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Some remarks on the stability of the "dead-ocean" steady-state in a plankton population model

by
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Abstract

We prove that, although unstable by Lyapunov, the equlibrium $(N_0, 0, 0)$ in the plankton population model given in [1] is stable with respect to a relevant set of solutions. This clarifies and completes the considerations in [2].

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The interaction between zooplankton population Z and phytoplankton population P in an ocean with varying nutrients N is modeled in [1] by the following nonlinear systems of ordinary differential equations

$$\begin{cases} \dot{N} = -N\overline{e} + Na\overline{b} + cPP + rP + \beta\lambda P^2\overline{\mu}^2 + P^2Z + \gamma dZ^2 + k(N_0 - N) \\ \dot{P} = N\overline{e} + Na\overline{b} + cPP - rP - \lambda P^2\overline{\mu}^2 + P^2Z - (s+k)P \\ \dot{Z} = \alpha\lambda P^2\overline{\mu}^2 + P^2Z - dZ^2 \end{cases} \tag{1}$$

All parameters in (1) have positive values a discussion of which is to be found in [1].

System (1) has $(N_0, 0, 0)$, called "dead-ocean" in [2] as one of its equilibria. It was proved in [3] that this equilibrium is unstable in Lyapunov sense. In [2] the author claims that the "dead-ocean" is stable when realistic conditions are taken into consideration. It is the purpose of this note to lay this statement in a proper mathematical setting and to give a complete proof.

As is a common practice we translate the equilibrium point to zero by introducing $\xi = N - N_0$. The following system replaces (1):

$$\begin{cases} \dot{\xi} = -\xi + N_0\overline{e} + N_0 + \xi a\overline{b} + cPP + rP + \beta\lambda P^2\overline{\mu}^2 + P^2Z + \gamma dZ^2 - k\xi \\ \dot{P} = \xi + N_0\overline{e} + N_0 + \xi a\overline{b} + cPP - rP - \lambda P^2\overline{\mu}^2 + P^2Z - (s+k)P \\ \dot{Z} = \alpha\lambda P^2\overline{\mu}^2 + P^2Z - dZ^2 \end{cases} \tag{2}$$

Under study is now the stability of the zero solution. General references for stability theory are [4] and [5]. We will sometimes refer to system (2) as $\dot{x} = F(x)$ with $x = (\xi, P, Z)$.

Definition *The solution φ of an autonomous system*

$$\dot{x} = f(x) \tag{3}$$

that is defined for all $t \geq 0$ is called (uniformly) stable with respect to a set $\mathcal{S}$ of solutions if, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $||x_0 - \varphi(0)|| < \delta(\varepsilon)$ and if the solution x of (3), with $x(0) = x_0$, belongs to $\mathcal{S}$ then x is defined for every $t \geq 0$ and $||x(t) - \varphi(t)|| < \varepsilon$ $\forall t \geq 0$.

If φ is stable with respect to $\mathcal{S}$ and there exists $\eta_0 > 0$ such that, for $||x_0 - \varphi(0)|| < \eta_0$, the solution x of (3) with $x(0) = x_0$ belongs to $\mathcal{S}$ and satisfies $\lim_{t\to\infty}[x(t) - \varphi(t)] = 0$, φ is called asymptotically stable with respect to $\mathcal{S}$. We use $||(x_1, x_2, x_3)|| \overset{def}{=} |x_1| + |x_2| + |x_3|$.

In the case of system (2), as remarked in [2] the set of solutions consistent with reality contains those solutions whose second (P) and third (Z) components start from positive initial values and remain positive throughout their domain of existence.

Denote $G = \{(\xi, P, Z) | \xi \in \mathbf{R},\ P \geq 0,\ Z \geq 0\}$.

Proposition 1 *G is invariant for system (2), that is, if $(\xi(0), P(0), Z(0)) \in G$ then $(\xi(t), P(t), Z(t)) \in G$ $\forall t \in [0, T)$ with T the right-hand side of the interval of existence for the maximal solutions.*

Proof: Introduce

$$g(\tau) = \frac{\sigma(\tau) + N_0}{e + N_0 + \xi(\tau)} \cdot \frac{a}{b + cP(\tau)} - r - s - k - \frac{\lambda P(\tau)Z(\tau)}{\mu^2 + P^2(\tau)}$$

and

$$h(\tau) = \frac{\alpha\lambda P^2(\tau)}{\mu^2 + P^2(\tau)} - dZ(\tau).$$

Then $P(t) = P(0)e^{\int_0^t g(\tau)d\tau}$ and $Z(t) = Z(0)e^{\int_0^t h(\tau)d\tau}$ and the Proposition is proved. □

Theorem 2. If $\lambda_2 \stackrel{def}{=} \frac{aN_0}{b(e+N_0)} - r - s - k < 0$ *then the zero solution of system (2) is asymptotically stable with respect to the set* $\mathcal{S}$ *of solutions with initial data in* G *so* $(N_0, 0, 0)$ *is an equilibrium point of (1), asymptotically stable with respect to* $\mathcal{S}$.

Proof: We know from Proposition 1 that the whole trajectory of a solution from $\mathcal{S}$ is contained in G. Consider the following functions that were also used, in a slightly different form, in [3]

$$W_1(\xi, P) = \frac{1}{2k}\xi^2 + \frac{N_1}{k(k-\lambda_2)}\xi P + \frac{k(k-\lambda_2) + N_1^2}{2(-\lambda_2)k(k-\lambda_2)}P^2$$

and

$$V_1 = (\xi, P, Z) = dZ + W_1(\xi, P)$$

where

$$N_1 \stackrel{def}{=} -\frac{aN_0}{b(N_0 + e)}.$$

Elementary computation gives $W_1(\xi, P) > 0$ for every $(\xi, P) \neq (0, 0)$ under the hypothesis $\lambda_2 < 0$ thus there exists a positive constant $C_1 > 0$ such that $W_1(\xi, P) \geq C_1(|\xi| + |P|) \quad \forall (\xi, P) \in \mathbf{R}^2$. It follows that there exists another constant $C_2 > 0$ such that $V_1(\xi, P, Z) \geq C_2(|\xi| + P + Z)$ for every $(\xi, P, Z) \in G$ so, for every solution $x \in \mathcal{S}$, we have

$$V_1[x(t)] \geq C_2||x(t)||. \tag{4}$$

The same reasonings, based on positive definitness of W_1 yields the existence of another constant $C_3 \geq 1$ such that, for every solution $x \in \mathcal{S}$,

$$V_1[x(t)] \leq C_3||x(t)|| \tag{5}$$

It is proved in ([3]) that the Lie derivative of V_1 along the system (2) is

$$L_F V = \frac{\partial V_1}{\partial \xi}\dot{\xi} + \frac{\partial V_1}{\partial P}\dot{P} + \frac{\partial V_1}{\partial Z}\dot{Z} = -d^2Z^2 - \xi^2 - P^2 + R_1(\xi, P, Z)$$

where R_1 contains terms of order equal or greater to three in its Taylor series around zero. Thus $L_F V$ is negatively defined if $||(\xi, P, Z)||$ is small enough (see also, for example, [5], ch. II, §7, Lema 4).

This, together with (4) and (5), makes possible the application of Lyapunov stability theorem with respect to $\mathcal{S}$ (see [4], ch. 1, §1.2, Theorem 1.5) and so Theorem 2 is proved.

□

From a biological point of vue, Theorem 2 asserts that, if the initial value of the concentration of nutrient N is closed to N_0, the initial values of concentrations of phytoplankton and zooplankton are small enough and the parameters obey $\lambda_2 < 0$, the plankton evolution goes to extinction.

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