorphic functions such that $E_{1)}(1; f) = E_{1)}(1; g)$ and $h \not\equiv 0$ then*

$$N(r,1; f \mid \leq 1) \leq N(r,0;h) \leq N(r,\infty;h) + S(r,f) + S(r,g).$$

Proof: The proof is obvious. □

Lemma 2.3. *If f, g be share $(1,1)^*$ and $h \not\equiv 0$. Then*

$$\begin{aligned} N^E(r,1; f, g \mid\leq 1) &\leq N(r,0;h) \\ &\leq N(r,\infty;h) + S(r,f) + S(r,g). \end{aligned}$$

Proof: The proof can be carried out in the line of proof of *Lemma 2.1.* □

Lemma 2.4. *If f, g share $(1,1)^*$ and $h \not\equiv 0$. Then*

$$\begin{aligned} N(r,\infty;h) \leq\ & \overline{N}(r,0;f \mid\geq 2) + \overline{N}(r,0;g \mid\geq 2) + \overline{N}(r,\infty;f \mid\geq 2) + \\ & \overline{N}(r,\infty;g \mid\geq 2) + \overline{N}_L(r,1;f) + \overline{N}_L(r,1;g) + \\ & \overline{N}_0(r,0;f^{'}) + \overline{N}_0(r,0;g^{'}) + S(r), \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ is the reduced counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g^{'})$ is similarly defined.

Proof: We can easily verify that possible poles of h occur at (i) multiple zeros of f and g, (ii) multiple poles of f and g,(iii) the common zeros of $f-1$ and $g-1$ whose multiplicities are different,(iii) those 1-points of f (g) which are not the 1-points of g (f), (iv) zeros of $f^{'}$ which are not the zeros of $f(f-1)$, (v) zeros of $g^{'}$ which are not zeros of $g(g-1)$. Since all the poles of h are simple the lemma follows from above. This proves the lemma. □

Lemma 2.5. *If for a positive integer k, $N_k(r,0;f^{'} \mid f \neq 0)$ denotes the counting function of those zeros of of $f^{'}$ which are not the zeros of f, where a zero of $f^{'}$ with multiplicity m is counted m times if $m \leq k$ and k times if $m > k$ then*

$$N_k(r,0;f^{'} \mid f \neq 0) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - \sum_{p=k+1}^{\infty} \overline{N}\left(r,0;\frac{f^{'}}{f} \mid\geq p\right) + S(r,f).$$

Proof: By the first fundamental theorem and Milloux theorem {p.55 [3]} we get

$$\begin{aligned} N(r,0;f^{'} \mid f \neq 0) &= N(r,0;\frac{f^{'}}{f}) \\ &\leq N(r,\infty;\frac{f^{'}}{f}) + S(r,f) \\ &= \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f). \end{aligned}$$

Now

$$\begin{aligned} & N_k(r,0;\frac{f^{'}}{f}) + \sum_{p=k+1}^{\infty} \overline{N}(r,0;\frac{f^{'}}{f} \mid\geq p) = N(r,0;f^{'} \mid f \neq 0) \\ \leq\ & \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f). \end{aligned}$$

The lemma follows from above. □

Lemma 2.6. *Let* $E_{2)}(1;f)=E_{2)}(1;g)$. *Then*

$$\overline{N}_{f\geq 3}(r,1;f\mid\ g\neq 1)\leq$$

$$\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}\sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{f^{'}}{f}\mid\geq p\right)-\frac{1}{2}N_0^2(r,0;f^{'})+S(r,f),$$

where by $N_0^2(r,0;f^{'})$ *we mean the counting function of those zeros of* $f^{'}$ *which are not the zeros of* $f(f-1)$ *where every simple zero is counted once and all other zeros are counted two times. The lemma can be proved for* g *instead of* f *also.*

Proof: Using *Lemma 2.5* we get

$$\begin{aligned}\overline{N}_{f\geq 3}(r,1;f\mid\ g\neq 1) &\leq \overline{N}(r,1;f\mid\geq 3)\\ &\leq \frac{1}{2}N_2(r,0;f^{'}\mid f=1)\\ &\leq \frac{1}{2}N_2(r,0;f^{'}\mid\ f\neq 0)-\frac{1}{2}N_0^2(r,0;f^{'})\\ &\leq \frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}\sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{f^{'}}{f}\mid\geq p\right)\\ &\quad -\frac{1}{2}N_0^2(r,0;f^{'})+S(r,f).\end{aligned}$$

□

Lemma 2.7. *Let* f, g *share* $(1,2)^*$. *Then*

$$\begin{aligned}&\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f\geq 3}(r,1;g\mid=1)\\ \leq\ & N(r,1;g)-\overline{N}(r,1;g)+S(r).\end{aligned}$$

Proof: Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. For each possible value of q, possible values of p is always ≥ 0. Since f, g share $(1,2)^*$ implies f, g share "(1,0)", the sum of the reduced counting functions corresponding to the 1 points of g for which $p=0$ are $S(r)$. Also from the definition of relaxed weighted sharing we note that the sum of the reduced counting functions corresponding to the common 1 points of f and g for which (i) q=1,p=0 (ii) q=1,p=2 and (iii) q=2, p=1 are $S(r)$, the lemma follows. □

Lemma 2.8. *Let* f, g *share* $(1,2)^*$. *Then*

$$\begin{aligned}\overline{N}_L(r,1;f)+\overline{N}_{f\geq 3}(r,1;g\mid=1) \leq\ & \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-\\ & \sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{f^{'}}{f}\mid\geq p\right)-N_0^2(r,0;f^{'})+S(r).\end{aligned}$$

Proof: Since

$$\begin{aligned}\overline{N}_L(r,1;f)+\overline{N}_{f\geq 3}(r,1;g\mid=1) &= \overline{N}(r,1;f\mid=2;g=1)+\\ &\quad 2\overline{N}(r,1;f\mid\geq 3)+S(r)\\ &= 2\overline{N}(r,1;f\mid\geq 3)+S(r),\end{aligned}$$

the rest of the proof can be carried out in the line of the proof of *Lemma 2.6*. □

Lemma 2.9. *Let f, g share "$(1,2)$" and $h\not\equiv 0$. Then*

$$\begin{aligned}T(r,f) &\leq N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\ &\quad -\sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{g^{'}}{g}\mid\geq p\right)+S(r,f)+S(r,g)\end{aligned}$$

Proof: Since f and g share "$(1,2)$" it follows that f and g share $(1,1)^*$. Also we note that $\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)\leq\overline{N}(r,1;g\mid\geq 3)$. So by the second fundamental theorem, *Lemmas 2.1* and *2.4* we get

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;f)-N_0(r,0;f^{'})+S(r,f)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+N(r,1;f\mid\leq 1)+\\ &\quad \overline{N}(r,1;f\mid\geq 2)-N_0(r,0;f^{'})+S(r,f)\\ &\leq N_2(r,0;f)+N_2(r,\infty;f)+\overline{N}(r,0;g\mid\geq 2)+\overline{N}(r,\infty;g\mid\geq 2)\\ &\quad +\overline{N}(r,1;g\mid\geq 2)+\overline{N}(r,1;g\mid\geq 3)+S(r,f)+S(r,g)\\ &\leq N_2(r,0;f)+N_2(r,\infty;f)+\overline{N}(r,0;g\mid\geq 2)+\overline{N}(r,\infty;g\mid\geq 2)\\ &\quad +N_2(r,0;g^{'}\mid g\neq 0)+S(r,f)+S(r,g)\\ &\leq N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\ &\quad -\overline{N}(r,1;g\mid\geq 4)-\sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{g'}{g}\mid\geq p\right)+S(r,f)+S(r,g)\end{aligned}$$

□

Lemma 2.10. *[2] Let $E_{2)}(1;f)=E_{2)}(1;g)$ and $h\not\equiv 0$. Then*

$$\begin{aligned}T(r,f) &\leq N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\ &\quad +2\overline{N}_{f\geq 3}(r,1;f\mid\ g\neq 1)-\overline{N}_E^{(3}(r,1;f)-\overline{N}_{g\geq 3}(r,1;g\mid\ f\neq 1)\\ &\quad -m(r,1;g)+S(r,f)+S(r,g).\end{aligned}$$

Lemma 2.11. *Let $E_{2)}(1;f)=E_{2)}(1;g)$ and $h\not\equiv 0$. Then*

$$\begin{aligned} T(r,g) \quad \leq \quad & N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)+\\ & +2\overline{N}_{g\geq 3}(r,1;g\mid\ f\neq 1)-\overline{N}^{(3}_{E}(r,1;g)-\overline{N}_{f\geq 3}(r,1;f\mid\ g\neq 1)-\\ & -m(r,1;f)+S(r,f)+S(r,g). \end{aligned}$$

Proof: We omit the proof since it can be carried out in the line of proof of *Lemma 2.10* □

Lemma 2.12. *Let f, g share $(1,2)^*$. Then*

$$\begin{aligned} T(r,f) \quad \leq \quad & N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)+\overline{N}(r,0;f)\\ & +\overline{N}(r,\infty;f)-m(r,1;g)+S(r). \end{aligned}$$

Proof: By the second fundamental theorem we get

$$\begin{aligned} T(r,f)+T(r,g) \quad \leq \quad & \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\\ & +\overline{N}(r,1;f)+\overline{N}(r,1;g)-N_0(r,0;f^{'})-N_0(r,0;g^{'})\\ & +S(r,f)+S(r,g). \end{aligned} \tag{2.1}$$

By *Lemmas 2.3, 2.4* and *2.7* we get from (2.1)

$$\begin{aligned} \overline{N}(r,1;f)+\overline{N}(r,1;g) \quad \leq \quad & N^{E}(r,1;f\mid\leq 1)+\overline{N}^{(2}_{E}(r,1;f)+\overline{N}_{L}(r,1;f)\\ & +\overline{N}_{L}(r,1;g)+\overline{N}(r,1;g)+S(r)\\ \leq \quad & \overline{N}(r,0;f\mid\geq 2)+\overline{N}(r,\infty;f\mid\geq 2)+\overline{N}(r,0;g\mid\geq 2)\\ & +\overline{N}(r,\infty;g\mid\geq 2)+2\overline{N}_{L}(r,1;f)+2\overline{N}_{L}(r,1;g)\\ & +\overline{N}^{(2}_{E}(r,1;f)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'})\\ & +N(r,1;g)-\overline{N}_{L}(r,1;f)-2\overline{N}_{L}(r,1;g)\\ & -\overline{N}^{(2}_{E}(r,1;f)+\overline{N}_{f\geq 3}(r,1;g\mid=1)+S(r)\\ \leq \quad & \overline{N}(r,0;f\mid\geq 2)+\overline{N}(r,\infty;f\mid\geq 2)+\overline{N}(r,0;g\mid\geq 2)\\ & +\overline{N}(r,\infty;g\mid\geq 2)+\overline{N}_{L}(r,1;f)+\overline{N}_{f\geq 3}(r,1;f\mid\ g=1)\\ & +T(r,g)-m(r,1;g)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'})+S(r). \end{aligned} \tag{2.2}$$

Using (2.2) in (2.1) the lemma follows from *Lemma 2.8*. This completes the proof of the lemma. □

Lemma 2.13. *[13] If $h\equiv 0$ and*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}}\frac{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)}{T(r)}<1,$$

then $f\equiv g$ or $f.g\equiv 1$.

Lemma 2.14. *{cf.[8],[9]} Let* f *be a nonconstant meromorphic function and* $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, *where* $a_0, a_1, a_2 \ldots, a_n$ *are constants and* $a_n \neq 0$. *Then* $T(r, P(f)) = nT(r, f) + O(1)$.

Lemma 2.15. *Let* f, g *be two nonconstant meromorphic functions* $F = \frac{f^n f^{'}}{a}$, $G = \frac{g^n g^{'}}{a}$ *and* n *is a positive integer. Then*
(i) $T(r, F) \geq (n-1)T(r, f) + N(r, \infty; f) + N(r, 0; f^{'}) + S(r, f)$.
(ii)$T(r, G) \geq (n-1)T(r, g) + N(r, \infty; g) + N(r, 0; g^{'}) + S(r, g)$.

Proof: Since

$$\begin{aligned} T(r, F) &= T(r, \frac{f^n f^{'}}{a}) \\ &\leq T(r, \frac{f^n}{a}) + T(r, f^{'}) \\ &\leq nT(r, f) + 2T(r, f) + S(r, f) \\ &= (n+2)T(r, f) + S(r, f) \end{aligned}$$

and

$$T(r, G) \leq (n+2)T(r, g) + S(r, g),$$

it follows that $S(r, F)$ can be replaced by $S(r, f)$ and $S(r, G)$ can be replaced by $S(r, g)$. Using *Lemma 2.14* we note that

$$\begin{aligned} T(r, F) + m(r, \frac{1}{f'}) &= N(r, \infty; \frac{f^n f^{'}}{a}) + m(r, \frac{f^n f^{'}}{a}) + m(r, \frac{1}{f'}) \\ &\geq N(r, \infty; f^n) + N(r, \infty; f^{'}) + m(r, f^n) \\ &= nT(r, f) + N(r, \infty; f^{'}) + O(1) \end{aligned}$$

i.e.

$$\begin{aligned} T(r, F) &\geq nT(r, f) - T(r, f^{'}) + N(r, 0; f^{'}) + N(r, \infty; f^{'}) + S(r, f) \\ &\geq nT(r, f) - T(r, f) + N(r, \infty; f) + N(r, 0; f^{'}) + S(r, f) \\ &= (n-1)T(r, f) + N(r, \infty; f) + N(r, 0; f^{'}) + S(r, f) \end{aligned}$$

In a similar manner we can obtain

$$T(r, G) \geq (n-1)T(r, g) + N(r, \infty; g) + N(r, 0; g^{'}) + S(r, g).$$

$\square$

Lemma 2.16. *[1] Let* F *and* G *be defined as in* Lemma 2.15. *Also let* $F_1 = \frac{f^{n+1}}{a(n+1)}$, $G_1 = \frac{g^{n+1}}{a(n+1)}$, *where* $n(> 2)$ *is an integer. Then* $F \equiv G$ *implies* $F_1 \equiv G_1$

Lemma 2.17. *[10] Let f, g be two nonconstant meromorphic functions and $n > 6$. If $f^n f^{'} g^n g^{'} = 1$, then $g = c_1 e^{cz}$, $f = c_2 e^{-cz}$ where c, c_1, c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -1$.*

Lemma 2.18. *[11] Let f be a nonconstant meromorphic function. Then*

$$N(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N(r, 0; f) + S(r, f).$$

3 Proofs of the theorems

Proof: [Proof of Theorem 1.1] Let F and G be defined as in *Lemma 2.15.* Since $f^n f^{'}$ and $g^n g^{'}$ share "$(a, 2)$", it follows that F and G share "$(1, 2)$". If possible, we suppose that $H \not\equiv 0$. Then by *Lemma 2.9* we obtain

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, \infty; F) + N_2(r, 0; G) + N_2(r, \infty; G) + S(r, F) + S(r, G). \tag{3.1}$$

We see that

$$N_2(r, 0; F) + N_2(r, \infty; F) \leq 2\overline{N}(r, 0; f) + N(r, 0; f^{'}) + 2\overline{N}(r, \infty; f)$$

$$N_2(r, 0; G) + N_2(r, \infty; G) \leq 2\overline{N}(r, 0; g) + N(r, 0; g^{'}) + 2\overline{N}(r, \infty; g)$$

So by *Lemmas 2.15* and *2.18* we get from (3.1) for $\varepsilon(> 0)$

$$\begin{aligned} T(r, F) \quad \leq \quad & 2\overline{N}(r, 0; f) + N(r, 0; f^{'}) + 2\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; g) \\ & + N(r, 0; g^{'}) + 2\overline{N}(r, \infty; g) + S(r, f) + S(r, g) \end{aligned}$$

i.e.

$$\begin{aligned} (n-1)T(r, f) \quad \leq \quad & 2\overline{N}(r, 0; f) + N(r, \infty; f \mid = 1) + 2\overline{N}(r, 0; g) \\ & + N(r, 0; g) + 3\overline{N}(r, \infty; g) + S(r, f) + S(r, g) \\ \leq \quad & 2T(r, f) + 3T(r, g) + (1 - \delta_{1)}(\infty; f) + \varepsilon)T(r, f) \\ & + (3 - 3\Theta(\infty; g) + \varepsilon)T(r, g) + S(r, f) + S(r, g) \\ \leq \quad & \{9 - \delta_{1)}(\infty; f) - 3\Theta(\infty; g) + 2\varepsilon\}T(r) + S(r) \end{aligned} \tag{3.2}$$

In a similar manner we obtain

$$(n-1)T(r, g) \leq \{9 - 3\Theta(\infty; f) - \delta_{1)}(\infty; g) + 2\varepsilon\}T(r) + S(r) \tag{3.3}$$

From (3.2) and (3.3) we obtain

$$[n - 10 + \min\{\delta_{1)}(\infty, f), \delta_{1)}(\infty; g)\} + 3\ \min\{\Theta(\infty; f), \Theta(\infty; g)\} - 2\varepsilon]T(r) \leq S(r). \tag{3.4}$$

Since $\varepsilon(>)0$ is arbitrary, (3.4) implies a contradiction. Hence $H \equiv 0$.
Since

$$\overline{N}(r, 0; f^{'}) \leq T(r, f^{'}) - m(r, \frac{1}{f'}) \leq 2T(r, f) - m(r, \frac{1}{f'}) + S(r, f),$$

we note that

$$\begin{aligned}
& \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G) && (3.5)\\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f^{'})+\overline{N}(r,0;g^{'})\\
\leq\ & 4T(r,f)+4T(r,g)-m(r,0;f^{'})-m(r,0;g^{'})+S(r)\\
\leq\ & 8T(r)-m(r,0;f^{'})-m(r,0;g^{'})+S(r)
\end{aligned}$$

Also using *Lemma 2.14* we get

$$\begin{aligned}
T(r,F)+m(r,\frac{1}{f'}) &= m(r,\frac{f^nf^{'}}{a})+m(r,\frac{1}{f'})+N(r,\infty;\frac{f^nf^{'}}{a}) && (3.6)\\
&\geq m(r,\frac{f^n}{a})+N(r,\infty;f^n)\\
&= T(r,f^n)+O(1)\\
&= nT(r,f)+O(1)
\end{aligned}$$

Similarly

$$T(r,G)+m(r,\frac{1}{g^{'}})\geq nT(r,g)+O(1) \tag{3.7}$$

From (3.6) and (3.7) we get

$$max\{T(r,F),T(r,G)\}\geq nT(r)-m(r,\frac{1}{f'})-m(r,\frac{1}{g^{'}})+O(1) \tag{3.8}$$

By (3.5) and (3.8) applying *Lemma 2.13* we get either $F\equiv G$ or $FG\equiv 1$.
If $F\equiv G$, then by *Lemma 2.16* we obtain $F\equiv G$ or $f\equiv dg$ where d is some $(n+1)$ th root of unity.
If $FG\equiv 1$ then $f^nf^{'}g^ng^{'}=a^2$. Set $f_1=a^{-\frac{1}{n+1}}f$ and $g_1=a^{-\frac{1}{n+1}}g$, then $f_1^nf_1^{'}g_1^ng_1^{'}=1$. So using *Lemma 2.17* we get $g=c_1e^{cz}$, $f=c_2e^{-cz}$ where c, c_1 and c_2 are constants and satisfy $(c_1c_2)^{n+1}c^2=-a^2$. This completes the proof of the theorem. □

Proof: [Proof of Theorem 1.2] Let F and G be defined as in *Lemma 2.15.* Clearly $E_{2)}(1;F)=E_{2)}(1;G)$. If possible, we suppose that $H\not\equiv 0$. Then by *Lemmas 2.10 and 2.11* we obtain

$$\begin{aligned}
T(r,F)+T(r,G) \leq\ & 2\{N_2(r,0;F)+N_2(r,\infty;F)+ && (3.9)\\
& +N_2(r,0;G)+N_2(r,\infty;G)\}\\
& +\overline{N}_{F\geq 3}(r,1;F\mid\ G\neq 1)+\overline{N}_{G\geq 3}(r,1;G\mid\ F\neq 1)\\
& +S(r,F)+S(r,G).
\end{aligned}$$

By *Lemmas 2.6*, *2.15* and *2.18* we see that

$$\begin{aligned}(n-1)[T(r,f)+T(r,g)] \quad &\leq \quad \frac{9}{2}\overline{N}(r,0;f)+\frac{3}{2}N(r,0;f^{'}) \\ &\quad +\frac{7}{2}\overline{N}(r,\infty;f)+\frac{9}{2}\overline{N}(r,0;g)+\frac{3}{2}N(r,0;g^{'}) \\ &\quad +\frac{7}{2}\overline{N}(r,\infty;g)+S(r,f)+S(r,g) \\ &\leq \quad 6T(r,f)+5\overline{N}(r,\infty;f)+6T(r,g) \\ &\quad +5\overline{N}(r,\infty;g)+S(r,f)+S(r,g).\end{aligned} \tag{3.10}$$

Let us choose

$$0<\varepsilon<n-12+5\min\{\Theta(\infty;f),\Theta(\infty;g)\}.$$

Then from (3.10) we obtain

$$\begin{aligned}(n-12+5\Theta(\infty;f)-\varepsilon)T(r,f)+(n-12+5\Theta(\infty;g)-\varepsilon)T(r,g) \\ \leq S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction. Hence $H\equiv 0$. The rest of the proof can be carried out in the line of the proof of *Theorem 1.1* $\square$

Proof: [Proof of Theorem 1.3] Let F and G be defined as in *Lemma 2.15*. Since $f^{n}f^{'}$ and $g^{n}g^{'}$ share $(a,2)^{*}$, it follows that F and G share $(1,2)^{*}$. If possible, we suppose that $H\not\equiv 0$. Then by *Lemma 2.12* we obtain

$$\begin{aligned}T(r,F) \quad &\leq \quad N_{2}(r,0;F)+N_{2}(r,\infty;F)+N_{2}(r,0;G)+N_{2}(r,\infty;G) \\ &\quad +\overline{N}(r,0:F)+\overline{N}(r,\infty;F)+S(r) \\ &\leq \quad 3\overline{N}(r,0;f)+2N(r,0;f^{'})+3\overline{N}(r;\infty;f) \\ &\quad +2\overline{N}(r,0;g)+N(r,0;g^{'})+2\overline{N}(r,\infty;g)+S(r)\end{aligned} \tag{3.11}$$

So by *Lemmas 2.15* and *2.18* we get from (3.11) for $\varepsilon(>0)$

$$\begin{aligned}(n-1)T(r,f) \quad &\leq \quad 3\overline{N}(r,0;f)+N(r,0;f)+3\overline{N}(r,\infty;f) \\ &\quad +2\overline{N}(r,0;g)+N(r,0;g)+3\overline{N}(r,\infty;g)+S(r) \\ &\leq \quad 4T(r,f)+3T(r,g)+3(1-\Theta(\infty;f)+\varepsilon)T(r,f) \\ &\quad +3(1-\Theta(\infty;g)+\varepsilon)T(r,g)+S(r) \\ &\leq \quad \{13-3\Theta(\infty;f)-3\Theta(\infty;g)+2\varepsilon\}T(r)+S(r)\end{aligned} \tag{3.12}$$

Similarly we can obtain

$$(n-1)T(r,g)\leq\{13-3\Theta(\infty;f)-3\Theta(\infty;g)+2\varepsilon\}T(r)+S(r) \tag{3.13}$$

Combining (3.12) and (3.13) we get

$$[n-14+3\Theta(\infty;f)+3\Theta(\infty;g)-2\varepsilon\}T(r)+S(r). \tag{3.14}$$

Since $\varepsilon(>0)$ is arbitrary (3.14) leads to a contradiction. So $H \equiv 0$. Now proceeding in the same way as done in *Theorem 1.1* the proof of the theorem can be carried out . ∎

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