Bull. Math. Soc. Sci. Math. Roumanie
Tome 50(98) No. 2, 2007, 149–156

A Hermite-Hadamard inequality for convex-concave symmetric functions *

by
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Abstract

In this paper we prove a Hermite-Hadamard type inequality for convex-concave symmetric functions, by considering integral mean values with respect to certain signed measures.

Key Words: Convex function, Borel measure, Hermite-Hadamard inequality.
2000 Mathematics Subject Classification: Primary 26A51, Secondary 26D15

The classical Hermite-Hadamard inequality gives us an estimate, from below and from above, of the mean value of a convex function $f:[a,b]\to\mathbb{R}$:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{HH}$$

See [8], pp. 50-51, for details. A weighted form of this inequality was proved by L. Fejér [5]. Precisely, if f is as above and $p:[a,b]\to\mathbb{R}$ is a nonnegative integrable function, symmetric with respect to the middle point $(a+b)/2$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx\leq\int_a^b f(x)\,p(x)\,dx\leq\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx. \tag{F}$$

However, it is Choquet's theory who provides the best understanding of the Hermite-Hadamard inequality in the framework of positive Radon measures on arbitrary compact convex sets. See [8], pp.192-195, for details. Many results are covered by this theory, in particular the main result in [1] (by using the technique of pushing forward measures).

*Research partially supported by CNCSIS Grant 80/2006

A. M. Fink [5] made an important remark concerning the generality of (HH), by noticing the possibility to consider certain signed measures. In order to recall it here we need a preparation.

Let μ be a real Borel measure on $[a,b]$, with $\mu([a,b]) > 0$. We say that μ is a *Hermite-Hadamard measure* if for every convex function $f:[a,b]\to\mathbb{R}$ the following inequalities hold,

$$\begin{aligned} f(x_\mu) &\leq \frac{1}{\mu([a,b])}\int_a^b f(x)\,d\mu(x) && \text{(LHH)} \\ &\leq \frac{b-x_\mu}{b-a}\cdot f(a)+\frac{x_\mu-a}{b-a}\cdot f(b), && \text{(RHH)} \end{aligned}$$

where

$$x_\mu=\frac{1}{\mu([a,b])}\int_a^b x\,d\mu(x)$$

is the *barycenter* of μ. Fejer's aforementioned result offers plenty of examples of nonnegative Hermite-Hadamard measures. Moreover, using the Hardy-Littlewood-Pólya majorization inequality, it is not difficult to built up examples within the class of signed-measures such as

$$\frac{5}{9}\delta_{-1/2}-\frac{1}{9}\delta_0+\frac{5}{9}\delta_{1/2} \quad \text{on } [-1,1].$$

A. M. Fink [5] gave a complete characterization of the real-valued Borel measures μ for which (LHH) works for every convex function, precisely the fulfillment of the following condition of end-positivity:

$$\int_a^t (t-x)\,d\mu(x)\geq 0 \quad \text{and} \quad \int_t^b (x-t)\,d\mu(x)\geq 0 \quad \text{for every } t\in[a,b]\,.$$

In the same paper, A. M. Fink proved a sufficient condition for (RHH), but his argument can be modified to get a complete characterization of the measures for which (RHH) works:

Theorem 1. *Let μ be a real Borel measure on $[a,b]$ with $\mu([a,b])>0$ and having the barycenter x_μ. Then the inequality*

$$\frac{1}{\mu([a,b])}\int_a^b f(x)d\mu(x)\leq\frac{b-x_\mu}{b-a}f(a)+\frac{x_\mu-a}{b-a}f(b) \qquad \text{(RHH)}$$

works for all continuous convex functions $f:[a,b]\to\mathbb{R}$ if and only if

$$\frac{b-t}{b-a}\int_a^t (x-a)d\mu(x)+\frac{t-a}{b-a}\int_t^b (b-x)d\mu(x)\geq 0 \quad \textit{for all } t\in[a,b].$$

Proof: An easy approximation argument shows that the formula (RHH) works for all continuous convex functions $f : [a,b] \to \mathbb{R}$ if and only if it works for all convex functions of class C^2 on $[a,b]$.

As well known, all such functions admit an integral representation of the form

$$f(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) + \int_a^b G(x,t)f''(t)dt,$$

where

$$G(x,t) = -\begin{cases} \frac{(x-a)(b-t)}{b-a} & \text{if } a \leq x \leq t \leq b \\ \frac{(t-a)(b-x)}{b-a} & \text{if } a \leq t \leq x \leq b \end{cases}$$

represents the Green function of the operator $\frac{d^2}{dt^2}$, with homogeneous boundary conditions $y(a) = y(b) = 0$.

Thus, for every convex function $f \in C^2([a,b])$ we have

$$\begin{aligned}
&\frac{1}{\mu([a,b])}\int_a^b f(x)d\mu(x) - \frac{b-x_\mu}{b-a}f(a) - \frac{x_\mu - a}{b-a}f(b) \\
&\qquad = \frac{1}{\mu([a,b])}\int_a^b \left[f(x) - \frac{b-x}{b-a}f(a) - \frac{x-a}{b-a}f(b)\right] d\mu(x) \\
&\qquad\quad = \frac{1}{\mu([a,b])}\int_a^b \left(\int_a^b G(x,t)f''(t)dt\right) d\mu(x) \\
&\qquad\quad = \frac{1}{\mu([a,b])}\int_a^b f''(t) \left(\int_a^b G(t,x)d\mu(x)\right) dt \\
&\qquad\qquad\qquad\qquad = \frac{1}{\mu([a,b])}\int_a^b f''(t)y(t)dt,
\end{aligned}$$

where $y(t) = \int_a^b G(t,x)d\mu(x)$ is a continuous function.

When f runs over the class of C^2 convex functions on $[a,b]$, its second derivative f'' will describe the class of continuous nonnegative functions on $[a,b]$. Consequently the inequality (HH) holds for all C^2 convex functions if and only if

$$y(t) = \int_a^b G(t,x)d\mu(x) \leq 0$$

for all $t \in [a,b]$. □

Using the above results we can easily check that $d\mu = (x^2 - \lambda)dx$ is a Hermite-Hadamard measure for all $\lambda \leq 1/6$.

Remark 1. *Using the technique of pushing-forward measures, the inequalities (LHH) and (RHH) above can be put in a more general form, that encompasses* *also the classical Jensen inequality. If ν is a signed Borel measure on X and $T : X \to [a, b]$ is a ν-integrable map, then the push-forward measure $\mu = T\#\nu$ is given by the formula $\mu(A) = \nu(T^{-1}(A))$. Assuming that $\mu = T\#\nu$ is a Hermite-Hadamard measure (which is always the case when ν is positive), we get*

$$f\left(\overline{T}\right) \leq \frac{1}{\nu(X)} \int_X f(T(x))\, d\nu(x) \tag{J}$$

$$= \frac{1}{\mu([a,b])} \int_{[a,b]} f(t)\, d\mu(t) \leq \frac{b - \overline{T}}{b-a} f(a) + \frac{\overline{T} - a}{b-a} f(b) \tag{CJ}$$

for every continuous convex function $f : [a, b] \to \mathbb{R}$. Here $\overline{T} = \frac{1}{\nu(X)} \int_X T(x)\, d\nu(x)$ represents the barycenter of μ. For even more general Jensen type inequalities see [8].

P. Czinder and Z. Páles [3] (see also P. Czinder [4]) have extended the Hermite-Hadamard inequality in another direction, by considering functions that mix the up and down convexity:

Theorem 2. *Suppose that I is an interval and $f : I \to \mathbb{R}$ is a function symmetric with respect to an element $m \in \overline{I}$, that is,*

$$f(x) + f(2m - x) = 2f(m) \quad \text{for all } x \in I \cap (-\infty, m]. \tag{S}$$

If f is convex over the interval $I \cap (-\infty, m]$ and concave over the interval $I \cap [m, \infty)$, then, for every interval $[a, b] \subset I$ with $(a + b)/2 \geq m$, the following inequalities hold true:

$$f\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a} \int_a^b f(x)\, dx \geq \frac{f(a) + f(b)}{2}. \tag{CP}$$

If $(a + b)/2 \leq m$, then the inequalities (CP) should be reversed.

An appropriate version is valid for functions that are *concave over the interval $I \cap (-\infty, m]$ and convex over $I \cap [m, \infty)$.*

Notice that the inequality (HH) can be derived from (CP), by choosing m as one of the endpoints of I.

The aim of this paper is to prove that Theorem 2 works actually in the general framework of Hermite-Hadamard measures.

Theorem 3. *Suppose that $f : I \to \mathbb{R}$ verifies the symmetry condition (S) and is convex over the interval $I \cap (-\infty, m]$ and concave over the interval $I \cap [m, \infty)$.*

If $(a + b)/2 \geq m$ and μ is a Hermite-Hadamard measure on each of the intervals $[a, 2m - a]$ and $[2m - a, b]$, and is invariant with respect to the map $T(x) = 2m - x$ on $[a, 2m - a]$, then

$$f(x_\mu) \geq \frac{1}{\mu([a,b])} \int_a^b f(x)\, d\mu \geq \frac{b - x_\mu}{b-a} f(a) + \frac{x_\mu - a}{b-a} f(b). \tag{GHH}$$

If $(a+b)/2 \leq m$, then the inequalities (GHH) work in a reverse way, provided μ is a Hermite-Hadamard measure on each of the intervals $[a, 2m-b]$ and $[2m-b, b]$, and is invariant with respect to the map $T(x) = 2m - x$ on $[2m-b, b]$.

Proof: Suppose first that $(a+b)/2 \geq m$. The case where $m \notin (a, b)$ is covered by our assumption on $d\mu$ (of being a Hermite-Hadamard measure on specific intervals), so we will concentrate only on the case where $a < m < b$. In order to prove the left hand side inequality in (GHH) we have to notice that

$$\begin{aligned} \int_a^b f(x)\,d\mu &= \int_a^{2m-a} f(x)\,d\mu + \int_{2m-a}^b f(x)\,d\mu \\ &= f(m)\int_a^{2m-a} d\mu + \int_{2m-a}^b f(x)\,d\mu, \end{aligned} \qquad \text{(LGHH)}$$

due to the invariance properties of f and μ. Since f is concave over the interval $[2m-a, b]$, the last integral in (LGHH) does not exceeds

$$f\left(\frac{1}{\mu([2m-a,b])}\int_{2m-a}^b x d\mu\right) \cdot \mu([2m-a,b])$$

and thus

$$\begin{aligned} \frac{1}{\mu([a,b])}\int_a^b f(x)\,d\mu &\leq \frac{\mu([a,2m-a])}{\mu([a,b])} \cdot f(m) \\ &\quad + \frac{\mu([2m-a,b])}{\mu([a,b])} \cdot f\left(\frac{\int_{2m-a}^b x d\mu}{\mu([2m-a,b])}\right) \\ &\leq f\left(\frac{\mu([a,2m-a])}{\mu([a,b])} \cdot m + \frac{\mu([2m-a,b])}{\mu([a,b])} \cdot \frac{\int_{2m-a}^b x d\mu}{\mu([2m-a,b])}\right) \\ &= f\left(\frac{1}{\mu([a,b])}\int_a^b x d\mu\right) = f(x_\mu). \end{aligned}$$

due to the symmetry properties of f and μ combined with the fact that f is concave on $[m, b]$.

Now we prove the right hand side inequality in (GHH). Using the symmetry of f and its property of being concave on the interval $[2m-a, b]$, we infer that

$$\begin{aligned} \int_a^b f(x)\,d\mu &= f(m)\int_a^{2m-a} d\mu + \int_{2m-a}^b f(x)\,d\mu \\ &\geq f(m) \cdot \mu([a, 2m-a]) \\ &\quad + \left(\frac{b - x'_\mu}{b+a-2m} f(2m-a) + \frac{x'_\mu - 2m + a}{b+a-2m} f(b)\right) \cdot \mu([2m-a, b]), \end{aligned}$$

where

$$\begin{aligned} x'_\mu &= \frac{1}{\mu([2m-a,b])}\int_{2m-a}^{b} x d\mu \\ &= \frac{1}{\mu([2m-a,b])}\left(\int_a^b x d\mu - \int_a^{2m-a} x d\mu\right) \\ &= \frac{x_\mu \cdot \mu([a,b]) - m\cdot \mu([a,2m-a])}{\mu([2m-a,b])}. \end{aligned}$$

To complete the proof of the right hand side of (GHH), it suffices to show that

$$\begin{aligned} & f(m)\cdot \mu([a,2m-a]) \\ & \quad + \left(\frac{b-x'_\mu}{b+a-2m} f(2m-a) + \frac{x'_\mu - 2m + a}{b+a-2m} f(b)\right)\cdot \mu([2m-a,b]) \\ & \qquad \geq \left(\frac{b-x_\mu}{b-a} f(a) + \frac{x_\mu - a}{b-a} f(b)\right)\cdot \mu([a,b]). \end{aligned}$$

Without loss of generality we may assume that $\mu([a,b]) = 1$. Put $\mu([a,2m-a]) = \lambda$. Then $\mu([2m-a,b]) = 1-\lambda$, and the last inequality becomes

$$\begin{aligned} & \lambda f(m) + (1-\lambda)\left(\frac{b - \frac{x_\mu - \lambda m}{1-\lambda}}{b+a-2m} f(2m-a) + \frac{\frac{x_\mu-\lambda m}{1-\lambda} - 2m + a}{b+a-2m} f(b)\right) \\ & \qquad \geq \frac{b-x_\mu}{b-a} f(a) + \frac{x_\mu - a}{b-a} f(b). \end{aligned}$$

Since $f(a) = 2f(m) - f(2m-a)$, this inequality can be restated as

$$\begin{aligned} & \lambda f(m) + \frac{(1-\lambda)b - x_\mu + \lambda m}{b+a-2m} f(2m-a) + \frac{x_\mu - \lambda m - (1-\lambda)(2m-a)}{b+a-2m} f(b) \\ & \qquad \geq \frac{b-x_\mu}{b-a}\left(2f(m) - f(2m-a)\right) + \frac{x_\mu - a}{b-a} f(b). \end{aligned}$$

Doing some algebra (including a simplification of both sides by $(2-\lambda)b + \lambda a - 2x_\mu$) we are led to

$$f(2m-a) \geq \frac{b+a-2m}{b-m} f(m) + \frac{m-a}{b-m} f(b),$$

which is indeed the case since f is concave on the interval $[m,b]$ and $2m-a$ is a convex combination of m and b :

$$2m - a = \frac{b+a-2m}{b-m} m + \frac{m-a}{b-m} b.$$

Notice that $(2-\lambda)b+\lambda a-2x_\mu \geq 0$ is a consequence of our hypotheses on x_μ. In fact,

$$\begin{aligned} x_\mu &= \int_a^b x d\mu = \int_a^{2m-a} x d\mu + \int_{2m-a}^b x d\mu \\ &= \lambda m + \int_{2m-a}^b x d\mu \\ &\leq \lambda m + (1-\lambda)b \leq \frac{(2-\lambda)b+\lambda a}{2}. \end{aligned}$$

The case where $(a+b)/2 \leq m$ can be treated in a similar way. □

Theorem 2 was applied by Czinder and Páles [3] to prove inequalities for the Gini and Stolarsky means. The weighted framework offered by Theorem 3 is suitable to get even more general inequalities, involving averages with respect to measures that are not necessarily positive.

Theorem 3 extends easily to the context of generalized convex functions in the sense of E. F. Beckenbach. These functions are attached to Chebyshev systems i.e., to pairs (ω_1, ω_2) of real-valued continuous functions on an interval I, such that

$$\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} > 0 \quad \text{for } x < y \text{ in } I.$$

Precisely, a function $f : I \to \mathbb{R}$ is *convex with respect to a Chebyshev system* (ω_1, ω_2) if

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} \geq 0 \quad \text{for } x < y < z \text{ in } I.$$

As was noticed by M. Bessenyei and Z. Páles [2], there is no loss of generality to assume that ω_1 is strictly positive and ω_2/ω_1 is increasing. But in this case f is (ω_1, ω_2)-convex if and only if $\frac{f}{\omega_1} \circ \left(\frac{\omega_2}{\omega_1}\right)^{-1}$ is convex in the usual way.

Acknowledgement 1. *The authors thank Prof. Zsolt Páles for several improvements on the original version of this paper.*

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Received: 11.12.2006.

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