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Verification theorems for infinite-horizon optimal control problems

by
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Abstract

Using a certain monotonicity property of the value function of an infinite-horizon optimal control problem, we derive first an "abstract verification theorem" for feasible selections of admissible trajectories. Further, using suitable monotonicity results and generalized derivatives we obtain several "practical" verification theorems under different regularity assumptions on the value function associated to a feasible selection of admissible trajectories.

Key Words: Optimal Control, Infinite-Horizon, Feasible Selection, Value Function, Verification Theorem.
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1 Introduction

The aim of this paper is to extend to infinite-horizon optimal control problems the results in V. Lupulescu and Şt. Mirică [8] and Şt. Mirică [14] concerning verification theorems of Dynamic Programming type.

In this setting one assumes that a "feasible selection" of admissible trajectories (i.e. a "generalized field of extremals") is given and the verification theorems contain sufficient conditions on the corresponding value functions that imply their optimality.

These sufficient conditions consist in regularity properties (i.e. differentiability, lipschitzianity, continuity, etc.) accompanied by suitable differential inequalities that imply the monotonicity of the value function along admissible trajectories.

Due to the rather irregular behavior of the value functions of optimal control problems, a reasonable approach seems to be that of attaching the simplest type of differential inequality to each type of regularity property of the value function.

We are not discussing here the important problem of finding a generalized "field of extremals" and the corresponding value function; this can be done either by using the available necessary optimality conditions (Pontriaghin's Minimum Principle (PMP)) or, even more efficiently, by using suitable extensions of Cauchy's Method of Characteristics for the generally non-smooth, Hamilton-Jacobi-Bellman equations associated to a problem.

The paper is organized as follows: in Section 2 we present the notations and definitions to be used in the rest of the paper. In Section 3 we recall the basic monotonicity and asymptotic properties in I. Mirică [9] and derive an "abstract verification theorem" for feasible selections of admissible trajectories".

The main results of the paper are contained in Section 4 where the required monotonicity property is proved to be implied by suitable differential inequalities and corresponding regularity properties of the value functions.

Finally, in the last section we present an example illustrating some of the theoretical aspects in the previous sections.

To the best of authors' knowledge, the only existing result of the type in Section 4 is the so called "Carathéodory's Method" in [2, 3] which, in fact, may be considered as a particular case of the "Elementary verification theorem" 4.1 below.

2 Notations and definitions

In this paper we are studying the *value function* of an infinite-horizon optimal control problem which consists in the *minimization of each of the functionals*

$$\mathcal{C}(s,y;x(\cdot)) := \int_s^\infty f_0(t,x(t),x'(t))dt,\ \ (s,y)\in E\subseteq \mathbb{R}\times\mathbb{R}^n \tag{2.1}$$

subject to:

$$x'(t)\in F(t,x(t))\ \ a.e.\ (s,\infty),\ \ x(s)=y \tag{2.2}$$

$$(t,x(t))\in E\subseteq \mathbb{R}\times\mathbb{R}^n\quad (\forall)\ t\in[s,\infty) \tag{2.3}$$

$$\begin{gathered}\widehat{x}(\cdot) := (x(\cdot),x_0(\cdot))\in\Omega_\alpha\subseteq AC^{loc}([s,\infty);\mathbb{R}^n\times\mathbb{R}),\\ x_0(t) := \int_s^t f_0(\sigma,x(\sigma),x'(\sigma))d\sigma\end{gathered} \tag{2.4}$$

$$f_0(\cdot,x(\cdot),x'(\cdot))\in L^\infty([s,\infty);\mathbb{R}), \tag{2.5}$$

where $\Omega_\alpha\subseteq AC^{loc}$ is a specified class of locally absolutely continuous mappings.

We note that *the data* of the problem are the following:

- the set $E\subseteq\mathbb{R}\times\mathbb{R}^n$ of *admissible phases* (time-state);

- the multifunction $F(\cdot,\cdot) : E \to \mathcal{P}(\mathbb{R}^n)$ defining the "dynamics" in (2.2) and which, in particular, may be of a "continuously parameterized" type:

$$F(t,x) = f(t,x,U), \quad f(\cdot,\cdot,u), \ u \in U, \ \textit{continuous}; \tag{2.6}$$

- the real function, $f_0(\cdot,\cdot,\cdot) : G(F(\cdot,\cdot)) \to \mathbb{R}$, $G(F(\cdot,\cdot)) := \{(t,x,x');\ (t,x) \in E,\ x' \in F(t,x)\}$, defining the cost functional in (2.1) and which, in the case of parameterized differential inclusion in (2.6) is replaced by $f_0(\cdot,\cdot,\cdot) : E \times U \to \mathbb{R}$ so that the cost functional in (2.1) takes the form:

$$\mathcal{C}(s,y;u(\cdot)) = \int_s^\infty f_0(t,x(t),u(t))dt, x'(t) = f(t,x(t),u(t)), u(t) \in U; \tag{2.7}$$

- the class $\Omega_\alpha \in \{\Omega_{pc}, \Omega_r, \Omega_p, p \in [1,\infty]\}$ of locally absolutely continuous admissible trajectories where $x(\cdot) \in \Omega_{pc}$ if $x'(\cdot)$ is piecewise continuous, $x(\cdot) \in \Omega_r$ if $x'(\cdot)$ is regulated and $x(\cdot) \in \Omega_p,\ p \in [1,\infty]$ if $x'(\cdot) \in L_p^{loc}([s,\infty);\mathbb{R}^n)$.

We point out the fact that if $h(\cdot) : [s,\infty) \to \mathbf{R}$ is measurable then we adopt the measure-theoretic definition (e.g. Dunford and Schwartz [5]) of the integral $\int_s^\infty h(t)dt$ so that it has the properties:

$$\int_s^\infty h(t)dt = \lim_{T\to\infty} \int_s^T h(t)dt, \quad \lim_{s\to\infty} \int_s^\infty h(t)dt = 0. \tag{2.8}$$

We note that in the theory of normal integrands in Rockafellar [17] as well as in Carlson et. al.[3], etc., one uses different definitions of this integral which may not have the properties in (2.8).

Denoting by $\Omega_\alpha(s,y)$ the set of trajectories $x(\cdot) : [s,\infty) \to \mathbb{R}^n$ satisfying (2.2)-(2.5), the *value function* of the problem is defined by:

$$W(s,y) := \inf_{x(\cdot)\in\Omega_\alpha(s,y)} \mathcal{C}(s,y;x(\cdot)),\ (s,y) \in E, \tag{2.9}$$

hence, using the convention $\inf\emptyset = +\infty$, one obtains: $W(s,y) = +\infty$ iff $\Omega_\alpha(s,y) = \emptyset$.

For each $(s,y) \in E$, the (possibly empty) set of *optimal trajectories* corresponding to the initial point (s,y) is defined by:

$$\widetilde{\Omega}_\alpha(s,y) := \{\widetilde{x}(\cdot) \in \Omega_\alpha(s,y);\ \mathcal{C}(s,y;\widetilde{x}(\cdot)) = W(s,y)\}. \tag{2.10}$$

Thus, the "phase space" $E \subseteq \mathbb{R} \times \mathbf{R}^n$ admits the partition:

$$\begin{gathered} E = E^{+\infty} \cup E^{-\infty} \cup E^R, \quad E^{\pm\infty} := \{(s,y) \in E;\ \ W(s,y) = \pm\infty\}, \\ E^R = dom(W(.,.)) := \{(s,y) \in E;\ \ W(s,y) \in \mathbb{R}\}. \end{gathered} \tag{2.11}$$

Moreover, the effective domain, E^R, of the value function may be partitioned by:

$$E^R = \widetilde{E} \cup E^i,\ \ \widetilde{E} := \{(s,y) \in E^R;\ \widetilde{\Omega}_\alpha(s,y) \neq \emptyset\}, E^i := E^R \setminus \widetilde{E}. \tag{2.12}$$

Throughout the paper we shall assume the following:

Hypothesis 2.1.

(i) The subset $E \subseteq \mathbb{R} \times \mathbb{R}^n$ is nonempty and $F(\cdot,\cdot) : E \to \mathcal{P}(\mathbb{R}^n)$ is a multifunction with nonempty closed values;

(ii) The real function $f_0(\cdot,\cdot,\cdot) : G(F(\cdot,\cdot)) \to \mathbb{R}$, is a "normal integrand" in the sense of Rockafellar [17].

3 Monotonicity and asymptotic properties and the abstract verification theorem

In this section we recall first the basic monotonicity and asymptotic properties in I. Mirică [9] of the value function in (2.9), which are expressed in terms of the larger class of *locally admissible trajectories* defined by:

$$\begin{gathered}\Omega^{loc}_\alpha(s,y) := \{x(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}^n;\ \widehat{x}(\cdot) \in \Omega^{loc}_\alpha,\\ x'(t) \in F(t,x(t)) a.e.,\ \ s \in I,\ \ x(s) = y,\ \ (t,x(t)) \in E\ \forall\ t \in I\}\end{gathered} \tag{3.1}$$

and of the associated "extended" real functions

$$\omega_x(t) := W(t,x(t)) + \int_s^t f_0(r,x(r),x'(r))dr,\ \ t \in I. \tag{3.2}$$

Theorem 3.1. ([9]) *If Hypothesis* 2.1 *is satisfied then the value function* $W(\cdot,\cdot)$ *in* (2.9) *has the following properties:*

(i) (**Monotonicity**). *For any* $(s,y) \in E,\ x(\cdot) \in \Omega^{loc}_\alpha(s,y),$ *the extended real function* $\omega_x(\cdot)$ *in* (3.2) *is increasing* (i.e. $\omega_x(t_1) \leq \omega_x(t_2),\ (\forall) s \leq t_1 < t_2 < +\infty,\ t_1,t_2 \in I$);

(ii) (**Asymptotic properties**). *For any* $(s,y) \in E \setminus E^{+\infty},\ x(\cdot) \in \Omega_\alpha(s,y),$ *there exists* $\lim\limits_{t\to+\infty} W(t,x(t))$ *and it satisfies the following relations:*

$$0 \geq \lim_{t\to+\infty} W(t,x(t)) \geq W(s,y) - \mathcal{C}(s,y;x(\cdot)); \tag{3.3}$$

(iii) (**Optimality**). *If* $(s,y) \in \widetilde{E}$ *and* $\widetilde{x}(\cdot) \in \Omega_\alpha(s,y)$ *then* $\widetilde{x}(\cdot) \in \widetilde{\Omega}_\alpha(s,y)$ (i.e. *it is optimal*) *iff the real function* $\omega_{\widetilde{x}}(\cdot)$ *in* (3.2) *is constant and satisfies:*

$$\omega_{\widetilde{x}}(t) = \omega_{\widetilde{x}}(s) = W(s,y), (\forall) t \in [s,\infty),\ \ \lim_{t\to+\infty} W(t,\widetilde{x}(t)) = 0. \tag{3.4}$$

Remark 3.2. As it is easy to prove, the monotonicity property (i) in Theorem 3.1 is equivalent with the so called "functional equation of Dynamic Programming",

$$W(s,y) = \inf_{x(\cdot)\in\Omega_\alpha^T(s,y)} [W(T,x(T)) + \int_s^T f_0(t,x(t),x'(t))dt], \ T > s, \tag{3.5}$$

where the set of "truncated" (locally admissible) trajectories is defined by:

$$\Omega_\alpha^T(s,y) := \{x(\cdot) \in \Omega_\alpha^{loc}(s,y);\ [s,T] \subset dom(x(\cdot)\}). \tag{3.6}$$

We recall that the "functional equation" in (3.5) ("Dynamic Programming Principle") is frequently used in the theory of viscosity solutions of Hamilton-Jacobi equations associated to the problem (2.1-(2.5) (e.g. Bardi and Cappuzo-Dolcetta [1], Crandall and Lions [4], etc.). However, as we shall see in what follows, the (equivalent) monotonicity property (i) in Theorem 3.1 may lead to stronger results than those in the theory of viscosity solutions.

Remark 3.3. The monotonicity property (i) in Theorem 3.1 implies the fact that for any $(s,y) \in E^{\mathbb{R}}$, $x(\cdot) \in \Omega_\alpha^{loc}(s,y)$ one has:

$$\frac{d}{dt}W(t,x(t)) + f_0(t,x(t),x'(t)) \geq 0 \ \ a.e.(s,\infty), \tag{3.7}$$

which, in the case $W(\cdot,\cdot)$ is *differentiable* at $(t,x(t))a.e.(s,\infty)$, takes the form:

$$\frac{\partial W}{\partial t}(t,x(t)) + \frac{\partial W}{\partial x}(t,x(t)) \cdot x'(t) + f_0(t,x(t),x'(t)) \geq 0 \ \ a.e.(s,\infty). \tag{3.8}$$

On the other hand, if $(s,y) \in \widetilde{E}$ and $\widetilde{x}(\cdot) \in \widetilde{\Omega}_\alpha(s,y)$ is optimal and the same condition is satisfied, then from (3.4) it follows:

$$\frac{\partial W}{\partial t}(t,\widetilde{x}(t)) + \frac{\partial W}{\partial x}(t,\widetilde{x}(t)) \cdot \widetilde{x}'(t) + f_0(t,\widetilde{x}(t),\widetilde{x}'(t)) = 0 \ \ a.e.(s,\infty). \tag{3.9}$$

The properties in (3.8), (3.9) suggest the fact that under certain hypotheses, at differentiability points, the value function satisfies the well known "Partial Differential Equation of Dynamic Programming":

$$\begin{aligned} &\tfrac{\partial W}{\partial t}(t,x) + H(t,x,\tfrac{\partial W}{\partial x}(t,x)) = 0 \\ &H(t,x,p) = \inf_{v\in F(t,x)} [< p,v > + f_0(t,x,v)] \end{aligned} \tag{3.10}$$

known also as the Hamilton-Jacobi-Bellman (HJB) equation associated to the problem (2.1)-(2.5).

However, as simple examples show, the value function in (2.9) has scarce regularity properties and, in particular, may not be differentiable at each point of its effective domain.

Though many authors apply the Dynamic Programming Method starting from a "viscosity solution" of the (HJB) equation in (3.10), a more realistic approach seems to be that of starting from a "feasible selection" of admissible trajectories ("generalized field of extremals") and the associated value function defined as follows.

Definition 3.4. *A multifunction* $E \ni (s,y) \mapsto \mathcal{A}(s,y) \subset \Omega_\alpha(s,y)$ *is said to be a* *feasible selection of admissible trajectories* *if:*

$$\mathcal{C}(s,y;\widetilde{x}_1(\cdot)) = \mathcal{C}(s,y;\widetilde{x}_2(\cdot)) \ \forall \ \widetilde{x}_1(\cdot), \widetilde{x}_2(\cdot) \in \mathcal{A}(s,y), \ (s,y) \in E \tag{3.11}$$

and the corresponding value function is defined by

$$W_\mathcal{A}(s,y) := \mathcal{C}(s,y;\widetilde{x}(\cdot)) \ if \ \widetilde{x}(\cdot) \in \mathcal{A}(s,y), \ (s,y) \in E. \tag{3.12}$$

The (multi)-selection $\mathcal{A}(\cdot,\cdot)$ *is said to be* *optimal* *if:*

$$W_\mathcal{A}(s,y) \leq \mathcal{C}(s,y;x(\cdot)) \ \forall \ x(\cdot) \in \Omega_\alpha(s,y), \ (s,y) \in E. \tag{3.13}$$

We note that such a selection may be "generated" by a "feedback control", $E \ni (s,y) \mapsto \widetilde{F}(s,y) \subset F(s,y)$, in the sense that for any initial point $(s,y) \in E$, the trajectories $\widetilde{x}(\cdot) \in \mathcal{A}(s,y)$ are solutions of the differential inclusion

$$x' \in \widetilde{F}(s,y), \ x(s) = y \tag{3.14}$$

that satisfy the constraints in (2.1)-(2.5).

Conversely, in some cases, a selection $\mathcal{A}(\cdot,\cdot)$ in Definition 3.4 could be "synthetized" , to obtain a feedback control of the form in (3.14).

As already stated, a multi-selection $\mathcal{A}(\cdot,\cdot)$ as well as its corresponding value function, $W_\mathcal{A}(\cdot,\cdot)$, in (3.12) may be obtained either by suitable extensions of Cauchy's Method of Characteristics for the generally non-smooth, (HJB) equations in (3.10) or by using the necessary optimality conditions (PMP), when available.

Due to the results in Theorem 3.1, we obtain the following *necessary and sufficient* optimality conditions:

Theorem 3.5. (**Abstract verification theorem**). *If Hypothesis* 2.1 *is satisfied,* $\mathcal{A}(\cdot,\cdot)$ *is a feasible selection and* $W_\mathcal{A}(\cdot,\cdot)$ *in* (3.12) *is the corresponding value function, then* $\mathcal{A}(\cdot,\cdot)$ *is optimal in the sense of* (3.13) *iff the following conditions are satisfied:*

(i) (**Asymptotic property**). *For any* $(s,y) \in E$

$$(\exists) \ \lim_{t\to+\infty} W_\mathcal{A}(t,x(t)) \leq 0, \ (\forall) \ x(\cdot) \in \Omega_\alpha(s,y) \tag{3.15}$$

(ii) (**Monotonicity**). *For any* $(s,y) \in E$, $x(\cdot) \in \Omega_\alpha(s,y)$ *the real function* $\omega_x(\cdot)$ *defined by:*

$$\omega_x(t) := W_\mathcal{A}(t,x(t)) + \int_s^t f_0(r,x(r),x'(r))dr, \ t \in [s,\infty) \tag{3.16}$$

is increasing. Moreover, in this case one has:

$$(\exists)\lim_{t\to+\infty} W_{\mathcal{A}}(t,\widetilde{x}(t))=0,\ (\forall)\ \widetilde{x}(\cdot)\in\mathcal{A}(s,y),\ (s,y)\in E. \tag{3.17}$$

Proof: " $\Rightarrow$ " : If $\mathcal{A}(\cdot,\cdot)$ is optimal in the sense of (3.13) then $W_{\mathcal{A}}(\cdot,\cdot)=W(\cdot,\cdot)$ has the properties in Theorem 3.1.

" $\Leftarrow$ " : Let us assume that $W_{\mathcal{A}}(\cdot,\cdot)$ has the properties (i), (ii) and let us consider $(s,y)\in E$, $x(\cdot)\in\Omega_\alpha(s,y)$ and $\omega_x(\cdot)$ the real function in (3.16); since $\omega_x(\cdot)$ is increasing, from the properties in (2.8) of the integral we infer that:

$$\begin{aligned}&\omega_x(t)=W_{\mathcal{A}}(t,x(t))+\int_s^t f_0(r,x(r),x'(r))dr\ge\omega_x(s)=W_{\mathcal{A}}(s,y),\\&\lim_{t\to+\infty}\int_s^t f_0(r,x(r),x'(r))dr=\int_s^\infty f_0(r,x(r),x'(r))dr=\mathcal{C}(s,y;x(\cdot))\end{aligned}\tag{3.18}$$

Moreover, since $\omega_x(\cdot)$ is increasing, it has one-sided limits hence:
$(\exists)\lim_{t\to+\infty}\omega_x(t)=\lim_{t\to+\infty}W_{\mathcal{A}}(t,x(t))+\mathcal{C}(s,y;x(\cdot))\ge W_{\mathcal{A}}(s,y)=\mathcal{C}(s,y;\widetilde{x}(\cdot))$
for any $\widetilde{x}(\cdot)\in\mathcal{A}(s,y)$ hence from the asymptotic property in (3.15) it follows:
$\lim_{t\to+\infty}\omega_x(t)=\lim_{t\to+\infty}W_{\mathcal{A}}(t,x(t))+\mathcal{C}((s,y);x(\cdot))\le\mathcal{C}(s,y;x(\cdot))$
and therefore
$W_{\mathcal{A}}(s,y)=\mathcal{C}(s,y;\widetilde{x}(\cdot))\le\mathcal{C}(s,y;x(\cdot)),\ (\forall)x(\cdot)\in\Omega_\alpha(s,y)$
which proves the optimality of $\mathcal{A}(\cdot,\cdot)$.

To prove that the optimal trajectories $\tilde{x}(\cdot)\in\mathcal{A}(s,y)$ have the asymptotic property in (3.17) we use first the monotonicity property in (ii) according to which $\omega_{\widetilde{x}}(t)\ge\omega_{\widetilde{x}}(s)=W_{\mathcal{A}}(s,y)$, $(\forall)t\ge s$ and, on the other hand, the properties in (2.8) and the asymptotic property in (3.15) from which it follows that
$(\exists)\lim_{t\to+\infty}\omega_{\widetilde{x}}(t)=\lim_{t\to+\infty}W_{\mathcal{A}}(t,\widetilde{x}(t))+\mathcal{C}(s,y;\widetilde{x}(\cdot))\ge W_{\mathcal{A}}(s,y)$,
hence $\lim_{t\to+\infty}W_{\mathcal{A}}(t,\widetilde{x}(t))\ge 0$ which, together with (3.15), implies (3.17). □

Remark 3.6. A different type of "abstract verification theorem" may be obtained in terms of the value functions associated the "finite-horizon truncated" optimal control problems:

$$W_T(s,y):=\inf_{x(\cdot)\in\Omega_\alpha^T(s,y)}\int_s^T f_0(t,x(t),x'(t))dt,\ (s,y)\in E,\ T>s \tag{3.19}$$

where
$\Omega_\alpha^T(s,y):=\{x(\cdot)\in\Omega_\alpha^{loc};\ (t,x(t))\in E,\ (\forall)t\in[s,T],\ x(s)=y\}$,
using the fact that

$$W(s,y)\le\limsup_{T\to+\infty}W_T(s,y),\ (\forall)(s,y)\in E. \tag{3.20}$$

On the other hand, the hypotheses of Theorem 3.5 are of "abstract type" since the monotonicity property (ii) is difficult to verify by "direct methods"; as the results in the following section show, this property may be implied by differential inequalities associated to corresponding regularity properties of the value functions $W_{\mathcal{A}}(\cdot,\cdot)$ in (3.12).

4 "Practical" verification theorems

The first case in which the monotonicity property in (ii) in Theorem 3.5 may be explicited is the particular one in which $E = Int(E) \subseteq \mathbb{R} \times \mathbb{R}^n$ (is open) and the value function $W_{\mathcal{A}}(\cdot,\cdot)$ in (3.12) is differentiable.

Theorem 4.1. (**The elementary verification theorem**). *Let the problem* (2.1)-(2.5) *be such that* $E = Int(E) \subseteq \mathbb{R} \times \mathbb{R}^n$ *and let* $\mathcal{A}(\cdot,\cdot)$ *be a feasible selection in the sense of Definition* 3.4 *such that the corresponding value function* $W_{\mathcal{A}}(\cdot,\cdot)$ *in* (3.12) *has the following properties:*

(i) (**Asymptotic properties**). *For any* $(s,y) \in E,\ x(\cdot) \in \mathcal{A}(s,y)$ *one has the asymptotic property in* (3.15).

(ii) (**Differential inequality**). *The usual* (Fréchet) *derivative of* $W_{\mathcal{A}}(\cdot,\cdot)$ *satisfies the inequality:*

$$DW_{\mathcal{A}}(s,y)\cdot(1,v) + f_0(s,y,v) \geq 0,\ (\forall)\ (s,y) \in E,\ v \in F(s,y) \tag{4.1}$$

(iii) *Either* $\Omega_\alpha \subseteq \Omega_r$ (i.e. *any admissible trajectory is at least regular*) *or* $W_{\mathcal{A}}(\cdot.\cdot)$ *it is locally-Lipschitz.*

Then the selection $\mathcal{A}(\cdot,\cdot)$ *is optimal in the sense of* (3.13).

Proof: According to Theorem 3.5, it is enough to prove that for any $(s,y) \in E,\ x(\cdot) \in \Omega_\alpha(s,y)$, the function $\omega_x(\cdot)$ in (3.16) is increasing; to this end we note that the function $\omega_x(\cdot)$ in (3.16) may be written as:

$$\omega_x(t) \equiv \omega_x^0(t) + x_0(t),\ \omega_x^0(t) := W_{\mathcal{A}}(t,x(t)), \tag{4.2}$$

and $x_0(.)$ is the function defined in (2.4). From the formulation in (2.2) of the problem it follows that there exists a null subset $J_x \subset [s,\ \infty)$ (that is at most countable if $\Omega_\alpha \subseteq \Omega_r$) such that

$$(\exists)\ x'(t) \in F(t,x(t)),\ x_0'(t) = f_0(t,x(t),x'(t)),\ \forall\ t \in [s,\infty) \setminus J_x. \tag{4.3}$$

Next, since $W_{\mathcal{A}}(\cdot,\cdot)$ is assumed to be differentiable, the functions $\omega_x^0(\cdot),\ x_0(\cdot)$ in (4.2) are differentiable at each point $t \in [s,\ \infty) \setminus J_x$ hence:

$$(\exists)\ \omega_x'(t) = DW_{\mathcal{A}}(t,x(t))\cdot(1,x'(t)) + f_0(t,x(t),x'(t)),\ t \notin J_x \tag{4.4}$$

and therefore, from the hypothesis in (4.1) it follows:

$$(\exists)\omega_x'(t) \geq 0,\ (\forall)\ t \in [s,\infty) \setminus J_x. \tag{4.5}$$

Finally, in the case $\Omega_\alpha \subseteq \Omega_r$, the "excepted" set J_x in (4.5) is at most countable and $\omega_x(\cdot)$ is continuous hence we may apply the so called "Corollary of the Zygmund's Lemma" to infer that $\omega_x(\cdot)$ is increasing; in the case $W_{\mathcal{A}}(\cdot,\cdot)$ is locally-Lipschitz, the function $\omega_x(\cdot)$ in (4.2) is locally AC and $J_x \subset (s,\infty)$ is a null subset hence, using the well-known Lebesgue monotonicity theorem, from (4.5) it follows that $\omega_x(\cdot)$ is increasing and Theorem 4.1 is proved. □

We note that in most papers the function $W_{\mathcal{A}}(\cdot,\cdot)$ is assumed to be of class C^2 (hence, in particular, locally-Lipschitz) overlooking the case in which $W_{\mathcal{A}}(\cdot,\cdot)$ is only differentiable but in which the optimality may be proved only in the case $\Omega_\alpha \subseteq \Omega_r$.

The result in Theorem 4.1 may be significantly extended to the case in which the value function $W_{\mathcal{A}}(\cdot,\cdot)$ is only "contingent differentiable" in the following sense:

Definition 4.2. *The mapping $g(\cdot) : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be "contingent differentiable" at $x \in X$ if for any vector in the right-contingent cone:*

$$K_x^+ X := \{v \in \mathbb{R}^n;\ \exists\ (s_m, v_m) \to (0+, v) :\ x + s_m \cdot v_m \in X\}, \tag{4.6}$$

there exists the limit:

$$g_K^+(x;v) := \lim_{\substack{(s,u)\to(0+,v)\\ x+s\cdot u\in X}} \frac{g(x+s\cdot u)-g(x)}{s},\ v \in K_x^+ X. \tag{4.7}$$

Equivalently, if the left-contingent cone is defined by

$$\begin{gathered} K_x^- X := \{v \in \mathbb{R}^n;\ (\exists)\ (s_m, v_m) \to (0-, v) :\ x + s_m \cdot v_m \in X, \}, \\ K_x^- X = -K_x^+ X \end{gathered} \tag{4.8}$$

there exists the limits:

$$g_K^-(x;v) := \lim_{\substack{(s,u)\to(0-,v)\\ x+s\cdot u\in X}} \frac{g(x+s\cdot u)-g(x)}{s},\ v \in K_x^- X. \tag{4.9}$$

As it is well known, $g(\cdot)$ is (Fréchet) differentiable at $x \in Int(X)$ iff it is contingent differentiable and $g_K^+(x;\cdot) :\ K_x^+ X = \mathbb{R}^n \to \mathbb{R}^m$ is linear; moreover, if $g(\cdot)$ is contingent differentiable then it is locally-radially Lipschitz, hence continuous; therefore the contingent differentiability it is a natural extension of the classical (Fréchet) differentiability to non-open subsets and to the case in which the contingent derivative $g_K^+(x;\cdot)$ in (4.7) is not linear.

With practically the same proof we obtain the following refinement of the "Elementary verification theorem" in Theorem 4.1.

Theorem 4.3. (**Verification theorem for contingent differentiable value functions**). *Let $\mathcal{A}(\cdot,\cdot)$ be a feasible selection in the sense of Definition*

3.4 *such that the corresponding value function* $W_{\mathcal{A}}(\cdot,\cdot)$ *in* (3.12) *has the following properties:*

(i) (**Asymptotic properties**). *For any* $(s,y)\in E$, $x(\cdot)\in\mathcal{A}(s,y)$ *one has the asymptotic property in* (3.15).

(ii) (**Differential inequality**). *The value function* $W_{\mathcal{A}}(\cdot,\cdot)$ *is contingent differentiable in the sense of Definition* 4.2 *and its contingent derivative satisfies the differential inequality:*

$$\begin{aligned}(W_{\mathcal{A}}(\cdot,\cdot))^{+}_{K}((s,y);(1,v))+f_0(s,y,v)\geq 0,\ (\forall)\ (s,y)\in E,\\ v\in F^{+}_{K}(s,y):=\{v\in F(s,y);\ (1,v)\in K^{+}_{(s,y)}E\}\end{aligned}\tag{4.10}$$

(iii) *Either* $\Omega_\alpha\subseteq\Omega_r$ (i.e. *any admissible trajectory is at least regular*) *or* $W_{\mathcal{A}}(\cdot,\cdot)$ *is locally-Lipschitz.*

Then the selection $\mathcal{A}(\cdot,\cdot)$ *is optimal in the sense of* (3.13).

Proof: The proof is practically the same as that of Theorem 4.1, noting that the inequality in (4.5) is replaced by the following one:

$$(W_{\mathcal{A}}(\cdot,\cdot))^{+}_{K}((t,x(t));(1,x'(t)))+f_0(t,x(t),x'(t))\geq 0,\tag{4.11}$$

$(\forall)\ t\in[s,\ \infty)\setminus J_x$ and also the fact that if $x(\cdot)\in\Omega_\alpha(s,y)$ then from the phase-constraints in (2.3) it follows that:

$$(1,x'(t))\in K^{\pm}_{(t,x(t))}E,\ (\forall)\ t\in[s,\ \infty)\setminus J_x,\tag{4.12}$$

hence (4.10), (4.11) imply (4.5), which, as in the case of Theorem 4.1, implies the fact that the function $\omega_x(\cdot)$ in (3.16) is increasing. □

Another natural extension of Theorem 4.1 may be obtained in the framework of "stratified sets and functions" defined as follows:

Definition 4.4. *A mapping* $g(\cdot):X\subseteq\mathbf{R}^n\to\mathbb{R}^m$ *is said to be* (*countably*) <u>*differentiably-stratified*</u> *if there exists a countable partition,* $\mathcal{S}_g$, *of* X *into differentiable submanifolds such that for any* $S\in\mathcal{S}_g$ *the restriction* $g_S(\cdot):=g(\cdot)|_S$ *is differentiable in the sense of Classical Analysis; in this case, the* <u>*stratified derivative*</u> *of* $g(\cdot)$ *at* $x\in X$ *is defined by:*

$$Dg(x):=Dg_S(x)\in L(T_xS;\ \mathbb{R}^m),\ if\ x\in S\in\mathcal{S}_g.\tag{4.13}$$

In particular, if $X=Int(X)$ is open and $g(\cdot)$ is differentiable then it is differentiable stratified in the sense above by $\mathcal{S}_g=\{X\}$.

Using certain natural properties concerning the behavior of AC mappings with respect to stratified sets and mappings (e.g. Şt. Mirică [13, 14]) we obtain the following extension of Theorem 4.1.

Theorem 4.5. (**Stratified value functions**). *Let* $\mathcal{A}(\cdot,\cdot)$ *be a feasible selection of admissible trajectories in the sense of Definition* 3.4 *such that the corresponding value function,* $W_{\mathcal{A}}(\cdot,\cdot)$ *in* (3.12), *has the following properties:*

(i) (**Asymptotic properties**). *For any* $(s,y)\in E$, $x(\cdot)\in\mathcal{A}(s,y)$ *one has the asymptotic property in* (3.15).

(ii) (**Differential inequalities**). *The value function* $W_{\mathcal{A}}(\cdot,\cdot)$ *is continuous, differentiably-stratified in the sense of Definition* 4.4 *and its stratified derivative satisfies the differential inequality:*

$$\begin{gathered} DW_{\mathcal{A}}(s,y)\cdot(1,v)+f_0(s,y,v)\geq 0,\ (\forall)\ (s,y)\in E,\\ v\in F_T(s,y):=\{v\in F(s,y);\ (1,v)\in T_{(s,y)}E\}\end{gathered}\tag{4.14}$$

(iii) *Either* $\Omega_\alpha\subseteq\Omega_r$ (i.e. *any admissible trajectories is at least regular*) *or* $W_{\mathcal{A}}(\cdot,\cdot)$ *it is locally-Lipschitz.*

Then the selection $\mathcal{A}(\cdot,\cdot)$ *is optimal in the sense of* (3.13).

Proof: As in the proof of Theorem 4.1, we have to prove that for any $(s,y)\in E$, $x(\cdot)\in\Omega_\alpha(s,y)$, the real function $\omega_x(\cdot)$ in (3.16) is increasing; in our case, according to the results in Şt. Mirică [13, 14], if $\mathcal{S}_E$ is the stratification of E in Definition 4.4, then there exists a null subset $J_x\subset(s,\ \infty)$ such that, in addition to (4.3) one has:

$$(1,x'(t))\in T_{(t,x(t))}E,\ (\forall)\ t\in(s,\ \infty)\setminus J_x\tag{4.15}$$

$$\overline{D}^{\pm}\omega_x^0(t)\geq DW_{\mathcal{A}}(t,x(t))\cdot(1,x'(t)),\ (\forall)\ t\in(s,\ \infty)\setminus J_x,\tag{4.16}$$

where $\omega_x^0(\cdot)=W_{\mathcal{A}}(\cdot,x(\cdot))$ is the function in (4.2) and $\overline{D}^{\pm}\omega(\cdot)$, $\underline{D}^{\pm}\omega(\cdot)$ denote the usual Dini derivatives of a real function $\omega(\cdot)$.

Therefore from (4.2), (4.14) and (4.16) it follows that
$\overline{D}^{\pm}\omega_x(t)\geq DW_{\mathcal{A}}(t,x(t))\cdot(1,x'(t))+f_0(t,x(t),x'(t))\geq 0$
hence

$$\overline{D}^{\pm}\omega_x(t)\geq 0,\ (\forall)\ t\in(s,\ \infty)\setminus J_x,\tag{4.17}$$

where J_x is at most countable if $\Omega_\alpha\subseteq\Omega_r$.

The same arguments as in the proof of Theorem 4.1 show that $\omega_x(\cdot)$ is increasing and Theorem 4.5 is proved. □

In the case of value functions with less regularity properties one may obtain similar verification theorems using *the extreme contingent derivatives* of a function $g(\cdot):X\subseteq\mathbb{R}^n\to\mathbb{R}$ defined by:

$$\begin{aligned}\overline{D}_K^{\pm}g(x;v)&:=\limsup_{\substack{(s,u)\to(0\pm,v)\\ x+s\cdot u\in X}}\frac{g(x+s\cdot u)-g(x)}{s},\ v\in K_x^{\pm}X\\ \underline{D}_K^{\pm}g(x;v)&:=\liminf_{\substack{(s,u)\to(0\pm,v)\\ x+s\cdot u\in X}}\frac{g(x+s\cdot u)-g(x)}{s},\ v\in K_x^{\pm}X,\end{aligned}\tag{4.18}$$

which are natural generalizations of the Fréchet derivatives.

The best result in this context may be obtained for locally-Lipschitz value functions that may be neither contingent differentiable nor stratified.

Theorem 4.6. (**Locally-Lipschitz value functions**). *Let* $\mathcal{A}(\cdot,\cdot)$ *be a feasible selection of admissible trajectories such that the corresponding value function,* $W_{\mathcal{A}}(\cdot,\cdot)$ *in* (3.12) *has the following properties:*

(i) (**Asymptotic properties**). *For any* $(s,y)\in E$, $x(\cdot)\in\mathcal{A}(s,y)$ *one has the asymptotic property in* (3.15).

(ii) (**Differential inequalities**). *The value function* $W_{\mathcal{A}}(\cdot,\cdot)$ *is locally-Lipschitz and its upper contingent derivatives in* (4.18) *satisfy the following differential inequalities:*

$$\begin{aligned}\max\{\overline{D}_K^+,\ \overline{D}_K^-\}W_{\mathcal{A}}((s,y);(1,v))+f_0(s,y,v)\geq 0,\ (\forall)\ (s,y)\in E,\\ v\in F_K(s,y):=F_K^+(s,y)\cap F_K^-(s,y),\end{aligned}\tag{4.19}$$

where $F_K^{\pm}(\cdot,\cdot)$ *are defined as in* (4.10).

Then $W_{\mathcal{A}}(\cdot,\cdot)$ *is an optimal selection.*

Proof: Proceeding like in the proof of Theorem 4.1, we note that in this case the real functions, $\omega_x(\cdot)$, in (3.16) are locally-AC hence from the Lebesgue monotonicity theorem it follows that condition (4.17) is necessary and sufficient for $\omega_x(\cdot)$ to be increasing.

On the other hand, since $W_{\mathcal{A}}(\cdot,\cdot)$ is locally-Lipschitz, from a natural "chain rule" (e.g. Şt. Mirică [14]) it follows that
$\overline{D}_K^{\pm}\omega_x^0(t;1)=\overline{D}_K^{\pm}W_{\mathcal{A}}((t,x(t));(1,x'(t))),\ (\forall)\ t\in(s,\ \infty)\setminus J_x,$
hence from (4.19) it follows
$\overline{D}_K^{\pm}\omega_x(t;1)=\overline{D}_K^{\pm}W_{\mathcal{A}}((t,x(t));(1,x'(t)))+f_0(t,x(t),x'(t))\geq 0,\ (\forall)\ t\notin J_x,$
and Theorem 4.6 is proved. □

In the case the value function is no more locally-Lipschitz the situation is much more complicated since the functions $\omega_x(\cdot)$ in (3.16) are no more locally-AC hence one should use more sophisticated monotonicity theorems.

For instance, in the case of lower semicontinuous (in particular, continuous) value functions we could use the monotonicity theorem in Şt. Mirică [12, 14] according to which if $\omega_x(\cdot)$ is *lower semicontinuous* then it is increasing iff:

$$\overline{D}_K^-\omega_x(t;1)\geq 0,\ (\forall)\ t\in(s,\ \infty).\tag{4.20}$$

On the other hand, the function $\omega_x(\cdot)$ in (3.16) may be written as follows

$$\omega_x(t)=\widehat{W}(\widehat{x}(t)),\ \widehat{W}(\widehat{x}):=W(x)+x_0,\ \widehat{x}=(x,x_0)\tag{4.21}$$

and the "extended" trajectory, $\widehat{x}(\cdot)=(x(\cdot),\ x_0(\cdot))$ in (2.4) is a solution of the "extended differential inclusion"

$$\widehat{x}'\in\widehat{F}(t,\widehat{x}):=\{(v,\ f_0(t,x,v));\ v\in F(t,x)\}\ if\ \widehat{x}=(x,x_0)\tag{4.22}$$

while the "set-valued contingent derivatives" of $\widehat{x}(\cdot)$ are defined by:

$$K^{\pm}\widehat{x}(t;1) := \{\widehat{v} = (v, v_0);\ \exists\ s_m \to 0\pm : \frac{\widehat{x}(t+s_m) - \widehat{x}(t)}{s_m} \to \widehat{v}\}. \tag{4.23}$$

Finally, we shall use the fact that the extreme contingent derivatives of the function $\omega_x(\cdot)$ in (4.21) may be evaluated as follows:

$$\overline{D}_K^{\pm}\omega_x(t;1) \geq \sup_{\widehat{v}=(v,v_0)\in K^{\pm}\widehat{x}(t;1)} [\underline{D}_K^{\pm}W(x(t);v) + v_0] \tag{4.24}$$

while the contingent derivatives in (4.23) are evaluated as follows (e.g. Şt. Mirică [11, 14, 15])):

$$K^{\pm}\widehat{x}(t;1) \subseteq \widehat{F}^{co}(t,\widehat{x}(t)),\ \ \widehat{F}^{co}(t,\widehat{x}) := \bigcap_{\delta>0} \overline{co}\widehat{F}(B_\delta(t,\widehat{x})) \tag{4.25}$$

where $B_r(y) := \{x \in R^n;\ \|x - y\| \leq r$ is the ball of radius $r > 0$ centered at $y \in R^n$.

Using these concepts and results we obtain:

Theorem 4.7. (**Lower semicontinuous value functions and Lipschitzian trajectories**). *Let* $\mathcal{A}(\cdot,\cdot)$ *be a feasible selection of admissible trajectories such that the corresponding value function in* (3.12) *has the following properties:*

(i) (**Asymptotic properties**). *For any* $(s,y) \in E$, $x(\cdot) \in \mathcal{A}(s,y)$ *one has the asymptotic property in* (3.15).

(ii) (**Differential inequalities**). $W_{\mathcal{A}}(\cdot,\cdot)$ *is lower semicontinuous and its lower left contingent derivatives in* (4.18) *satisfy the following differential inequalities:*

$$\begin{gathered}\underline{D}_K^{-}W_{\mathcal{A}}((s,y);(1,v)) + v_0 \geq 0,\ (\forall)\ (s,y) \in E,\\ \widehat{v} = (v,v_0) \in \widehat{F}^{co}(s,y),\ (1,v) \in K^{-}_{(s,y)}E.\end{gathered} \tag{4.26}$$

Then $\mathcal{A}(\cdot,\cdot)$ *is an optimal selection in the class* Ω_∞ *of locally-Lipschitz trajectories.*

Proof: Proceeding like in the proof of Theorem 4.1, we note that in this case the real function $\omega_x(\cdot)$ in (3.16) is lower semicontinuous and its upper left contingent derivatives are evaluated by:

$\overline{D}_K^{-}\omega_x(t;1) \geq \sup_{\widehat{v}=(v,v_0)\in K^{-}\widehat{x}(t;1)} [\underline{D}_K^{-}W_{\mathcal{A}}((t,x(t));(1,v)) + v_0].$

Since in the case of locally-Lipschitz admissible trajectories the relation in (4.25) holds, from (4.26) it follows (4.20) hence $\omega_x(\cdot)$ is increasing.

□

Exactly in same way one obtains the symmetric result for upper semicontinuous value functions.

Theorem 4.8.(**Upper semicontinuous value functions and Lipschitzian trajectories**). *Let* $\mathcal{A}(\cdot,\cdot)$ *be a selection of admissible trajectories such that the corresponding value function in* (3.12) *has the following properties:*

(i) (**Asymptotic properties**). *For any* $(s,y)\in E,\ x(\cdot)\in\mathcal{A}(s,y)$ *one has the asymptotic property in* (3.15).

(ii) (**Differential inequalities**). $W_{\mathcal{A}}(\cdot,\cdot)$ *is upper semicontinuous and its lower right contingent derivatives in* (4.18) *satisfy the following differential inequalities:*

$$\begin{gathered}\underline{D}_K^+ W_{\mathcal{A}}((s,y);(1,v))+v_0\geq 0,\ (\forall)\ (s,y)\in E,\\ \widehat{v}=(v,v_0)\in \widehat{F}^{co}(s,y),\ (1,v)\in K^+_{(s,y)}E,\end{gathered} \tag{4.27}$$

where $\widehat{F}^{co}(\cdot,\cdot)$ *is the u.s.c. convexified limit in* (4.25) *of the extended orientor field* $\widehat{F}(\cdot,\cdot)$ *in* (4.22).

Then $\mathcal{A}(\cdot,\cdot)$ *is an optimal selection* (*in the sense of Definition* 3.4) *in the class* Ω_∞ *of locally-Lipschitz trajectories.*

Remark 4.9. Using the results in Şt. Mirică [13] concerning the invariance and monotonicity with respect to differential inclusions one may obtain other verification theorems expressed in terms of the upper contingent derivatives but using certain additional properties on the data, $\widehat{F}(\cdot,\cdot)$ and the subset $E\subseteq \mathbb{R}\times\mathbb{R}^n$.

Remark 4.10. The so called "Carathéodory's Method", as extended to infinite horizon optimal control problems by D.A. Carlson [2, 3] may be considered as a variant of the "elementary verification theorem" 4.1.

In fact, in this approach one starts from a smooth solution, $W(\cdot,\cdot)$, of the (HJB) equation in (3.10) and from a (possibly optimal) feedback control, $\widetilde{v}(\cdot,\cdot)$, satisfying the relation:

$$DW((t,x);(1,\widetilde{v}(t,x)))+f_0(t,x,\widetilde{v}(t,x))=0,\ (\forall)\ (t,x)\in E. \tag{4.28}$$

Further, one assumes that each of the Cauchy problems

$$x'=\widetilde{v}(t,x),\ x(s)=y,\ (s,y)\in E \tag{4.29}$$

has at least a solution that is an admissible trajectory, $\widetilde{x}_{s,y}(\cdot):[s,\ \infty)\to\mathbb{R}^n$ that satisfies the phase-space constraints in (2.3), the asymptotic properties in (3.3) and the relation:

$$W(s,y)=\mathcal{C}(s,y;\widetilde{x}_{s,y}(\cdot))=\int_s^\infty f_0(t,\widetilde{x}_{s,y}(t),\widetilde{x}'_{s,y}(t))dt,\ (s,y)\in E. \tag{4.30}$$

Under these conditions, Theorem 4.1 implies the fact that the selection $\mathcal{A}(s,y)=\{\widetilde{x}_{s,y}(\cdot)\},\ (s,y)\in E$ is optimal.

5 Halkin's example

To illustrate some of the theoretical aspects in the previous sections we consider in a very succinct manner Halkin's example in Halkin [7]; the complete solution of this problem may justify the fact that in this case the so called "asymptotic transversality condition" may not be satisfied.

The problem consists in the minimization of each of the functionals

$$\mathcal{C}(s,y;u(\cdot)) := \int_s^\infty (x(t)-1)\cdot u(t)dt,\ (s,y)\in E=\mathbb{R}^2 \tag{5.1}$$

subject to:

$$\begin{cases} x'(t)=(1-x(t))\cdot u(t),\ x(s)=y,\ u(t)\in U:=[0,1]\ a.e.(s,\infty) \\ u(\cdot)\in\mathcal{U}_1(s)=L_1^{loc}([s,\infty);[0,1]). \end{cases} \tag{5.2}$$

Using the fact that the Hamiltonian in (3.10), given in our case by:

$$H(t,x,p)=\begin{cases} 0, & if\ (1-x)\cdot(p-1)\geq 0 \\ (1-x)\cdot(p-1), & if\ (1-x)\cdot(p-1)<0 \end{cases} \tag{5.3}$$

is a stratified function in the sense of Definition 4.4 and the extensions in I. Mirică [10], Şt. Mirică [14] of Cauchy's Method of Characteristics, one obtains the following selection of admissible controls:

$$\mathcal{U}_{s,y}(t)=\begin{cases} \{u(t)\equiv l\}\ if\ (s,y)\in E,\ y>1 \\ \mathcal{U}_1(s),\ if\ (s,y)\in E,\ y=1 \\ \mathcal{U}_1^\infty(s):=\{u(\cdot)\in\mathcal{U}_1(s);\ \int_s^\infty u(t)dt=+\infty\}\ if\ y<1, \end{cases}$$

while the corresponding selection of admissible trajectories is given by:

$$\mathcal{A}(s,y):=\begin{cases} \{x_{s,y}(t)\equiv y\}, & if\quad (s,y)\in E,\ y\geq 1 \\ \{x_u(t)\equiv 1+(y-1)e^{-\int_s^t u(r)dr};u(\cdot)\in\mathcal{U}_1^\infty(s)\}, & if\ y<1 \end{cases} \tag{5.4}$$

Elementary computations and arguments show that $\mathcal{A}(\cdot,\cdot)$ in (5.4) is a feasible selection in the sense of Definition 3.4 and its corresponding value function in (3.12) is given by:

$$W_{\mathcal{A}}(s,y)\equiv\begin{cases} 0, & if\ \ (s,y)\in E=\mathbb{R}^2,\ y\geq 1 \\ y-1, & if\ \ (s,y)\in E,\quad y<1 \end{cases} \tag{5.5}$$

and it is contingent differentiable, locally-Lipschitz and also C^∞-differentiably stratified. In this case one may use either Theorem 4.5 (for stratified value functions) or Theorem 4.6 (for locally-Lipschitz value functions) to prove the optimality of the selection in (5.4).

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References

[1] Bardi, M., Capuzzo-Dolcetta, I., Optimal Control and Viscosity Solutions for Hamilton-Jacobi-Bellman Equations, Birkhäuser, Boston, 1997.

[2] Carlson, D. A., A Carathéodory Hamilton-Jacobi Theory for Infinite Horizon Optimal Control Problems, *Journal of Optimization Theory and Applications*, 1986, 48, 265-287.

[3] Carlson, D. A., Haurie, A., Leizarowitz, A., Infinite Horizon Optimal Control, Deterministic and Stochastic Systems, Springer-Verlag, New York, 1991.

[4] Crandall, M. G., Lions, P. L., Viscosity Solutions of Hamilton-Jacobi Equations, *Trans. A.M.S.*, 1983, 277, 1-42.

[5] Dunford, N., Schwartz, J. T., Linear Operators, Part I: General Theory, Interscience, New York, 1958.

[6] Fleming, W. H., Rishel, R.W., Deterministic and Stochastic Optimal Control, Springer-Verlag, New York, 1975.

[7] Halkin, H., Necessary Conditions for Optimal Control Problems with Infinite Horizons,*Econometrica*, 1974, 42, 267-272.

[8] Lupulescu, V., Mirică, Şt., Verification theorems for discontinuous value functions in optimal control, *Math. Reports*, 2000, 2(52), 399-326.

[9] Mirică, I., Basic Properties of the Value Functions of Infinite-Horizon Optimal Control Problems, *Studii Cerc. Şt., Univ. Bacău, Seria Mat.*, 1997, 7, 75-89.

[10] Mirică, I., Asymptotic Characteristic Solutions of Autonomous Hamilton-Jacobi Equations, *Anal.Şt. Univ. "Al.I. Cuza" Iaşi, Ser. Mat.*, 1999, XLV, 257-274.

[11] Mirică, Şt., Tangent and Contingent Directions to Trajectories of Differential Inclusions, *Revue Roum. Math. Pures Appl.*, 1992, 37, 421-444.

[12] Mirică, Şt., Monotonicity of Real Functions without Regularity Assumptions, *Anal. Univ. Bucureşti, Mat.*, 1995, 44, 49-60.

[13] Mirică, Şt., Invariance and Monotonicity for Autonomous Differential Inclusions, *Studii Cerc. Mat.*, 1995, 47, 179-204.

[14] Mirică, Şt., Constructive Dynamic Programming in Optimal Control-Autonomous Problems, Ed. Academiei Române, Bucureşti, 2004.

[15] MIRICĂ, ŞT., Tangent, Quasitangent and Peritangent Directions to the Trajectories of Differential Inclusions, *J. Dyn. Control Systems*, 2004, 10, 227-245.

[16] MIRICĂ, ŞT., MIRICĂ, I., Distinct Types of Intrinsic Tangent Cones to Arbitrary Subsets of Euclidian Spaces, *Anal. Univ. Bucureşti, Mat.*, 1989, 38, 1, 21-39.

[17] ROCKAFELLAR, R. T., Integral Functionals, Normal Integrands and Measurable Selections in Nonlinear Operators and the Calculus of Variations, in Lect. Notes in Math., Vol. **543**, Springer-Verlag, Berlin, 1986, p. 157-207.

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