A remarkable transformations group on the tangent bundle

by

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Abstract

In the present paper starting from the notion of metrical structure on the tangent bundle, we determine all metrical d-linear connections in the case when the nonlinear connection is arbitrary and we find important particular cases. We study the role of the torsion tensor fields: T and S in this theory. We find the group of transformations of semi-symmetric metrical d-linear connections, corresponding to the same nonlinear connection N, and its important invariants.

Key Words: Tangent bundle, d-linear connection, curvature, torsion, metrical structure, metrical d-linear connection, semi-symmetric metrical d-linear connection, invariants.

2000 Mathematics Subject Classification: Primary 53C05, Secondary: 53C20, 53C60, 53B40, 58B20.

1 Introduction

The geometry of the tangent bundle \((TM, \pi, M)\) has been studied by M.Matsumoto in [4], by R.Miron and M.Anastasiei in [5], [6], by R.Miron and M.Hashiguchi in [7], by V.Oproiu in [8], by Gh.Atanasiu and I.Ghinea in [1], by R.Bowman in [2], by K.Yano and S.Ishihara in [10], etc.

In the present section we recall the basic notions which are needed. For more details see [5-6].

Let \(M\) be a real \(C^\infty\)-differentiable manifold with dimension \(n\), and \((TM, \pi, M)\) its tangent bundle. If \((x^i)\) is a local coordinates system on a domain \(U\) of a chart on \(M\), the induced system of coordinates on \(\pi^{-1}(U)\) is \((x^i, y^j), (i = 1, \ldots, n)\).

Let \(V(TM) = ker \pi \subset T(TM)\) be the vertical bundle, spanned locally by \(\{\frac{\partial}{\partial y^i}\}\). A nonlinear connection \(N\) determines a supplementary subbundle to \(V(TM)\) in \(T(TM)\), i.e. \(T(TM) = H(TM) \oplus V(TM)\). The adapted basis is \(\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - N^j \frac{\partial}{\partial y^j}\) where \(N^j(x, y)\) are the coefficients of the nonlinear connection.
A metric structure on $TM$ is a tensor field $G$ which satisfies the conditions: it is nondegenerate, symmetric and with constant signature. In the adapted basis, the metric structure $G$ is:

$$G(x, y) = \frac{1}{2} g_{ij}(x, y) dx^i \otimes dx^j + \frac{1}{2} \tilde{g}_{ij}(x, y) dy^i \otimes dy^j,$$

where $\{dx^i, \delta y^i\}$, is the dual basis of $\{\delta x^i, \partial / \partial y^i\}$, and $(g_{ij}(x, y), \tilde{g}_{ij}(x, y))$ is a pair of given d-tensor fields on $TM$, of the type $(0,2)$, each of them symmetric and nondegenerate.

The Obata’s operators associated to the metric structure $G$ are $\Omega^{ir}_{ij}$, $\tilde{\Omega}^{ir}_{ij}$, $\Omega^{*ir}_{ij}$, $(\text{see } [5] \text{ p.96})$, which have the same properties as the ones associated with a Finsler space $[7]$.

Let $D$ be a d-linear connection on $TM$ with the local coefficients:

$$D\Gamma(N) = (L^i_{jk}, \tilde{L}^i_{jk}, C^i_{jk}, C^i_{jk}).$$

It is called metrical d-linear connection with respect to $G$ if $D$ preserves by parallelism the vertical distribution $V(TM)$ and $DG=0$. In locally coordinate, these mean:

$$g_{ij;k} = 0, \quad g_{ij;k} = 0, \quad \tilde{g}_{ij;k} = 0, \quad \tilde{g}_{ij;k} = 0,$$

where $\big| \big|$ denote the h- and respective v-covariant derivatives with respect to $D$.

A d-linear connection, $D$, on $TM$, is called semi-symmetric d-linear connection if the torsion tensor fields $T_{(0)}^i_{jk}$ and $S^i_{jk}$ have the form:

$$T_{(0)}^i_{jk} = \frac{1}{n-1} (T_j \delta_k^i - T_k \delta_j^i) = \sigma_j \delta_k^i - \sigma_k \delta_j^i;$$
$$S^i_{jk} = \frac{1}{n-1} (S_j \delta_k^i - S_k \delta_j^i) = \tau_j \delta_k^i - \tau_k \delta_j^i,$$

where $T_j = T_{(0)}^i_{ji}$, $S_j = S^i_{ji}$ and $\sigma_j = \frac{T_j}{n-1}, \tau_j = \frac{S_j}{n-1}$.

### 2 Metrical d-linear connections on $TM$.

We shall determine the set of all metrical d-linear connections with respect to $G$.

Let $\Gamma(N)$ be another nonlinear connection on $TM$, with the coefficients:

$$N^i_{\ j}(x, y), (i, j = 1, ..., n).$$

Let $D\Gamma(N) = (L^i_{jk}, \tilde{L}^i_{jk}, C^i_{jk}, C^i_{jk})$ be the local coefficients of a fixed d-linear connection $D$ on $TM$. Then any d-linear connection, $D$, on $TM$, with local coefficients: $D\Gamma(N) = (L^i_{jk}, \tilde{L}^i_{jk}, C^i_{jk}, C^i_{jk})$, can be expressed...
in the form (see [9]):

\[ \begin{align*}
N_i^i &= N_j^j - A_i^i, \quad A_i^0 = 0, \\
L_i^j &= L_j^i + A_j^i C_j^i_l - B_i^l, \quad C_i^j = C_{0}^j = D_i^j, \\
\tilde{L}_i^j &= \tilde{L}_j^i + A_j^i C_j^i_l - \tilde{B}_i^l, \quad \tilde{C}_i^j = \tilde{C}_j^i = \tilde{D}_i^j = \tilde{D}_j^i,
\end{align*} \]

where \((A_i^j, B_i^j, \tilde{B}_i^j, \tilde{D}_i^j, D_i^j)\) are components of the difference tensor fields of \(D\Gamma(N)\) from \(D\Gamma(0)\).

By extension of the R. Miron-M. Hashiguchi method given for the case of Finsler connections in [7], from (4) and (2) we have one of the most important results concerning to the metrical d-linear connections:

**Theorem 2.1.** The set of all metrical d-linear connections on \(TM\), with local coefficients \(D\Gamma(N) = (L_i^j, \tilde{L}_i^j, \tilde{C}_i^j, C_i^j)\) is given by:

\[ \begin{align*}
N_i^i &= N_j^j - X_i^j, \\
L_i^j &= L_j^i + \tilde{C}_j^i m X_m^k + \frac{1}{2} g^{ij}(g_{jk} + g_{kj}) X_m^k + \Omega_{hk}^r X_r^k, \\
\tilde{L}_i^j &= \tilde{L}_j^i + C_j^i m X_m^k + \frac{1}{2} \tilde{g}^{ij}(\tilde{g}_{jk} + \tilde{g}_{kj}) X_m^k + \tilde{\Omega}_{hk}^r \tilde{X}_r^k, \\
\tilde{C}_i^j &= \tilde{C}_j^i + \frac{1}{2} g^{ij} g_{js} X_s^k + \Omega_{hk}^r \tilde{Y}_r^k, \\
C_i^j &= C_j^i + \frac{1}{2} \tilde{g}^{ij} \tilde{g}_{js} X_s^k + \tilde{\Omega}_{hk}^r Y_r^k, \\
X_i^0 &= 0,
\end{align*} \]

where \(X_i^j, X_j^i, \tilde{X}_i^j, \tilde{Y}_i^j, Y_j^i\) are arbitrary tensor fields on \(TM\).

**Particular cases:**

1. If we take \(X_i^j = X_j^i = \tilde{X}_i^j = \tilde{Y}_i^j = Y_j^i = 0\) in Theorem 2.1., we obtain an example of metrical d-linear connection on \(TM\), given in (1.12) p.96 from [5].

2. If we take a metrical d-linear connection on \(TM\) (e.g. canonical d-linear connection of \(G\), with local coefficients: \(\Gamma(N) = (L_i^j, \tilde{L}_i^j, \tilde{C}_i^j, C_i^j)\), (see (1.11) p.96 from [5]) as \(D\), in Theorem 2.1., we have:

**Proposition 2.1.** The set of all metrical d-linear connections on \(TM\), with local coefficients: \(D\Gamma(N) = (L_i^j, \tilde{L}_i^j, \tilde{C}_i^j, C_i^j)\) is given by:
Proposition 2.2. \( R.Miron \) and \( M.Anastasiei \) in (2.6) p.98 from [5].

Conversely, given \( \sigma \) a transformation of a semi-symmetric metrical d-linear connection \( \sigma \).

The set of all semi-symmetric metrical d-linear connections

3. If we take \( X^i_j = 0 \) in Proposition 2.1 we obtain: Theorem 1.3 p.96 from [5].

4. If we shall try replace the arbitrary tensor fields \( X^i_j, Y^i_j \) in Theorem 2.1 with \( X^i_j = 0 \), by the torsion fields \( T^i_{(0)jk}, S^i_{jk} \) we find a result obtained by \( R.Miron \) and \( M.Anastasiei \) in (2.6) p.98 from [5].

Taking into account (3) and (2.6) p.98, [5] we obtain:

**Proposition 2.2.** The set of all semi-symmetric metrical d-linear connections with local coefficients: \( D\Gamma(N) = (L^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, \tilde{C}^i_{jk}) \), is given by:

\[
\begin{align*}
L^i_{jk} &= L^i_{jk} + C^i_{jk} X^m_k + \Omega^i_{jk} X^h r_k, \\
\tilde{L}^i_{jk} &= \tilde{L}^i_{jk} + \tilde{C}^i_{jk} X^m_k + \tilde{\Omega}^i_{jk} X^h r_k, \\
\tilde{C}^i_{jk} &= \tilde{C}^i_{jk} + \tilde{\Omega}^i_{jk} X^h r_k, \\
C^i_{jk} &= C^i_{jk} + \Omega^i_{jk} X^h r_k, \\
X^i_{j|k} &= 0.
\end{align*}
\]

(6)

\[
\begin{align*}
L^i_{jk} &= L^i_{jk} + C^i_{jk} X^m_k + \Omega^i_{jk} X^h r_k, \\
\tilde{L}^i_{jk} &= \tilde{L}^i_{jk} + \tilde{C}^i_{jk} X^m_k + \tilde{\Omega}^i_{jk} X^h r_k, \\
\tilde{C}^i_{jk} &= \tilde{C}^i_{jk} + \tilde{\Omega}^i_{jk} X^h r_k, \\
C^i_{jk} &= C^i_{jk} + \Omega^i_{jk} X^h r_k.
\end{align*}
\]

(7)

3 The group of transformations of semi-symmetric metrical d-linear connections.

We study the transformations \( t(\sigma_j, \tau_j) : D\Gamma(N) \rightarrow D\tilde{\Gamma}(N) \) of the semi-symmetric metrical d-linear connections, on \( TM \), with respect to \( G \).

Let \( N \) be a given nonlinear connection. Then any semi-symmetric metrical d-linear connection with local coefficients \( D\tilde{\Gamma}(N) = (\tilde{L}^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, \tilde{C}^i_{jk}) \) is given by (7). We have:

**Proposition 3.1.** Two semi-symmetric metrical d-linear connections with local coefficients \( D\Gamma(N) = (L^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, \tilde{C}^i_{jk}) \) and \( D\tilde{\Gamma}(N) = (\tilde{L}^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, \tilde{C}^i_{jk}) \) are related as follows:

\[
\begin{align*}
\tilde{L}^i_{jk} &= L^i_{jk} + \sigma_j \delta^i_k - g_{jk} g^{im} \sigma_m, \\
\tilde{L}^i_{jk} &= \tilde{L}^i_{jk}, \\
\tilde{C}^i_{jk} &= \tilde{C}^i_{jk}, \\
\tilde{C}^i_{jk} &= C^i_{jk} + \tau_j \delta^i_k - \tilde{g}_{jk} g^{im} \tau_m, \\
\tilde{C}^i_{jk} &= C^i_{jk}.
\end{align*}
\]

(8)

Conversely, given \( \sigma_j \in \mathcal{X}^*(M), \tau_j \in \mathcal{X}^*(M) \) the above (8) is thought to be a transformation of a semi-symmetric metrical d-linear connection \( D \), with local
coincide with the nonlinear connection $N$, transitively.

We shall denote this transformation by: $t(\sigma_j, \tau_j)$.

Thus we have:

**Proposition 3.2.** The set $\mathcal{T}_N$ of all transformations $t(\sigma_j, \tau_j) : D(N) \to D(N)$ of semi-symmetric metrical d-linear connections given by $(8)$ is an abelian group, together with the mapping product:

$$t(\sigma_j, \tau_j) \circ t(\sigma_j, \tau_j) = t(\sigma_j + \sigma_j, \tau_j + \tau_j).$$

This group acts on the set of all semi-symmetric metrical d-linear connections, corresponding to the same nonlinear connection $N$, transitively.

In order to find invariants of the group $\mathcal{T}_N$, let us consider the transformation formulas of the torsion and the curvature tensor fields by a transformation of d-linear connections corresponding to the same nonlinear connection $N (A^i_j = 0)$:

$$t(0, B^i_jk, \bar{B}^i_jk, \tilde{D}^i_jk, D^i_jk) : D(N) \to D(N)$$

$$N^i_j = N^i_j, \quad \bar{L}^i_jk = L^i_jk - B^i_jk, \quad \tilde{L}^i_jk = \tilde{L}^i_jk - \bar{B}^i_jk,$$

$$\bar{C}^i_jk = C^i_jk - \bar{D}^i_jk, \quad \bar{S}^i_jk = S^i_jk - D^i_jk.$$  \hspace{1cm} (9)

**Proposition 3.3.** By a transformation $(9)$ of d-linear connections, corresponding to the same nonlinear connection $N, D(N) \to D(N)$, the torsion and curvature tensor fields, $T_{(0)}^i jak, T_{(1)}^i jak, P_{(1)}^i jak, P_{(2)}^i jak, S^i jkl, S^i_{(0)} jkl, S^i_{(1)} jkl, P^i_{(1)} jkl, P^i_{(2)} jkl, S^i_{(0)} jkl, S^i_{(1)} jkl$ are transformed as follows:

$$\bar{T}_{(0)}^i jak = T_{(0)}^i jak + (B^i_{kj} - B^i_{jk}), \quad \bar{T}_{(1)}^i jak = T_{(1)}^i jak,$$  \hspace{1cm} (10)

$$\bar{P}_{(1)}^i jak = P_{(1)}^i jak - \bar{D}^i jk, \quad \bar{P}_{(2)}^i jak = P_{(2)}^i jak + \bar{B}^i jk,$$  \hspace{1cm} (11)

$$\bar{S}_{\bar{S}}^i jak = S^i jak + (D^i_{kj} - D^i_{jk}).$$  \hspace{1cm} (12)

$$\bar{R}_{(0)}^i jkl = R_{(0)}^i jkl - \bar{D}^i jk + \bar{D}^i_{hk} + B^i_{jh} T_{(0)}^h jkl + A_{kl} \{-B^i_{jkl} + B^h_{jh} B^i_{hlk}\},$$  \hspace{1cm} (13)

$$\bar{R}_{(1)}^i jkl = R_{(1)}^i jkl - D^i_{jh} + \bar{D}^i_{hk} + B^i_{jh} T_{(0)}^h jkl + A_{kl} \{-B^i_{jkl} + B^h_{jh} B^i_{hlk}\},$$  \hspace{1cm} (14)

$$\bar{P}_{(0)}^i jkl = P_{(0)}^i jkl - \bar{D}^i jh + \bar{D}^i_{hk} + B^i_{jh} \bar{C}^h_{kl} - B^i_{jklh} + \bar{D}^i_{jhl},$$  \hspace{1cm} (15)

$$+ B^h_{jh} \bar{D}^i_{hl},$$

$$\bar{P}_{(1)}^i jkl = P_{(1)}^i jkl - D^i jh + \bar{D}^i_{hk} + B^i_{jh} \bar{C}^h_{kl} - B^i_{jklh} + \bar{D}^i_{jhl},$$  \hspace{1cm} (16)

$$+ B^h_{jh} \bar{D}^i_{hl},$$

$$\bar{S}_{(0)}^i jkl = S_{(0)}^i jkl - \bar{D}^i jh + \bar{D}^i_{hl} + A_{kl} \{-D^i_{jkl} + B^h_{jh} \bar{D}^i_{hl}\},$$  \hspace{1cm} (17)

$$\bar{S}_{(1)}^i jkl = S_{(1)}^i jkl - D^i jh + \bar{D}^i_{hl} + A_{kl} \{D^i_{jkl} + B^h_{jh} \bar{D}^i_{hl}\}.$$  \hspace{1cm} (18)
We consider the tensor fields:

\[ K_{i(0)}^jkl = R_{i(0)}^jkl - \tilde{C}_{ih}^j R_{hkl}^i, \]  
(19)  

\[ K_{i(1)}^jkl = R_{i(1)}^jkl - C_{ih}^j R_{hkl}^i, \]  
(20)  

\[ P_{i(0)}^jkl = A_{kl} \left\{ P_{i(0)}^jkl - \tilde{C}_{ih}^j \partial N_h^i \partial y^l \right\}, \]  
(21)  

\[ P_{i(1)}^jkl = A_{kl} \left\{ P_{i(1)}^jkl - C_{ih}^j \partial N_h^i \partial y^l \right\}. \]  
(22)  

**Proposition 3.4.** By a transformation (9) of d-linear connections corresponding to the same nonlinear connection \( N \), the tensor fields \( K_{i(0)}^jkl, K_{i(1)}^jkl, P_{i(0)}^jkl, P_{i(1)}^jkl \) are transformed as follows:

\[ \bar{K}^i_{(0)}^jkl = K_{i(0)}^jkl + 2A_{kl} \left\{ -B_{ij}^l + B_{ij}^h \tilde{B}_h^i \right\}, \]  
(23)  

\[ \bar{K}^i_{(1)}^jkl = K_{i(1)}^jkl - \tilde{B}_{ij}^l + \tilde{B}_{ij}^h \tilde{B}_h^i, \]  
(24)  

\[ \bar{P}_{(0)}^i_jkl = P_{(0)}^i_jkl + A_{kl} \left\{ -B_{ij}^l + \tilde{B}_{ij}^l + B_{ij}^h \tilde{D}_h^i \right\}, \]  
(25)  

\[ \bar{P}_{(1)}^i_jkl = P_{(1)}^i_jkl + A_{kl} \left\{ -\tilde{B}_{ij}^l + D_{ij}^l + \tilde{B}_{ij}^h \tilde{D}_h^i \right\}. \]  
(26)  

Substituting in (9):

\[ B_{ij}^l = -2\tilde{\Omega}_{kij}^m \sigma_m, \quad \tilde{B}_{ij}^l = 0, \quad \tilde{D}_{ij}^l = 0, \quad D_{ij}^l = -2\tilde{\Omega}_{kij}^ml, \]  
(27)  

we have the transformation (8)

**Proposition 3.5.** By a transformation (8) of semi-symmetric metrical d-linear connections corresponding to the same nonlinear connection \( N \), the tensor fields \( K_{i(0)}^jkl, K_{i(1)}^jkl, S_{i(0)}^jkl, S_{i(1)}^jkl \) are transformed as follows:

\[ \bar{K}_{(0)}^i_jkl = K_{(0)}^i_jkl + 2A_{kl} \left\{ \Omega_{kij}^m \sigma_m \right\}, \]  
(28)  

\[ \bar{K}_{(1)}^i_jkl = K_{(1)}^i_jkl, \]  
(29)  

\[ \bar{S}_{(0)}^i_jkl = S_{(0)}^i_jkl, \]  
(30)  

\[ \bar{S}_{(1)}^i_jkl = S_{(1)}^i_jkl + 2A_{kl} \tilde{\Omega}_{kij}^ml. \]  
(31)  

where:
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\[
\sigma_{ml} = \sigma_{m|l} - \sigma_{m} \sigma_{l} + \frac{1}{2} g_{ml} \sigma - \frac{g^{rm} \sigma_{r} \sigma_{m}}{n-1}, \quad (\sigma = g^{rm} \sigma_{r} \sigma_{m}), \tag{32}
\]

\[
\tau_{ml} = \tau_{m|l} - \tau_{m} \tau_{l} + \frac{1}{2} \tilde{g}_{ml} \tau - \frac{\tau_{m} s_{n}}{n-1}, \quad (\tau = g^{rm} \tau_{r} \tau_{m}). \tag{33}
\]

Using this results we can determine the invariants of the group \( \mathcal{T}^{n} \) using a well-known elimination method:

**Theorem 3.1.** For \( n > 2 \) the following tensor fields:

\[
H_{(0)}^{i} j kl, \quad H_{(1)}^{i} j kl, \quad M_{(0)}^{i} j kl, \quad M_{(1)}^{i} j kl
\]

are invariants of the group \( \mathcal{T}^{n} \) of transformations, of semi-symmetric metrical d-linear connections on TM, corresponding to the same nonlinear connection N:

\[
H_{(0)}^{i} j kl = \mathcal{K}_{(0)}^{i} j kl + \frac{2}{n-2} A_{kl} \{ \mathcal{K}_{(0)}^{i} r l (\mathcal{K}_{(0)}^{r} r l - \frac{\mathcal{K}_{(0)}^{r} g_{rl}}{2(n-1)} ) \}, \tag{34}
\]

\[
H_{(1)}^{i} j kl = \mathcal{K}_{(1)}^{i} j kl, \tag{35}
\]

\[
M_{(0)}^{i} j kl = S_{(0)}^{i} j kl, \tag{36}
\]

\[
M_{(1)}^{i} j kl = S_{(1)}^{i} j kl + \frac{2}{n-2} A_{kl} \{ \mathcal{K}_{(1)}^{i} r l (S_{(1)}^{r} r l - \frac{S_{(1)}^{r} \tilde{g}_{rl}}{2(n-1)} ) \}, \tag{37}
\]

where:

\[
\mathcal{K}_{(0)}^{i} j k l = \mathcal{K}_{(0)}^{i} j k l, \quad \mathcal{K}_{(0)} = g^{ik} \mathcal{K}_{(0)}^{i} k l, \quad S_{(1)}^{i} j k l = S_{(1)}^{i} k l, \quad S_{(1)} = g^{ik} S_{(1)}^{i} k l.
\]

We note that the results obtained from Theorem 3.1 in the particular case of the normal d-linear connections support the findings of R. Miron and M. Hashiguchi in their paper [7].

**References**


Received: 06.06.2005.

Revised: 29.05.2006.