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Monomial ideals with linear quotients whose Taylor resolutions are minimal

by

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Abstract

We study when Taylor resolutions of monomial ideals are minimal, particularly for ideals with linear quotients.

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Introduction

Let $S = K[X_1, \ldots, X_n]$ be a polynomial ring over a field K and consider a monomial ideal $I \subset S$. Let $G(I) = \{u_1, \ldots, u_r\}$ be the minimal set of monomial generators of I. Then the Taylor resolution $(T_\bullet(I), d_\bullet)$ of I is defined as follows (cf. [1] Exer. 17.11): $T_q(I) = \bigwedge^{q+1} L$ for $q = 0, \ldots, r-1$ where L is the S-free module with the basis $\{e_1, \ldots, e_r\}$ and $d_q : T_q(I) \longrightarrow T_{q-1}(I)$, for $q = 1, \ldots, r-1$, is defined by

$$d_q(e_{i_0} \wedge \cdots \wedge e_{i_q}) = \sum_{s=1}^{q} (-1)^s \frac{\mathrm{lcm}(u_{i_0}, \ldots, u_{i_q})}{\mathrm{lcm}(u_{i_0}, \ldots, \check{u}_{i_s}, \ldots u_{i_q})} e_{i_0} \wedge \cdots \overset{i_s}{\vee} \cdots \wedge e_{i_q}$$

and the augumentation $\epsilon : T_0 \longrightarrow I$ is defined by $\epsilon(e_i) = u_i$ for $i = 1, \ldots, r$. In general, $T_\bullet(I)$ is far from minimal and the aim of this paper is to determine some of the cases in which this resolution is minimal.

A monomial ideal $I \subset S$ is said to be an *ideal with linear quotients* if, for some specified order $u_1, \ldots, u_r$ of the minimal set of generators, the colon ideals $(u_1, \ldots, u_{j-1}) : u_j$ are generated by a subset of $\{X_1, \ldots, X_n\}$, for $j = 1, \ldots, r$. When we consider such an ideal $I = (u_1, \ldots, u_r)$, we will always assume that I has linear quotients with this order of the minimal set of generators $u_1, \ldots, u_r$. We also set $\mathrm{set}(u_j) = \{i_1, \ldots, i_s\}$ when $(u_1, \ldots, u_{j-1}) : u_j = (X_{i_1}, \ldots, X_{i_s})$.

Stable ideals, squarefree stable ideals and (poly)matroidal ideals are all ideals with linear quotients and they have Eliahou-Kervaire type minimal resolutions [4].

We will show that an ideal $I = (u_1, \dots, u_r)$ with linear quotients has a minimal Taylor resolution if and only if $|\mathrm{set}(u_i)| = i - 1$ for $i = 1, \dots, r$ (Theorem 1.1), where $|A|$ denotes the cardinality of the set A. In the case of stable ideal, this is precisely when $1 \leq r \leq n$ and u_i are in the form of $u_i = X_i(\prod_{k=1}^{i} X_k^{n_k})$, $i = 1, \dots, r$, for some integers $n_1, \dots, n_r \geq 0$ (Theorem 2.2). On the other hand, for a monomial ideal $I \subset S$ with a linear resolution, I has the minimal Taylor resolution precisely when I is in the form of $I = u \cdot (X_{i_1}, \dots, X_{i_k})$, where u is a monomial and $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ (Theorem 3.1). Such an ideal also has linear quotients. We also give several examples such as matroidal ideals and squarefree stable ideal having minimal Taylor resolutions.

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1 Ideal with linear quotients

This section recalls some general facts on ideal with linear quotients and give a condition for such ideals to have the minimal Taylor resolutions.

Lemma 1. *Let $I = (u_1, \dots, u_r)$ be a monomial ideal with linear quotients. Then,*

$$|\mathrm{set}(u_i)| \leq i - 1 \qquad \text{for } i = 1, \dots, r$$

Proof: Since $\mathrm{set}(u_i) = \{j \mid X_j \in \bigcup_{k=1}^{i-1}(u_k) : u_i\} = \bigcup_{k=1}^{i-1}\{j \mid X_j \in (u_k) : u_i\}$ and each $(u_k) : u_i$ is generated by a single variable, we obtain the desired result. ∎

Lemma 2 (cf. lemma 1.5 [4]). *Let $I = (u_1, \dots, u_r)$ be a monomial ideal with linear quotients and assume that we have $\deg u_1 \leq \dots \leq \deg u_r$. Then the Betti numbers $\beta_q(I)$ of I are as follows:*

$$\beta_q(I) = \sum_{u \in G(I)} \binom{|\mathrm{set}(u)|}{q} \quad \textit{for all } q \geq 0.$$

Remark 1. Recall that a monomial ideal $I \subset S$ is stable if, for an arbitrary monomial $w \in I$, we have $X_i w / X_{m(w)} \in I$ for all $i < m(w)$ where $m(u) = \max\{j \mid X_j \text{ divides } u\}$. If $I = (u_1, \dots, u_r)$ is stable, we have

$$\mathrm{set}(u_i) = \{1, \dots, m(u_i) - 1\}$$

for $i = 1, \dots, r$ if $\deg u_1 \leq \dots \leq \deg u_r$ and $u_i > u_{i+1}$ by reverse lexicographical order if $\deg u_i = \deg u_{i+1}$. Then we can recover the well-known Eliahou-Kervaire formula [2] from Lemma 2.

Theorem 1.1. *Let $I = (u_1, \dots, u_r)$ be a monomial ideal with linear quotients and assume that we have $\deg u_1 \le \dots \le \deg u_r$. Then I has the minimal Taylor resolution if and only if $|\mathrm{set}(u_i)| = i - 1$ for $i = 1, \dots, r$.*

Proof: We have

$$\beta_q(I) \le \sum_{i=1}^{r} \binom{i-1}{q} \quad \text{for all } q \ge 0 \tag{1}$$

by Lemma 1 and 2. On the other hand, the Taylor resolution $T_\bullet(I)$ is minimal if and only if $\beta_q(I) = \binom{r}{q+1} = \sum_{i=1}^{r} \binom{i-1}{q}$. Thus the inequality in (1) must be equality, which implies $|\mathrm{set}(u_i)| = i - 1$ for all i by Lemma 1. □

2 Stable ideals having the minimal Taylor resolutions

The goal of this section is to determine precisely the stable ideals that have the minimal Taylor resolutions.

We first prepare a formal characterization of such ideals.

Proposition 2.1. *Let I be a stable ideal of S. Then the following conditions are equivalent :*

(i) I has the minimal Taylor resolution;

(ii) $\max\{m(u) \mid u \in G(I)\} = |G(I)|$;

(iii) $m_i(I) = \begin{cases} 1 & \text{for } 1 \le i \le |G(I)| \\ 0 & \text{for } |G(I)| < i \le n, \end{cases}$

where we define $m_i(I) = |\{u \in G(I) \mid m(u) = i\}|$,

Proof: We first show that $m_i(I) \ge 1$ for $i = 1, \dots, b_0$, where

$$b_0 := \max\{m(u) \mid u \in G(I)\}.$$

Suppose $m_i(I) \ge 1$ for all $i > j$ but $m_j(I) = 0$ for some $j \ge 1$. Then there exists $v \in G(I)$ such that $v = wX_{j+1}^{\alpha}$ with $m(w) < j + 1$ and $\alpha > 0$. Since I is stable, we have $v' = wX_j^{\alpha} \in I$ and there exists $u \in G(I)$ that divides v'. As $m_j(I) = 0$, we see that u divides w. This implies u divides v, which is a contradiction since $u, v \in G(I)$. Thus $m_i(I) \ge 1$ for $1 \le i \le b_0$ and we have $b_0 \le |G(I)|$.

Now we show (i) $\Rightarrow$ (ii) : Assume that I has the minimal Taylor resolution. Since I is stable, we have by Theorem 1.1 (see also Remark 1) $m(u_i) = i$ for $i = 1, \dots, r$. Thus we have $b_0 = |G(I)|$. (ii) $\Rightarrow$ (iii) is clear from the inequality $b_0 \le |G(I)|$. Finally we show (iii) $\Rightarrow$ (i) : By (iii), we have $G(I) = \{u_1, \dots, u_r\}$ with $m(u_i) = i$ for $i = 1, \dots, r$. We claim that, with this order of the generators, I is an ideal with linear quotients with $\mathrm{set}(u_i) = \{1, \dots, m(u_i) - 1\}$. Since I is stable, $m(u_i) = i$ for $i = 1, \dots, r$ implies that $u_i > u_{i+1}$ by reverse lexicographical order when $\deg u_i = \deg u_{i+1}$. Thus, to prove the claim we have only to show that $\deg u_i \leq \deg u_{i+1}$ for all i (see Remark 1). We can set $u_i = vX_i^p$ and $u_{i+1} = v'X_{i+1}^q$ for some monomials v and v' with $m(v) < i$ and $m(v') < i+1$ and integers $p, q > 0$. Then, since I is stable, we have $w := v'X_i^q \in I$ so that there exists $u_j \in G(I)$ that divides w. In particular, $m(u_j) \leq i$ and $\deg u_j \leq \deg w = \deg u_{i+1}$. If $m(u_j) < i$, then u_j divides v', which implies that u_j divides u_{i+1}, a contradiction. Thus $m(u_j) = i$ so that by (iii) we must have $u_j = u_i$. Then $\deg u_i \leq \deg u_{i+1}$ as required. Now we have $|\mathrm{set}(u_i)| = i - 1$, so that I has the minimal Taylor resolution by Theorem 1.1. □

Using above proposition we can determine the stable ideals with the minimal Taylor resolutions.

Theorem 2.2. *Let $I \subset S$ be a stable ideal. Then I has the minimal Taylor resolution if and only if it is in the following form: $I = (u_1, \dots, u_r)$ for some $r \leq n$, where*

$$u_i = X_i \cdot \prod_{k=1}^{i} X_k^{a_k} \quad (i = 1, \dots, r)$$

for some integers $a_1, \dots, a_r \geq 0$.

Proof: We can easily check that an ideal in the above form is stable and it has the minimal Taylor resolution by Proposition 2.1. Now we show the converse. By Proposition 2.1 we can assume that $G(I) = \{u_1, \dots, u_r\}$ with $m(u_i) = i$ for $i = 1, \dots, r$ and $r \leq n$. Thus we can write u_i as follows

$$u_i = X_i \cdot \prod_{k=1}^{i} X_k^{a_{i,k}} \quad (i = 1, \dots, r)$$

for some integers $a_{i,k} \geq 0$, $1 \leq i \leq r$ and $1 \leq k \leq i$. We will show that each $a_{i,k}$ is constant with regard to i.

Since I is stable, we have

$$w := u_r X_{r-1}/X_r = X_{r-1} \cdot \textstyle\prod_{k=1}^{r} X_k^{a_{r,k}} \in I,$$

so that there exists $u \in G(I)$ that divides w. We claim that $u = u_{r-1}$. In fact, we have $m(u) \leq r-1$ since $u \neq u_r$. If $m(u) < r-1$, then u divides $w/X_{r-1} = \prod_{k=1}^{r} X_k^{a_{r,k}}$, which implies that u divides u_r, contradicting the assumption that both u and u_r are from $G(I)$. Thus we must have $m(u) = r-1$, which implies $u = u_{r-1}$. Then we know that

$$a_{r-1,k} \leq a_{r,k} \quad \text{for all } 1 \leq k \leq r-1.$$

Thus, at least the exponents of X_1 to X_{r-2} in u_{r-1} are less than or equal to those of u_r. Moreover, if $a_{r-1,r-1} < a_{r,r-1}$, then the exponent $1 + a_{r-1,r-1}$ of X_{r-1}

in u_{r-1} is less than or equal to the exponent $a_{r,r-1}$ of X_{r-1} in u_r so that u_{r-1} divides u_r, a contradiction. Consequently, we must have $a_{r-1,r-1} = a_{r,r-1}$.

Now assume that there exists $1 \leq k \leq r-2$ such that $a_{r-1,k} < a_{r,k}$. Then since $w' := X_k u_{r-1}/X_{r-1} \in I$, there exists $u' \in G(I)$ dividing w' such that $u' \neq u_{r-1}, u_r$. Then we easily have $\mathrm{lcm}(u', u_{r-1}, u_r) = \mathrm{lcm}(u_{r-1}, u_r)$, which contradicts the minimality of the Taylor resolution. Consequently, we must have $a_{r-1,k} = a_{r,k}$ for all $1 \leq k \leq r-1$.

Now since $(u_1, \ldots, u_{r-1})$ is also stable and has the minimal Taylor resolution by Proposition 2.1, an inductive argument shows that for all $i = 1, \ldots, r$ and $j = 1, \ldots, r-1$ we have $a_{i,k} = a_{j,k}$ for all $1 \leq k \leq j$, which means that $u_1, \ldots, u_r$ are in the form stated above. □

Corollary 1. *Assume that $I \subset S$ is a stable ideal generated by monomials of the same degree $d > 1$. Then I has the minimal Taylor resolution if and only if I is in the form of $X_1^{d-1}(X_1, X_2, \ldots, X_r)$ for some $r \leq n$.*

3 Linear minimal Taylor resolutions

In this section, we consider non-stable cases of ideals with linear quotients.

Theorem 3.1. *Let I be a monomial ideal with a linear resolution. Then the following conditions are equivalent:*

(i) I has the minimal Taylor resolution:

(ii) $I = u \cdot (X_{i_1}, \ldots, X_{i_k})$ for some $1 \leq i_1 < \cdots < i_k \leq n$ and a monomial u.

In this case, I is an ideal with linear quotients.

Proof: It is clear that ideals in the form of (ii) have linear quotients. We only have to show (i) $\Rightarrow$ (ii), and the converse is clear. We prove by induction on $|G(I)|$. Let $G(I) = \{u_1, \ldots, u_r\}$ $(r \geq 2)$ and let $T_\bullet(I)$ be the linear minimal Taylor resolution. Since we have

$$\frac{\mathrm{lcm}(u_1, u_{i_1}, \ldots, u_{i_q})}{\mathrm{lcm}(u_1, u_{i_1}, \ldots, \check{u}_{i_s}, \ldots u_{i_q})} = \frac{\mathrm{lcm}(u_{i_1}, \ldots, u_{i_q})}{\mathrm{lcm}(u_{i_1}, \ldots, \check{u}_{i_s}, \ldots u_{i_q})}$$

for all $1 < i_1 < \ldots, i_q \leq r$, by truncating all the bases in the form of $e_1 \wedge \cdots$ from $T_\bullet(I)$, we obtain the linear minimal Taylor resolution of $J = (u_2, \ldots, u_r)$. Thus by the induction hypothesis we have $u_k = uX_{i_j}$, $k = 2, \ldots, r$, for some monomial u and $1 \leq i_1 < \cdots < i_j \leq n$. Now we show that u_1 is in the form of uX_{i_0} for some $i_0 \notin \{i_1, \ldots, i_j\}$. Since $T_\bullet(I)$ is linear, both $\mathrm{lcm}(u_1, u_i)/u_1$ and $\mathrm{lcm}(u_1, u_i)/u_i$ must be linear for $i = 2, \ldots, r$. Then we easily know that u must divide u_1 and conclusion follows. □

Remark 2. Notice that we can also give a more naive proof for Theorem 3.1 based on the observation that both $\mathrm{lcm}(u_1, u_i)/u_1$ and $\mathrm{lcm}(u_1, u_i)/u_i$ must be linear for $i = 2, \ldots, r$.

Now we show some examples produced by Theorem 3.1.

Example 1. *Let $I = X_1^p X_{k+1}^q (X_1, X_2, \ldots, X_k)$ with $1 < k \leq n$ and $p, q \geq 1$. Then I is a non-stable ideal with linear quotients whose Taylor resolution is minimal.*

Example 2. *A Stanley-Reisner ideal $I \subset S$ generated by squarefree monomials with the same degree is called matroidal if it satisfies the following exchange property: For all $u, v \in G(I)$ and all i with $\nu_i(u) > \nu_i(v)$, there exists an integer j with $\nu_j(u) < \nu_j(v)$ such that $X_j(u/X_i) \in G(I)$, where we define $\nu_i(u) = a_i$ for $u = X_1^{a_1} \cdots X_n^{a_n}$. A matroidal ideal I has a linear resolution (cf. [4]). If it has the minimal Taylor resolution, we know that I is in the form of*

$$I = X_{i_1} \cdots X_{i_p}(X_{j_1}, \ldots, X_{j_q})$$

for $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} = \emptyset$ with $p + q \leq n$.

Example 3. *A squarefree stable ideal is a Stanley-Reisner ideal $I \subset S$ satisfying the condition that, for all $i < m(u)$ such that X_i does not divide u, one has $X_i(u/X_{m(u)}) \in I$. Let I be a squarefree stable ideal generated by monomials with the same degree. If I has the minimal Taylor resolution, then I is in the form of $I = (u_1, \ldots, u_r)$ with $u_1 = vX_s$ and $u_i = vX_{p+i-1}$ for $i \geq 2$ where $v = X_1 \cdots \overset{s}{\vee} \cdots X_p$ for some $1 \leq s \leq p \leq n$.*

Remark 3. After finishing this work, one of the authors and his colleagues found that Proposition 2.1 holds in a more general setting. Namely, instead of a stable monomial ideal, we can assume $I \subset S$ to be a componentwise linear monomial ideal. Then it turns out that I is a Gotzmann ideal if its Taylor resolution is minimal. See Theorem 1.5 in [3].

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