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A viability result for a class of nonconvex differential inclusions with memory

by
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Abstract

We prove the existence of viable solutions for an autonomus functional differential inclusion in the case when the multifunction that define the inclusion is upper semicontinuous compact valued and contained in the Clarke subdifferential of an uniformly regular function.

Key Words: Differential inclusion with memory, uniform regular function, viable solution, Clarke's subdifferential.
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1 Introduction

Differential inclusions with memory, known also as functional differential inclusions, express the fact that the velocity of the system depends not only on the state of the system at a given instant but depends upon the history of the trajectory until this instant.

The class of functional differential inclusions encompasses a large variety of differential inclusions and control systems. In particular, this class covers the differential inclusions, the differential-difference inclusions and the Volterra inclusions. For a detailed discussion on this topic we refer to [1].

Let $\mathbf{R}^m$ be the m-dimensional euclidian space. We consider $C(-\infty,0;\mathbf{R}^m)$ the space of continuous functions from $(-\infty,0)$ to $\mathbf{R}^m$ supplied with the topology of uniform convergence on compact intervals. For $t\in\mathbf{R}$, the operator $T(t):C(-\infty,\infty;\mathbf{R}^m)\to C(-\infty,0;\mathbf{R}^m)$ is defined by $(T(t)x)(s):=x(t+s)$, $s\in(-\infty,0)$. If K is a given nonempty subset in $\mathbf{R}^m$ then we introduce the following set $\mathcal{K}=:\{\varphi\in C(-\infty,0;\mathbf{R}^m);\varphi(0)\in K\}$.

For a given multifunction $F:\mathcal{K}\to\mathcal{P}(\mathbf{R}^m)$ we consider the following functional differential inclusion

$$x'\in F(T(t)x) \tag{1.1}$$

and we are interested in finding sufficient conditions for such that for each $\varphi \in \mathcal{K}$ there exist $\tau > 0$ and a solution $x(.) : [-\infty, \tau] \to \mathbf{R}^m$ of (1.1) satisfying the initial condition

$$T(0)x = \varphi \tag{1.2}$$

and the viability constraint

$$x(t) \in K \quad \forall t \geq 0. \tag{1.3}$$

The existence of solutions of problem (1.1)-(1.3), well known as viable solutions, in the case when F is single valued were studied by many authors. For results and references in this framework we refer to [9].

In general the results concerning differential inclusions defined by upper semicontinuous multifunctions can be extended to functional differential inclusions. The first viability result for functional differential inclusions was given by Haddad ([7], [8]) in the case when F is upper semicontinuous with convex compact values.

Recently in [4], the situation when the multifunction is not convex valued is considered. More exactly, in [4] it is proved the existence of solutions of problem (1.1)-(1.3) when $F(.)$ is an upper semicontinuous multifunction contained in the subdifferential of a proper convex function $V(.)$. Afterwards, the convexity of $V(.)$ was relaxed in [5], [6] for special classes of functions $V(.)$.

The aim of the present paper is to provide an alternative improvement of the result in [4]. More precisely, we relax the convexity assumption on the function $V(.)$ that appear in [4], in the sense that we assume that $F(.)$ is contained in the Clarke subdifferential of an uniform regular function.

The proof of our result follows the general ideas in [1] and [8] and is essentially based on the corresponding viability result for differential inclusions ([2]).

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2 Preliminaries

For $x \in \mathbf{R}^m$ and for a closed subset $A \subset \mathbf{R}^m$ we denote by $\mathrm{d}(x, A)$ the distance from x to A given by $\mathrm{d}(x, A) := \inf\{||y - x||; y \in A\}$. By $co(A)$ we denote the convex hull of A and by $\overline{co}(A)$ we denote the closed convex hull of A

Consider a locally Lipschitz continuous function $V : \mathbf{R}^m \to \mathbf{R}$. For every direction $v \in \mathbf{R}^m$ its *Clarke directional derivative* at x in the direction v is defined by

$$\mathrm{D}_C V(x; v) = \limsup_{y \to x, t \to 0+} \frac{V(y + tv) - V(y)}{t}.$$

The *Clarke subdifferential (generalized gradient)* of V at x is defined by

$$\partial_C V(x) = \{q \in \mathbf{R}^m, \quad < q, v > \leq \mathrm{D}_C V(x; v) \quad \forall v \in \mathbf{R}^m\}.$$

We recall that the *proximal subdifferential* of $V(.)$ is defined by

$$\partial_P V(x) = \{q \in \mathbf{R}^m, \quad \exists \delta, \sigma > 0 \quad \text{such that}$$

$$V(y) - V(x) + \sigma ||y - x||^2 \geq < q, y - x > \quad \forall y \in B(x, \delta)\}.$$

Definition 2.1 *Let $V : \mathbf{R}^m \to \mathbf{R} \cup \{\infty\}$ be a lower semicontinuous function and let $O \subset dom(V)$ be a nonempty open subset. We say that V is uniform regular over O if there exists a positive number $\beta > 0$ such that for all $x \in O$ and for all $\xi \in \partial_P V(x)$ one has*

$$< \xi, y - x > \leq V(y) - V(x) + \beta ||y - x||^2 \quad \forall y \in O.$$

We say that V is uniformly regular over the closed set K if there exists an open set O containing K such that V is uniformly regular over O.

In [2] there are several examples and properties of such maps. For example, according to [2], any lower semicontinuous proper convex function V is uniformly regular over any nonempty compact subset of its domain with $\beta = 0$; any lower - C^2 function V is uniformly regular over any nonempty convex compact subset of its domain.

The next lemma will be used in the proof of our main result.

Lemma 2.2 ([2]) *Let $V : \mathbf{R}^m \to \mathbf{R}$ be a locally Lipschitz function and let $\emptyset \neq K \subset dom(V)$ a closed set. If V is uniformly regular over K then*

i) $\partial_C V(x) = \partial_P V(x)$ $\forall x \in K$.

ii) If $z(.) : [0, \tau] \to \mathbf{R}^m$ is absolutely continuous and $z'(t) \in \partial_C V(z(t))$ a.e. $([0, \tau])$, then

$$(V \circ z)'(t) = ||z'(t)||^2, \quad a.e.\ ([0, \tau]).$$

We recall that the contingent cone to the set $K \subset \mathbf{R}^m$ at $x \in K$ is defined by

$$T_K(x) = \{v \in \mathbf{R}^m; \quad \liminf_{h \to 0+} \frac{1}{h} d(x + hv, K) = 0\}.$$

The following viability result for differential inclusions is due to Bounkhel ([2]).

Theorem 2.3 *Let $K \subset \mathbf{R}^m$ be a nonempty closed set, $G : K \to \mathcal{P}(\mathbf{R}^m)$ be an upper semicontinuous multifunction with nonempty closed values and $V(.) : \mathbf{R}^m \to \mathbf{R}$ be a locally Lipschitz function that is uniform regular over K such that*

$$G(x) \subset \partial_C V(x) \quad \forall x \in K,$$

$$G(x) \cap T_K(x) \neq \emptyset \quad \forall x \in K.$$

Then, for any $x_0 \in K$, there exists $\tau > 0$ such that the following problem

$$x' \in G(x), \quad x(t_0) = x_0 \in K,$$

$$x(t) \in K \quad \forall t \geq t_0$$

admits a solution on $[t_0, \tau]$.

If $K \subset \mathbf{R}^m$ we introduce the following set

$$\mathcal{K} =: \{\varphi \in C(-\infty, 0; \mathbf{R}^m); \varphi(0) \in K\}.$$

In what follows we consider the functional differential inclusion (1.1) under the following assumptions.

Hypothesis 2.4 a) *$K \subset \mathbf{R}^m$ is a locally closed set and $F : \mathcal{K} \to \mathcal{P}(\mathbf{R}^m)$ is an upper semicontinuous multifunction with nonempty compact values.*

b) *There exists a locally Lipschitz function $V : \mathbf{R}^m \to \mathbf{R}$ uniform regular over K such that the following conditions are satisfied*

$$F(\varphi) \cap T_K(\varphi(0)) \neq \emptyset \quad \forall \varphi \in \mathcal{K}, \tag{2.1}$$

$$F(\varphi) \subset \partial_C V(\varphi(0)) \quad \forall \varphi \in \mathcal{K}. \tag{2.2}$$

3 The main result

We are now able to prove our main result.

Theorem 3.1 *We assume that Hypothesis 2.4 is satisfied.*
Then, for any $\varphi \in \mathcal{K}$ there exists a solution to

$$x' \in F(T(t)x) \quad a.e. \quad t \geq 0,$$

$$T(0)x = \varphi, \quad x(t) \in K \quad \forall t \geq 0.$$

Proof: We divide the interval $[0, \infty)$ into the subintervals $[\frac{j}{n}, \frac{j+1}{n}]$, $j \in \mathbf{N}$. For any $j \in \mathbf{N}$, $x \in \mathbf{R}^m$ and $\varphi(.) \in C(-\infty, \frac{j}{n}; \mathbf{R}^m)$ we consider the function $f_j^\varphi(x)(.) \in C(-\infty, 0; \mathbf{R}^m)$ defined by

$$f_j^\varphi(x)(s) := \begin{cases} \varphi(\frac{j+1}{n} + s) & \text{if} \quad s \leq -\frac{1}{n} \\ \varphi(\frac{j}{n} + (sn+1)(x - \varphi(\frac{j}{n})) & \text{if} \quad -\frac{1}{n} \leq s \leq 0. \end{cases} \tag{3.1}$$

Obviously,

$$f_j^\varphi(x)(-\frac{j}{n}) := \varphi(\frac{j}{n}), \quad f_j^\varphi(x)(0) = x. \tag{3.2}$$

We define the set-valued maps $G_j^\varphi : K \to \mathcal{P}(\mathbf{R}^m)$ by

$$G_j^\varphi(x) := F(f_j^\varphi(x)) \tag{3.3}$$

and we note that $G_j^\varphi(.)$ is upper semicontinuous with nonempty closed values and satisfy, via (2.1), (2.2) and (3.2),

$$\begin{array}{l} G_j^\varphi(x) \subset \partial_C V(x) \quad \forall x \in K, \\ G_j^\varphi(x) \cap T_K(x) \neq \emptyset \quad \forall x \in K. \end{array} \tag{3.4}$$

Consequently, by (3.4) the assumptions of Theorem 2.3 are satisfied and therefore there exists a viable solution $x_j^{\varphi}(.) : [\frac{j}{n}, \frac{j+1}{n}] \to \mathbf{R}^m$ to the problem

$$\begin{aligned} & x' \in G_j^{\varphi}(x) \quad a.e.([\tfrac{j}{n}, \tfrac{j+1}{n}]), \quad x(\tfrac{j}{n}) = \varphi(\tfrac{j}{n}) \in K \\ & x(t) \in K \quad \forall t \in [\tfrac{j}{n}, \tfrac{j+1}{n}]. \end{aligned} \tag{3.5}$$

We construct next by induction a sequence of approximate solutions $x_n(.)$ to (1.1). If $t \leq 0$ we define $x_n(t) := \varphi(t)$.

Assume now that $x_n(.)$ is defined on $(-\infty, \frac{j}{n}]$. We take for $t \in [\frac{j}{n}, \frac{j+1}{n}]$ the solution $x_n(.) := x_j^{x_n}(.)$ of problem (3.5) with $\varphi(.) := x_n(.)$.

We prove that for almost all $t \in \mathbf{R}$

$$(T(t)x_n, x_n'(t)) \in \text{graph}(F) + 2\frac{1}{n}||ImF||.B \times \{0\}, \tag{3.6}$$

where B denotes the unit ball of $C(-1, 0; \mathbf{R}^m)$.

If $t \in [\frac{j}{n}, \frac{j+1}{n}]$, by (3.3) and (3.5), $(f_j^{x_n}(x_n(t)), x_n'(t)) \in \text{graph}(F)$. On the other hand, one has

$$T(t)x_n(t) - f_j^{x_n}(x_n(t))(s) =$$

$$\begin{cases} 0 & \text{if} \quad s \leq -\frac{1}{n} \\ (sn+1)(x_n(t+s) - x_n(s)) - sn(x_n(t+s) - x_n(\frac{j}{n})) & \text{if} \quad -\frac{1}{n} \leq s \leq 0 \end{cases}$$

and thus, it belongs to $\frac{2}{n}\overline{co}(F(\mathcal{K}))$, since

$$x_n(t+s) - x_n(t) = \int_t^{t+s} x_n'(s)ds \in \frac{1}{n}\overline{co}(F(\mathcal{K})),$$

$$x_n(t+s) - x_n(\frac{j}{n}) = \int_{\frac{j}{n}}^{t+s} x_n'(s)ds \in \frac{1}{n}\overline{co}(F(\mathcal{K})).$$

Hence

$$||T(t)x_n - f_j^{x_n}(x_n(t))||_{C(-1,0;\mathbf{R}^m)} \leq \frac{2}{n}||\overline{co}(F(\mathcal{K}))||.$$

We infer that

$$\begin{aligned} & ||x_n'(t)|| \leq ||co(F(\mathcal{K}))||, \\ & x_n(t) \in \varphi(0) + tco(F(\mathcal{K}) \quad \text{which is relatively compact.} \end{aligned}$$

We apply Theorem 0.3.4 in [1] to deduce that $x_n(.)$ converges uniformly over compact intervals to some function $x(.)$ in $C(-\infty, \infty; \mathbf{R}^m)$, so that, for all $t \geq 0$, $T(t)x_n$ converges to $T(t)x$ in $C(-\infty, 0; \mathbf{R}^m)$ and $x_n'(.)$ converges weakly to $x'(.)$ in $L^\infty(0, \infty; \mathbf{R}^m)$, i.e for all $\tau > 0$ $x_n'(.)$ converges weakly to $x'(.)$ in $L^2([0, \tau], \mathbf{R}^m)$.

Since the assumptions of Theorem 1.4.1 in [1] are satisfied we obtain that

$$x'(t) \in coF(T(t)x) \subset \partial_C V(x(t)) \quad a.e. \ ([0, \tau]). \tag{3.7}$$

On the other hand, by construction

$$x_n'(t) \in \partial_C V(x_n(t)) \quad a.e. \ ([0,\tau]). \tag{3.8}$$

We apply Lemma 2.2 and we get

$$\lim_{n\to\infty} \int_0^\tau ||x_n'(t)||^2 dt = \lim_{n\to\infty} [(V \circ x_n)(\tau) - (V \circ x_n)(0)] =$$

$$= (V \circ x)(\tau) - (V \circ x)(0) = \int_0^\tau ||x'(t)||^2 dt,$$

i.e. $x_n'(.)$ converges strongly in $L^2([0,\tau], \mathbf{R}^m)$ to $x'(.)$. Hence, there exists a subsequence (still denoted) $x_n'(.)$ that converges pointwise to $x'(.)$.

On the other hand, from Hypothesis 2.4 graph(F) is closed and from (3.6) it follows that for almost all t, $x'(t) \in F(T(t)x)$; at the same time $T(0)x = \varphi$ and $x(t) \in K$ and the proof is complete. □

Remark 3.2 If in Theorem 2.3 $V(.)$ is assumed to be a convex function then from Theorem 3.1 we obtain Theorem 2.2 in [4].

The result in Theorem 3.1 remains valid if instead of $\mathbf{R}^m$ we work on a real Hilbert space X. The main tool in the proof will be, this time, an infinite dimensional version of Theorem 2.3, namely Theorem 2.3 in [3].

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