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Sequentially Cohen-Macaulay monomial ideals of embedding dimension four

by
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Abstract

Let I be a monomial ideal of the polynomial ring $S = K[x_1, \ldots, x_4]$ over a field K. Then S/I is sequentially Cohen-Macaulay if and only if S/I is pretty clean. In particular, if S/I is sequentially Cohen-Macaulay then I is a Stanley ideal.

Key Words: Monomial Ideals, Prime Filtrations, Pretty Clean Filtrations, Stanley Ideals.
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Introduction

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and $I \subset S$ a monomial ideal. If S/I is Gorenstein of codimension three then a description of I is given in [1, Theorem 6.1] in terms of the minimal system of monomial generators. Here we are interested to describe monomial ideals I when $n = 4$ and S/I is Cohen-Macaulay of codimension two in terms of the primary decomposition of I. As a consequence we get a particular form of [4, Proposition 1.4] for $n = 4$, which says that if S/I is Cohen-Macaulay of codimension two then S/I is *clean*, that is (after [3]) there exists a prime filtration $I = F_0 \subset F_1 \subset \ldots \subset F_r = S$ of monomial ideals such that $F_i/F_{i-1} \cong (S/P_i)(a_i)$ for some prime ideals P_i of S with $ht(P_i) = \dim(S/I)$ and $a_i \in \mathbb{Z}$, $i = 1, \ldots, r$.

More general, given a monomial ideal I of S then S/I is called *pretty clean* after [5] if there exists a prime filtration $I \subset F_1 \subset \ldots \subset F_r = S$ of monomial ideals such that $F_i/F_{i-1} \cong S/P_i(a_i)$ for some prime ideals P_i of S with the property that $P_i \subset P_j$ and $i \leq j$ implies $P_i = P_j$, that is, roughly speaking, "bigger primes come first" in the filtration. [5, Corollary 4.3] says that if S/I is pretty clean then S/I is sequentially Cohen-Macaulay, that is the non-zero factors of the dimension filtration of [8] (see next section) are Cohen-Macaulay.

Our Theorem 1.3 says that for $n = 4$ it is true also the converse, namely that if S/I is sequentially Cohen-Macaulay then S/I is pretty clean.

A decomposition of S/I as a direct sum of linear K-spaces of the form $S/I = \oplus_{i=1}^r u_i K[Z_i]$, where u_i are monomials of S and $Z_i \subset \{x_1, \ldots, x_n\}$ are subsets, is called a *Stanley decomposition.* Stanley [10] conjectured that there always exists such a decomposition such that $|Z_i| \geq \operatorname{depth}(S/I)$. If Stanley conjecture holds for S/I then I is called a *Stanley ideal.* Our Corollary 1.4 says that if $n = 4$, I is monomial and S/I is sequentially Cohen-Macaulay then I is a Stanley ideal (this follows because I is a Stanley ideal whenever S/I is pretty clean as says [5, Theorem 6.5]).

1 Sequentially Cohen-Macaulay monomial ideals of embedding dimension four are pretty clean

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. The following result [4, Proposition 1.4] is essential in this section.

Theorem 1.1 (Herzog-Soleyman Jahan-Yassemi). *Let $I \subset S$ be a monomial ideal of height two such that S/I is Cohen-Macaulay. Then S/I is clean.*

The proof of Herzog, Soleyman Jahan and Yassemi passes the problem to the polarization, where they could use strong tools from simplicial complex theory. In the next section we give a direct proof in the case $n = 4$, which uses just elementary theory of monomial ideals. With this occasion we give also a complete description of all monomials ideals I of height 2 in the case $n = 4$ with S/I Cohen-Macaulay. The conditions given in this description are sometimes difficult but they could easily give nice examples of monomial ideals I with S/I not Cohen-Macaulay, but having all associated primes of height 2 and with $S/\sqrt{I}$ Cohen-Macaulay (see Example 2.7). Certainly if S/I is Cohen-Macaulay then $S/\sqrt{I}$ is too by [6, Theorem 2.6] (this holds only for monomial ideals). We mention that special descriptions of some monomial Cohen-Macaulay ideals of codimension 2 are given in [6, Theorem 3.2].

Let $I \subset S$ be a monomial ideal and $I = \bigcap_{p \in \operatorname{Ass}(S/I)} P_p$, $\sqrt{P_p} = p$, an irredundant primary decomposition of I. Set $D_i(I) = \bigcap_{p \in \operatorname{Ass}^{>i}(S/I)} P_p$, for $-1 \leq i < n$, where $\operatorname{Ass}^{>i}(S/I) = \{p \in \operatorname{Ass}(S/I) : \dim(S/p) > i\}$. We get in this way the dimension filtration of S/I

$$I = D_{-1}(I) \subset D_0(I) \subset \ldots \subset D_{n-2}(I) \subset D_{n-1}(I) = S,$$

introduced by Schenzel [8] (n is the number of variables of S). S/I is *sequentially Cohen-Macaulay* if all non-zero factors of this filtration are Cohen-Macaulay. In the monomial case, the notions of "sequentially Cohen-Macaulay" and "pretty clean" are connected by the following result of [5, Corollary 4.3].

Theorem 1.2 (Herzog-Popescu). *Let $I \subset S$ be a monomial ideal and*

$$I = D_{-1}(I) \subset D_0(I) \subset \ldots \subset D_{n-2}(I) \subset D_{n-1}(I) = S$$

the dimension filtration of S/I. Then the following statements are equivalent:

1. *S/I is pretty clean,*
2. *S/I is sequentially Cohen-Macaulay and all non-zero factors of the dimension filtration are clean,*
3. *all non-zero factors of the dimension filtration are clean.*

From now on $S = K[x, y, z, w]$, that is the case $n = 4$. The above theorems are main tools in proving the following:

Theorem 1.3. *Let $I \subset S = K[x, y, z, w]$ be a monomial ideal. Then S/I is pretty clean if and only if I is sequentially Cohen-Macaulay.*

Proof: By Theorem 1.2 it is enough to show that the non-zero factors of the dimension filtration

$$I = D_{-1}(I) \subset D_0(I) \subset D_1(I) \subset D_2(I) \subset D_3(I) = S,$$

are clean if they are Cohen-Macaulay. Since S is factorial ring and $D_2(I)$ is an intersection of primary height one ideals we get $D_2(I) = (u)$ for a certain monomial $u \in S$. Clearly $S/(u)$ is clean (see e. g. [9, Lemma 1.9]). As $D_2(I)/D_1(I) \cong S/(D_1(I) : u)$ is Cohen-Macaulay of dimension 2 we get $D_2(I)/D_1(I)$ clean by Theorem 1.1. Now note that $D_1(I)/D_0(I)$ and $D_1(I)/I$ are clean by [7, Corollary 2.2] because the prime ideals associated to those modules are of height ≥ 3. □

Corollary 1.4. *Let $I \subset S = K[x, y, z, w]$ be a monomial ideal. If S/I is sequentially Cohen-Macaulay then I is a Stanley ideal.*

Proof: By the above theorem S/I is pretty clean and it is enough to apply [5, Theorem 6.5]. □

2 Proof of Theorem 1.1 in the case $n = 4$

Let K be a field and $S = K[x, y, z, w]$ be the polynomial ring in four variables. We denote $G(I)$ to be the set of minimal monomial generators for an ideal I in S. First next lemmas , which involve ideals generated in 3 variables are easy and contained somehow in [9], but we prove them for the sake of our completeness.

Lemma 2.1. *Let $I \subset S$ be a monomial ideal such that* $\operatorname{Ass}(S/I) = \{(x, y), (x, z)\}$. *Then S/I is clean.*

Proof: Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant decomposition of I in irreducible monomial ideals (see [11]). Let $Q_1 = (x^a, y^b)$ and $J = \bigcap_{i=2}^{s} Q_i$, where b is the maximum power of y, which enters in $G(Q_i)$. Then we have the filtration $I \subset (I, x^a) \subset Q_1 \subset S$.
Clearly S/Q_1 is clean. Apply induction on s. We have $Q_1/(I, x^a) \cong S/((I, x^a) : y^b)$. As $G((I, x^a) : y^b)$ contains only monomials in $\{x, z\}$ we see that $((I, x^a) : y^b)$ is primary because it is the intersection of those (Q_i, x^a) with $\sqrt{Q_i} = (x, z)$. Thus $Q_1/(I, x^a)$ is clean.
On the other hand $(I, x^a)/I \cong S/(I : x^a)$ and $(I : x^a) = \bigcap_{i=2}^{s} (Q_i : x^a)$. We are done by induction hypothesis on $s \geq 2$, case $s = 2$ being trivial since $(I : x^a)$ is irreducible. Thus $(I, x^a)/I$ is clean. □

Lemma 2.2. *Let $I \subset S$ be a monomial ideal such that*

$$\mathrm{Ass}(S/I) = \{(x, y), (x, z), (y, z)\}.$$

Then S/I is clean.

Proof: Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant decomposition of I in irreducible monomial ideals. Let $Q_1 = (x^a, y^b)$ and $J = \bigcap_{i=2}^{s} Q_i$, where b is the maximum such that y^b enter in $G(Q_i)$. Then we have the filtration $I \subset (I, x^a) \subset Q_1 \subset S$.
Clearly S/Q_1 is clean. Apply induction on s. We have $Q_1/(I, x^a) \cong S/((I, x^a) : y^b)$. As $G((I, x^a) : y^b)$ contains only monomials in $\{x, z\}$ we see that $((I, x^a) : y^b)$ is primary and its radical is (x, z). Thus $Q_1/(I, x^a)$ is clean.
On the other hand $S/(I : x^a) \cong (I, x^a)/I$ and $(I : x^a) = \bigcap_{i=2}^{s} (Q_i : x^a)$. We apply induction hypothesis on $s \geq 3$, $(I : x^a)$ being in the case $s = 3$ just an irreducible ideal. Thus $(I, x^a)/I$ is clean. □

Lemma 2.3. *Let $I \subset S$ be a monomial ideal such that* $\mathrm{Ass}(S/I) = \{(x, y), (z, w)\}$. *Then S/I is not Cohen-Macaulay.*

Proof: Let $I = P_1 \cap P_2$ be the irredundant decomposition of I in monomial primary ideals, let us say $\sqrt{P_1} = (x, y), \sqrt{P_2} = (z, w)$. Then $S/(P_1 + P_2)$ has dimension 0 and from the exact sequence $0 \to S/I \to S/P_1 \oplus S/P_2 \to S/(P_1 + P_2) \to 0$, we get $depth(S/I) = 1$ by Depth Lemma (see e. g. [2, Proposition 1.2.9]). Thus S/I is not Cohen-Macaulay. □

Remark 2.4. The above lemma is trivial when I is a reduced ideal because the simplicial complex associated to I is not connected and so not Cohen-Macaulay. If I is Cohen-Macaulay then $\sqrt{I}$ is too by [6, Theorem 2.6], which gives another proof of this lemma.

Lemma 2.5. *Let $I \subset S$ be a monomial ideal such that*

$$\mathrm{Ass}(S/I) = \{(x,y),(x,z),(x,w)\}$$

and let $I = P_1 \cap P_2 \cap P_3$ be the irredundant monomial primary decomposition of I, where $\sqrt{P_1} = (x,y)$, $\sqrt{P_2} = (x,z)$, $\sqrt{P_3} = (x,w)$. Then (S/I) is clean.

Proof: Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant monomial irreducible decomposition of I. Apply induction on s. If $s = 3$, then $(P_i)_i$ must be irreducible and so P_1 has the form (x^a, y^b). We consider the filtration
$I \subset (I, x^a) \subset (x^a, y^b) \subset S$. Note that $P_1/(I, x^a) \cong S/((I, x^a) : y^b)$. But $((I, x^a) : y^b) = (P_1 \cap (P_2, x^a) \cap ((P_3, x^a) : y^b)) = ((P_2, x^a) : y^b) \cap ((P_3, x^a) : y^b) = (P_2, x^a) \cap (P_3, x^a)$ is clean by Lemma 2.1. Thus $P_3/(I, x^a)$ is clean.
Now note that $(I, x^a)/I \cong S/(I : x^a)$. We have $(I : x^a) = (P_2 : x^a) \cap (P_3 : x^a)$ and so $S/(I : x^a)$ is clean by Lemma 2.1. Gluing together the clean filtrations obtained above we get a clean filtration of S/I for $s = 3$.
Assume $s > 3$. After renumbering Q_i we may suppose that $Q_1 = (x^a, y^b)$ for some a, b. Moreover we may suppose that b is the biggest power of y which can enter in $\bigcup_{i=1}^{s} G(Q_i)$. Consider the filtration as above $I \subset (I, x^a) \subset Q_1 = (x^a, y^b) \subset S$. We have $Q_1/(I, x^a) \cong S/((I, x^a) : y^b)$ and $(I, x^a) : y^b = (P_2, x^a) \cap (P_3, x^a)$ as above. Thus $Q_1/(I, x^a)$ is clean. Now note that $(I, x^a)/I \cong S/(I : x^a)$ and $(I : x^a) = \bigcap_{i=2}^{s} (Q_i : x^a)$ and $S/(I : x^a)$ is clean by induction hypothesis. As above gluing the obtained clean filtrations we get S/I clean. □

Lemma 2.6. *Let $I \subset S$ be a monomial ideal such that*

$$\mathrm{Ass}(S/I) = \{(x,y),(x,z),(z,w)\}$$

and let $I = P_1 \cap P_2 \cap P_3$ be the irredundant monomial primary decomposition of I, where $\sqrt{P_1} = (x,y)$, $\sqrt{P_2} = (x,z)$, $\sqrt{P_3} = (z,w)$. Then the following statements are equivalent:
i) S/I is clean.
ii) S/I is Cohen-Macaulay.
iii) $P_2 \subset P_1 + P_3$.

Proof: $i) \Rightarrow ii)$: By [5, Corollary 4.3], we get S/I sequentially Cohen-Macaulay. Since all primes from $\mathrm{Ass}(S/I)$ have the same dimension it follows that S/I is

Cohen-Macaulay.
$ii) \Rightarrow iii)$: Let $J = P_1 \cap P_2$. As in the proof of Lemma 2.2, from the exact sequence $0 \to S/I \to S/J \oplus S/P_3 \to S/(J+P_3) \to 0$, we get that $depth(S/I) = 1$ if $depth(S/(J+P_3)) = 0$. But $J+P_3 = (P_1+P_3) \cap (P_2+P_3)$ and P_1+P_3 is primary of height 4 and P_2+P_3 is primary of height 3. Thus $depth(S/(J+P_3)) = 0$ if and only if $P_2+P_3 \not\subset P_1+P_3$, that is $P_2 \not\subset P_1+P_3$. Therefore if $P_2 \not\subset P_1+P_3$ then S/I is not Cohen-Macaulay, which proves $ii) \Rightarrow iii)$.
$iii) \Rightarrow i)$: Suppose now that $iii)$ holds and let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant monomial irreducible decomposition of I. Apply induction on s. If $s = 3$, then $(P_i)_i$ must be irreducible and so P_3 has the form (z^r, w^t). We consider the filtration
$I \subset (I, z^r) \subset (z^r, w^t) \subset S$. Note that $P_3/(I, z^r) \cong S/((I, z^r) : w^t)$. But $((I, z^r) : w^t) = (((P_1, z^r) \cap (P_2, z^r) \cap P_3) : w^t) = ((P_1, z^r) : w^t) \cap ((P_2, z^r) : w^t) = (P_1, z^r) \cap (P_2, z^r)$. As $P_2 \subset P_1 + P_3$ it follows that $P_2 \subset (P_1, z^r)$ and so $(I, z^r) : w^t = (P_2, z^r)$ which is primary with $\sqrt{(P_2, z^r)} = (x, z)$. Thus $P_3/(I, z^r)$ is clean. Now note that $(I, z^r)/I \cong S/(I : z^r)$. We have $(I : z^r) = (P_1 : z^r) \cap (P_2 : z^r)$ and so $S/(I : z^r)$ is clean by Lemma 2.1. Gluing together the clean filtrations obtained above we get a clean filtration of S/I, that is $iii) \Rightarrow i)$ for $s = 3$.

Assume $s > 3$. After renumbering Q_i we may suppose that $Q_1 = (z^r, w^t)$ for some r, t. Moreover we may suppose that t is the biggest power of w which can enter in $\bigcup_{i=1}^{s} G(Q_i)$. Consider the filtration as above $I \subset (I, z^r) \subset Q_1 = (z^r, w^t) \subset S$. We have $Q_1/(I, z^r) \cong S/((I, z^r) : w^t)$ and $((I, z^r) : w^t) = ((P_1, z^r) : w^t) \cap ((P_2, z^r) : w^t) \cap ((P_3, z^r) : w^t) = (P_1, z^r) \cap (P_2, z^r) = (P_2, z^r)$ as above. Thus $Q_1/(I, z^r)$ is clean. Now note that $(I, z^r)/I \cong S/(I : z^r)$ and $(I : z^r) = \bigcap_{i=2}^{s} (Q_i : z^r)$ and we apply the induction hypothesis for $(I : z^r)$ if we see that $iii)$ holds for it. Clearly $iii)$ implies $(P_2 : z^r) \subset (P_1 : z^r) + (P_3 : z^r)$ which is enough (note that $(P_3 : z^r)$ can be a proper ideal in this case). As above gluing the obtained clean filtrations we get S/I clean. □

Example 2.7. Let $I = (x^2, y) \cap (x, z) \cap (z, w)$. Then S/I is not Cohen-Macaulay by the above lemma, but $S/\sqrt{I}$ is Cohen-Macaulay, because the simplicial complex associated to $\sqrt{I}$ is shellable.

Lemma 2.8. *Let $I \subset S$ be a monomial ideal such that*

$$\mathrm{Ass}(S/I) = \{(x, y), (x, w), (y, w), (x, z)\}$$

and let $I = P_1 \cap P_2 \cap P_3 \cap P_4$ be the irredundant monomial primary decomposition of I, where $\sqrt{P_1} = (x, y)$, $\sqrt{P_2} = (x, w)$, $\sqrt{P_3} = (y, w)$, $\sqrt{P_4} = (x, z)$. Then the following statements are equivalent:
i) S/I is clean.
ii) S/I is Cohen-Macaulay.
iii) $P_1 \subset P_3 + P_4$ or $P_2 \subset P_3 + P_4$.

Proof: $i) \Rightarrow ii)$ as in Lemma 2.6.
$ii) \Rightarrow iii)$: Let $J = P_1 \cap P_2 \cap P_4$. From the exact sequence $0 \to S/I \to S/J \oplus S/P_3 \to S/(J+P_3) \to 0$, we get that $depth(S/I) = 1$ if $depth(S/J+P_3) = 0$. But $(J+P_3) = (P_1+P_3) \cap (P_2+P_3) \cap (P_4+P_3)$, where (P_4+P_3) is primary of height 4 and $(P_1+P_3), (P_2+P_3)$ are primary of height 3. Thus $depth(S/(J+P_3)) = 0$ if and only if $P_1+P_3 \not\subset P_4+P_3$ and $P_2+P_3 \not\subset P_4+P_3$, that is $P_1 \not\subset P_4+P_3$ and $P_2 \not\subset P_4+P_3$. Therefore if $P_1 \not\subset P_4+P_3$ and $P_2 \not\subset P_4+P_3$ then S/I is not Cohen-Macaulay, which proves $ii) \Rightarrow iii)$.
$iii) \Rightarrow i)$: Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant monomial irreducible decomposition of I. Applying induction on s. If $s = 4$, then (P_i) must be irreducible and so P_1 has the form (x^a, y^b). Let $iii)$ holds, let us say $P_1 \subset P_3 + P_4$. Consider the filtration $I \subset (I, x^a) \subset (x^a, y^b) \subset S$. Note that $P_1/(I, x^a) \cong S/((I, x^a) : y^b)$. But $((I, x^a) : y^b) = (P_1 \cap (P_2, x^a) \cap (P_3, x^a) \cap ((P_4, x^a) : y^b) = ((P_2, x^a) : y^b) \cap ((P_3, x^a) : y^b) \cap ((P_4, x^a) : y^b) = (P_2, x^a) \cap ((P_3, x^a) : y^b) \cap (P_4, x^a)$.
As $P_1 \subset P_4 + P_3$ it follows that b is the biggest power of y appearing in

$$\{G(P_1), G(P_3)\}$$

and so $(I, x^a) : y^b$ is generated by the variables in x, z, w only, and hence clean by Lemma 2.1.
Now note that $(I, x^a)/I \cong S/(I : x^a)$. We have $(I : x^a) = (P_2 : x^a) \cap (P_3 : x^a) \cap (P_4 : x^a)$, again since by hypothesis $(I : x^a) = P_2 \cap P_3$, and so $S/(I : x^a)$ is clean by Lemma 2.2. Gluing together the filtration described above we get a clean filtration of S/I.
Similarly, if $P_2 \subset P_3 + P_4$, and $P_2 = (x^n, w^p)$, then the filtration $I \subset (I, x^n) \subset (x^n, w^p) \subset S$ is refined to a clean one. That is $iii) \Rightarrow i)$ for $s = 4$.
Assume $s > 4$. After renumbering Q_i we may suppose that $Q_1 = (x^a, y^b)$ for some a, b. Moreover we may suppose that b is the biggest power of y which can enter in $G(Q_i)$ with $\sqrt{Q_i} = (x, y)$. Consider the filtration as above $I \subset (I, x^a) \subset Q_1 = (x^a, y^b) \subset S$. We have $Q_1/(I, x^a) \cong S/((I, x^a) : y^b)$ and $(I, x^a) : y^b = (P_2, x^a) \cap ((P_3, x^a) : y^b) \cap (P_4, x^a)$. As $P_1 \subset P_3 + P_4$ it follows that b is the biggest power of y, which appear in $G(P_3)$. Thus $(I, x^a) : y^b = (P_2, x^a) \cap (P_4, x^a)$ and so $Q_1/(I, x^a)$ is clean by Lemma 2.1. Now note that $(I, x^a)/I \cong S/(I : x^a)$ and $(I : x^a) = \bigcap_{i=2}^{s} (Q_i : x^a)$ and we apply the induction hypothesis for $(I : x^a)$ if we see that $iii)$ holds for it. Clearly $iii)$ implies $(P_1 : x^a) \subset (P_3 : x^a) + (P_4 : x^a)$ which is enough. As above gluing the described clean filtration we get S/I clean. Similarly for $P_2 \subset P_3 + P_4$, choosing $Q_1 = (x^n, w^p)$, we complete the proof as above. □

Lemma 2.9. *Let $I \subset S$ be a monomial ideal such that*

$$\mathrm{Ass}(S/I) = \{(x, y), (x, z), (z, w), (y, w)\}$$

and let $I = P_1 \cap P_2 \cap P_3 \cap P_4$ be the irredundant monomial primary decomposition of I, where $\sqrt{P_1} = (x,y)$, $\sqrt{P_2} = (x,z)$, $\sqrt{P_3} = (z,w)$, $\sqrt{P_4} = (y,w)$. Then the following statements are equivalent:
i) S/I is clean.
ii) S/I is Cohen-Macaulay.
iii) $\{P_1 \subset P_2 + P_4$ or $P_3 \subset P_2 + P_4\}$ and $\{P_2 \subset P_1 + P_3$ or $P_4 \subset P_1 + P_3\}$.

Proof: $i) \Rightarrow ii)$ as in Lemma 2.6.
$ii) \Rightarrow iii)$: Let $J = P_1 \cap P_2 \cap P_3$. From the exact sequence $0 \to S/I \to S/J \oplus S/P_4 \to S/(J+P_4) \to 0$, we get that $depth(S/I) = 1$ if $depth(S/J+P_4) = 0$. But $(J+P_4) = (P_1+P_4) \cap (P_2+P_4) \cap (P_4+P_3)$, where (P_2+P_4) is primary of height 4 and $(P_1+P_4), (P_3+P_4)$ are primary of height 3. Thus $depth(S/J+P_4) = 0$ if and only if $P_1+P_4 \not\subset P_2+P_4$ and $P_3+P_4 \not\subset P_2+P_4$, that is $P_1 \not\subset P_2+P_4$ and $P_3 \not\subset P_2+P_4$. Therefore if $P_1 \not\subset P_2+P_4$ and $P_3 \not\subset P_2+P_4$ then S/I is not Cohen-Macaulay.
On the other hand if $J = P_1 \cap P_2 \cap P_4$ then the exact sequence $0 \to S/I \to S/J \oplus S/P_3 \to S/(J+P_3) \to 0$ gives the other conditions i.e. $P_2 \subset P_1+P_3$ or $P_4 \subset P_1+P_3$. Remaining choices for J, are equivalent to these two cases, which proves $ii) \Rightarrow iii)$.
$iii) \Rightarrow i)$: Suppose now that $iii)$ holds, let us say $\{P_1 \subset P_2+P_4, P_2 \subset P_1+P_3\}$ holds. Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant monomial irreducible decomposition of I. Apply induction on s. If $s = 4$, then (P_i) must be irreducible and so P_1 has the form (x^a, y^b). We consider the filtration
$I \subset (I, x^a) \subset (x^a, y^b) \subset S$. Note that $P_1/(I, x^a) \cong S/((I, x^a) : y^b)$. But $(I, x^a) : y^b = (P_1 \cap (P_2, x^a) \cap (P_3, x^a) \cap (P_4, x^a)) : y^b = ((P_2, x^a) : y^b) \cap ((P_3, x^a) : y^b) \cap ((P_4, x^a) : y^b) = (P_2, x^a) \cap (P_3, x^a) \cap ((P_4, x^a) : y^b)$.
As $P_1 \subset P_2+P_4$, so b is the biggest power of y in $\{G(P_1), G(P_4)\}$. It follows that $(I, x^a) : y^b = (P_2, x^a) \cap (P_3, x^a)$. Since $P_2 \subset P_1+P_3$ it follows that $(P_2, x^a) = P_2 \subset (P_3, x^a)$. Thus $(I, x^a) : y^b$ is primary and so clean.
Now note that $(I, x^a)/I \cong S/(I : x^a)$. We have $(I : x^a) = (P_2 : x^a) \cap (P_3 : x^a) \cap (P_4 : x^a)$. As above a is the biggest power of x in $G(P_2)$ because $P_1 \subset P_2+P_4$. Thus $I : x^a = P_3 \cap P_4$ and so $S/(I : x^a)$ is clean by again Lemma 2.1. Gluing together the clean filtrations obtained above we get a clean filtration of S/I, that is when $s = 4$, then $i)$ holds for $\{P_1 \subset P_2+P_4, P_2 \subset P_1+P_3\}$.
Assume $s > 4$. After renumbering Q_i we may suppose that $Q_1 = (x^a, y^b)$ for some a, b. Moreover we may suppose that b is the biggest power of y which can enter in $G(Q_i)$ with $\sqrt{Q_i} = (x,y)$. Consider the filtration as above $I \subset (I, x^a) \subset Q_1 = (x^a, y^b) \subset S$. We have $Q_1/(I, x^a) \cong S/((I, x^a) : y^b)$ and $(I, x^a) : y^b = (P_2, x^a) \cap (P_3, x^a) \cap ((P_4, x^a) : y^b) = (P_2, x^a) \cap (P_3, x^a)$ as above because $P_1 \subset P_2+P_4$. Since $P_2 \subset P_1+P_3$ we see that $(P_2, x^a) \subset (P_3, x^a)$ and so $(I, x^a) : y^b = (P_2, x^a)$ is primary. Thus $Q_1/(I, x^a)$ is clean. Now note that $(I, x^a)/I \cong S/(I : x^a)$ and $(I : x^a) = \bigcap_{i=2}^{s} (Q_i : x^a)$ and we apply the induction hypothesis for $(I : x^a)$ if we see that $(P_1 : x^a) \subset (P_2 : x^a) + (P_4 : x^a)$ and

$(P_2 : x^a) \subset (P_1 : x^a) + (P_3 : x^a)$ which is clear. As above gluing the obtained clean filtrations we get S/I clean.
Other cases from $iii)$, i.e. $\{P_1 \subset P_2 + P_4, P_4 \subset P_1 + P_3\}$, $\{P_3 \subset P_2 + P_4, P_2 \subset P_1 + P_3\}$ and $\{P_3 \subset P_2 + P_4, P_4 \subset P_1 + P_3\}$ are similar. □

Lemma 2.10. *Let $I \subset S$ be a monomial ideal such that*

$$\mathrm{Ass}(S/I) = \{(x,y),(x,z),(z,w),(y,w),(y,z)\}$$

and let $I = P_1 \cap P_2 \cap P_3 \cap P_4 \cap P_5$ be the irredundant monomial primary decomposition of I, where $\sqrt{P_1} = (x,y)$, $\sqrt{P_2} = (x,z)$, $\sqrt{P_3} = (z,w)$, $\sqrt{P_4} = (y,w)$, $\sqrt{P_5} = (y,z)$. Then the following statements are equivalent:
i) S/I is clean.
ii) S/I is Cohen-Macaulay.
iii) $\{P_1 \subset P_2 + P_4$ or $P_3 \subset P_2 + P_4$ or $P_5 \subset P_2 + P_4\}$ and $\{P_2 \subset P_1 + P_3$ or $P_4 \subset P_1 + P_3$ or $P_5 \subset P_1 + P_3\}$.

Proof: $i) \Rightarrow ii)$ as in Lemma 2.6.
$ii) \Rightarrow iii)$: Let $J = P_1 \cap P_2 \cap P_3 \cap P_5$. From the exact sequence $0 \to S/I \to S/J \oplus S/P_4 \to S/(J+P_4) \to 0$, we get that $depth(S/I) = 1$ if $depth(S/J+P_4) = 0$. But $(J+P_4) = (P_1+P_4) \cap (P_2+P_4) \cap (P_3+P_4) \cap (P_5+P_4)$, where (P_2+P_4) is primary of height 4 and $(P_1+P_4), (P_3+P_4), (P_4+P_5)$ are primary of height 3. Thus $depth(S/J+P_4) = 0$ if and only if $P_1+P_4 \not\subset P_2+P_4$ and $P_3+P_4 \not\subset P_2+P_4$ and $P_3+P_4 \not\subset P_5+P_4$, that is $P_1 \not\subset P_2+P_4$ and $P_3 \not\subset P_2+P_4$ and $P_5 \not\subset P_2+P_4$. Therefore if $P_1 \not\subset P_2+P_4$ and $P_3 \not\subset P_2+P_4$ and $P_5 \not\subset P_2+P_4$ then S/I is not Cohen-Macaulay.
On the other hand if $J = P_1 \cap P_2 \cap P_4 \cap P_5$ then the exact sequence $0 \to S/I \to S/J \oplus S/P_3 \to S/(J+P_3) \to 0$ gives the other conditions i.e. $P_2 \subset P_1 + P_3$ or $P_4 \subset P_1 + P_3$ or $P_5 \subset P_1 + P_3$. Remaining choices for J, are equivalent to these two cases, which proves $ii) \Rightarrow iii)$.
$iii) \Rightarrow i)$: Suppose now that $iii)$ holds, let us say $\{P_1 \subset P_2 + P_4, P_2 \subset P_1 + P_3\}$ holds. Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant monomial irreducible decomposition of I. Apply induction on s. If $s = 5$, then (P_i) must be irreducible and so P_1 has the form (x^a, y^b).
Here we can suppose b to be the biggest power of y in $\{P_1, P_4\}$ because $P_1 \subset P_2 + P_4$. If $y^b \in G(P_5)$ then we consider the filtration $I \subset (I, x^a) \subset (x^a, y^b) \subset S$. Note that $P_1/(I, x^a) \cong S/((I, x^a) : y^b)$. But $((I, x^a) : y^b) = (P_2, x^a) \cap (P_3, x^a) = (P_2, x^a)$ because $P_2 \subset P_1 + P_3$. Thus $P_1/(I, x^a)$ is clean. Also note that $(I, x^a)/I \cong S/(I : x^a)$ and $I : x^a = P_3 \cap P_4 \cap P_5$ because $P_1 \subset P_2 + P_4$. Thus $(I, x^a)/I$ is clean by Lemma 2.2. If $y^b \notin G(P_5)$ then let $P_5 = (y^r, z^t)$ and we consider the filtration $I \subset (I, z^t) \subset (y^r, z^t) \subset S$. As above we have $P_5/(I, z^t) \cong S/((I, z^t) : y^r)$ and $((I, z^t) : y^r) = (P_2, z^t) \cap (P_3, z^t)$. Thus $P_5/(I, z^t)$ is clean by Lemma 2.1. Also note that $(I, z^t)/I \cong S/(I : z^t)$. Since $I : z^t =$

$P_1 \cap (P_2 : z^t) \cap (P_3 : z^t) \cap P_4$ we see that $(I, z^t)/I$ is clean by Lemma 2.9. Gluing together the clean filtrations obtained above we get a clean filtration of S/I, that is when $s = 5$, then $i)$ holds for $\{P_1 \subset P_2 + P_4, P_2 \subset P_1 + P_3\}$.
Assume $s > 5$. After renumbering Q_i we may suppose that $Q_1 = (x^a, y^b)$ for some a, b. Moreover we may suppose that b is the biggest power of y which can enter in $G(Q_i)$ with $\sqrt{Q_i} = (x, y)$. If $y^b \in G(P_5)$ consider the filtration as above $I \subset (I, x^a) \subset Q_1 = (x^a, y^b) \subset S$. We have $Q_1/(I, x^a) \cong S/((I, x^a) : y^b)$ and $(I, x^a) : y^b = (P_2, x^a) \cap (P_3, x^a)$ because $P_1 \subset P_2 + P_4$. Also we get $x^a \in G(P_2)$. Since $P_2 \subset P_1 + P_3$ we have $P_2 \subset (P_3, x^a)$ and so $(I, x^a) : y^b = P_2$ is primary. Thus $Q_1/(I, x^a)$ is clean. Now note that $(I, x^a)/I \cong S/(I : x^a)$ and $(I : x^a) = \bigcap_{i=2}^{s} (Q_i : x^a)$ and we apply the induction hypothesis because $(I : x^a)$ satisfies the condition similar to $iii)$. Gluing the obtained clean filtrations we get S/I clean.
If $y^b \notin G(P_5)$ then $y^r \in G(P_5)$ for some $r > b$. After renumbering Q_i we may suppose that $Q_1 = (y^r, z^t)$. We consider the filtration $I \subset (I, z^t) \subset Q_1 \subset S$. We have $Q_1/(I, z^t) \cong S/((I, z^t) : y^r)$ and $((I, z^t) : y^r) = (P_2, z^t) \cap (P_3, z^t)$ and applying Lemma 2.1 we get $Q_1/(I, z^t)$ clean. Now the proof goes as above.
Other cases from $iii)$ are similar. □

Lemma 2.11. *Let $I \subset S$ be a monomial ideal such that*

$$\mathrm{Ass}(S/I) = \{(x, y), (x, z), (z, w), (y, w), (y, z), (x, w)\}$$

and let $I = P_1 \cap P_2 \cap P_3 \cap P_4 \cap P_5 \cap P_6$ be the irredundant monomial primary decomposition of I, where $\sqrt{P_1} = (x, y)$, $\sqrt{P_2} = (x, z)$, $\sqrt{P_3} = (z, w)$, $\sqrt{P_4} = (y, w)$, $\sqrt{P_5} = (y, z)$, $\sqrt{P_6} = (x, w)$. Then the following statements are equivalent:
i) S/I is clean.
ii) S/I is Cohen-Macaulay.
iii) $\{P_1 \subset P_5 + P_6$ or $P_2 \subset P_5 + P_6$ or $P_3 \subset P_5 + P_6$ or $P_4 \subset P_5 + P_6\}$
and $\{P_1 \subset P_2 + P_4$ or $P_3 \subset P_2 + P_4$ or $P_5 \subset P_2 + P_4$ or $P_6 \subset P_2 + P_4\}$
and $\{P_2 \subset P_1 + P_3$ or $P_4 \subset P_1 + P_3$ or $P_5 \subset P_1 + P_3$ or $P_6 \subset P_1 + P_3\}$.

Proof: $i) \Rightarrow ii)$ as in Lemma 2.6.
$ii) \Rightarrow iii)$: Let $J = P_1 \cap P_2 \cap P_3 \cap P_4 \cap P_5$. From the exact sequence $0 \to S/I \to S/J \oplus S/P_6 \to S/(J + P_6) \to 0$, we get that $depth(S/I) = 1$ if $depth(S/J + P_6) = 0$. But $(J + P_6) = (P_1 + P_6) \cap (P_2 + P_6) \cap (P_3 + P_6) \cap (P_4 + P_6) \cap (P_5 + P_6)$, where $(P_5 + P_6)$ is primary of height 4 and $\{(P_1 + P_6), (P_2 + P_6), (P_3 + P_6), (P_4 + P_6)\}$ are primary of height 3. Thus $depth(S/J + P_6) = 0$ if and only if $P_1 + P_6 \not\subset P_5 + P_6$ and $P_2 + P_6 \not\subset P_5 + P_6$ and $P_3 + P_6 \not\subset P_5 + P_6$ and $P_4 + P_6 \not\subset P_5 + P_6$, that is $P_1 \not\subset P_5 + P_6$ and $P_2 \not\subset P_5 + P_6$ and $P_3 \not\subset P_5 + P_6$ and $P_4 \not\subset P_5 + P_6$. So this gives one condition of $iii)$.
On the other hand if $J = P_1 \cap P_2 \cap P_3 \cap P_5 \cap P_6$ then the exact sequence $0 \to S/I \to S/J \oplus S/P_4 \to S/(J + P_4) \to 0$ gives the second condition of $iii)$. And finally if $J = P_1 \cap P_2 \cap P_4 \cap P_5 \cap P_6$ then the exact sequence $0 \to S/I \to S/J \oplus S/P_3 \to$

$S/(J+P_3) \to 0$ gives the second condition of $iii)$. Remaining choices for J, are equivalent to these three cases, which proves $ii) \Rightarrow iii)$.
$iii) \Rightarrow i)$: Suppose now that $iii)$ holds, let us say $\{P_1 \subset P_5 + P_6, P_1 \subset P_2 + P_4, P_2 \subset P_1 + P_3\}$ holds. Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant monomial irreducible decomposition of I. Apply induction on s. If $s = 6$, then (P_i) must be irreducible and so P_1 has the form (x^a, y^b). We consider the filtration
$I \subset (I, x^a) \subset (x^a, y^b) \subset S$. Note that $P_1/(I, x^a) \cong S/((I, x^a) : y^b)$. But $(I, x^a) : y^b = (P_2, x^a) \cap (P_3, x^a) \cap ((P_4, x^a) : y^b) \cap ((P_5, x^a) : y^b) \cap (P_6, x^a)$.
As $P_1 \subset P_2 + P_4$ and $P_1 \subset P_5 + P_6$, b is biggest power of y in $\{G(P_1), G(P_4), G(P_5)\}$ and thus $(I, x^a) : y^b = (P_2, x^a) \cap (P_3, x^a) \cap (P_6, x^a)$. Also since $P_2 \subset P_1 + P_3$ it follows that $(P_2, x^a) = P_2 \subset (P_3, x^a)$. Thus $(I, x^a) : y^b = (P_2, x^a) \cap (P_6, x^a)$ and $P_1/(I, x^a)$ is clean by Lemma 2.1.
Now note that $(I, x^a)/I \cong S/(I : x^a)$. We have $(I : x^a) = (P_2 : x^a) \cap (P_3 : x^a) \cap (P_4 : x^a) \cap (P_5 : x^a) \cap (P_6 : x^a)$. As above a is the biggest power of x in $G(P_1), G(P_2), G(P_6)$. It follows $I : x^a = P_3 \cap P_4 \cap P_5$, so $S/(I : x^a)$ is clean by Lemma 2.2. Gluing together the clean filtrations obtained above we get a clean filtration of S/I, that is when $s = 5$, then $i)$ holds for $\{P_1 \subset P_5 + P_6, P_1 \subset P_2 + P_4, P_2 \subset P_1 + P_3\}$.
Assume $s > 5$. After renumbering Q_i we may suppose that $Q_1 = (x^a, y^b)$ for some a, b. Moreover we may suppose that b is the biggest power of y which can enter in $G(Q_i)$ with $\sqrt{Q_i} = (x, y)$. Consider the filtration as above $I \subset (I, x^a) \subset Q_1 = (x^a, y^b) \subset S$. We have $Q_1/(I, x^a) \cong S/((I, x^a) : y^b)$ and $(I, x^a) : y^b = (P_2, x^a) \cap (P_3, x^a) \cap (P_6, x^a)$ because $P_1 \subset P_2 + P_4$, $P_1 \subset P_5 + P_6$. We get also $x^a \in P_2$. Since $P_2 \subset P_1 + P_3$ we have $(P_2, x^a) \subset (P_3, x^a)$ and so $(I, x^a) : y^b = (P_2, x^a) \cap (P_6, x^a)$. Thus $Q_1/(I, x^a)$ is clean by Lemma 2.1. Now note that $(I, x^a)/I \cong S/(I : x^a)$ and $(I : x^a) = \bigcap_{i=2}^{s} (Q_i : x^a)$ and we apply the induction hypothesis for $(I : x^a)$ because the condition $iii)$ are fulfilled in this case. As above gluing the obtained clean filtrations we get S/I clean.
Other cases from $iii)$, are similar. □

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