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A formula for the development of the permanent of a matrix

by
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Abstract

In this paper a formula for the development of the permanent of a matrix and several identities and inequalities deduced from it are pproposed. We mention for example the identity of Hurwitz and Muirhead, the mean's inequality, the inequality of Chebyshev, the identity of Binet-Cauchy, the identity of Lagrange and the identity of Tiberiu Popoviciu.

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1 Definitions and notations

Let S be a finite set and k be a natural number. We denote by $|S|$ the number of elements from S, by $S^{(k)}$ the set of k-subsets of S and by $\overrightarrow{S}^{(k)}$ the set of words $i_1 \dots i_k$ with k different letters from S. For a natural number $n \in \mathbb{N}_{\geq 1}$ we denote by $[n] = \{1, 2, 3, \dots, n\}$ and by $n^{\underline{k}} = n(n-1)(n-2)\dots(n-k+1)$ the falling factorial. We consider by convention $[0] = \phi$ and $n^{\underline{0}} = 1$. Obviously we have $\left|\overrightarrow{S}^{(k)}\right| = |S|^{\underline{k}}$.

For n indeterminates $a_1, \dots, a_n$ and $k \in \{1, 2, 3, \dots .n\}$ we denote by $s_k(a_1, a_2, a_3, \dots, a_n)$ the Viètte sum

$$s_k(a_1, a_2, a_3, \dots, a_n) = \sum_{\{i_1, \dots, i_k\} \in [n]^{(k)}} a_{i_1} \dots a_{i_k}. \qquad (1)$$

For a finite set $V \neq \phi$ of *vertices,* an n-uniform hypergraph on V is a pair $G = (V, E)$ where $E \in V^{(n)}$ is a set of n-parts of V called *edges.* For a vertex $k \in V$ the number of edges containing k is called *the degree* in G of k and is denoted by $d_G(k)$.

Let $A = (a_{i,j})_{i\in P j\in N}$ be a real matrix where $P = \{1, 2, 3, \ldots, p\}$, $N = \{1, 2, 3, \ldots, n\}$ and $p, n \in \mathbb{N}_{\geq 1}$, $p \leq n$. For $\phi \neq L \subseteq P$, $\phi \neq K \subseteq N$ we denote by $A_{L,K}$ the submatrix induced by the set L of rows of and the set K of columns of A, $A_{L,K} = (a_{i,j})_{i\in L j\in K}$.

The permanent of the matrix A, denoted by per(A), is the number equal to the sum of all products of p elements from A, any two of them are not lying on the same row or on the same column:

$$\operatorname{per}(A) = \sum_{i_1,\ldots,i_p \in \overrightarrow{N}^{(p)}} a_{1,i_1} \ldots a_{p,i_p} \tag{2}$$

For $\ell \in \{1, 2, 3, \ldots, p\}$ and L a fixed ℓ-part of P, $L \in P^{(\ell)}$, the analogous of Laplace's development for the permanent holds:

$$\operatorname{per}(A) = \sum_{K\in N^{(\ell)}} \operatorname{per}(A_{L,K}) \operatorname{per}\left(A_{\overline{L},\overline{K}}\right), \tag{3}$$

where we denoted by $\overline{L} = P - L$ and $\overline{K} = N - K$ and we considered by definition per$(A_{\phi,K}) = 1$. We say that the formula (3) is *the development of the permanent of the matrix A on the set L of rows.*

2 The main result

The following lemma immediately follows by straightforward computation:

Lemma 2.1. *For any square 2×2 matrix, $Z = \begin{pmatrix} x_1a_1 & x_2a_2 \\ y_1b_1 & y_2b_2 \end{pmatrix}$ where $x_1, x_2, y_1, y_2, a_1, a_2, b_1, b_2$ are real nonnull numbers, we have:*

$$\operatorname{per}\begin{pmatrix} x_1a_1 & x_2a_2 \\ y_1b_1 & y_2b_2 \end{pmatrix} = \operatorname{per}\begin{pmatrix} x_1 & x_2 \\ y_1a_1b_1 & y_2a_2b_2 \end{pmatrix} - (a_1 - a_2)\left(\frac{y_1}{x_1}b_1 - \frac{y_2}{x_2}b_2\right), \tag{4}$$

$$\operatorname{per}\begin{pmatrix} x_1a_1 & x_2a_2 \\ y_1b_1 & y_2b_2 \end{pmatrix} = \operatorname{per}\begin{pmatrix} x_1b_1 & x_2b_2 \\ y_1a_1 & y_2a_2 \end{pmatrix} + \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}. \square \tag{5}$$

• We remark that the identities (4) and (5) are equivalent. For example the identity (5) results from the identity (4) by the transformation

$$\operatorname{per}\begin{pmatrix} x_1a_1 & x_2a_2 \\ y_1b_1 & y_2b_2 \end{pmatrix} = \operatorname{per}\begin{pmatrix} x_1b_1 \cdot \dfrac{a_1}{b_1} & x_2b_2 \cdot \dfrac{a_2}{b_2} \\ y_1a_1 \cdot \dfrac{b_1}{a_1} & y_2a_2 \cdot \dfrac{b_2}{a2} \end{pmatrix}$$

substituting in (4) x_i by x_ib_i, y_i by y_ia_i, a_i by $\dfrac{a_i}{b_i}$ and b_i by $\dfrac{b_i}{a_i}$ for $i \in \{1, 2\}$.

Theorem 2.2. *We consider two natural numbers* $p, n \in \mathbb{N}_{\geq 2},\ p \leq n$ *and the sets* $P = \{1, 2, 3, \ldots, p\}$ *and* $N = \{1, 2, 3, \ldots, n\}$.

Let $A = (a_{i,j})_{i \in P j \in N}$ *be a real matrix,* $X = (x_{i,j})_{i \in P j \in N}$ *a real matrix consisting of nonull numbers and the matrix* $Z = (\mathcal{Z}_{i,j})_{i \in P j \in N}$ *defined by* $\mathcal{Z}_{i,j} = x_{i,j} a_{i,j}$ *for any* $i \in P$ *and* $j \in N$.

For $L \subseteq P$ *we denote by* Z^L *the matrix obtained from the matrix* Z *by the substitution of the rows with the indices from* L *with the corresponding rows from the matrix* X :

$$\mathcal{Z}_{i,j} = \begin{cases} x_{i,j} & \text{for } i \in L \text{ and } j \in N \\ x_{i,j} a_{i,j} & \text{for } i \in \overline{L} \text{ and } j \in N. \end{cases}$$

We have

$$\operatorname{per}(Z) = \operatorname{per}\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ \vdots & \vdots & & \vdots \\ x_{p-1,1} & x_{p-1,2} & \cdots & x_{p-1,n} \\ x_{p,1} a_{1,1} \cdots a_{p,1} & x_{p,2} a_{1,2} \cdots a_{p,2} & \cdots & x_{p,n} a_{1,n} \cdots a_{p,n} \end{pmatrix} -$$

$$-\sum_{t=2}^{p} \sum_{1 \leq i < j \leq n} (a_{1,i} \cdots a_{t-1,i} - a_{1,j} \cdots a_{t-1,j}) \tag{6}$$

$$\left(\frac{x_{t,i}}{x_{t-1,i}} a_{t,i} - \frac{x_{t,j}}{x_{t-1,j}} a_{t,j} \right) x_{t-1,i} x_{t-1,j} \operatorname{per}\left(Z^{[t-2]}_{\overline{\{t-1,t\}}, \overline{\{i,j\}}} \right).$$

Proof: We apply the identity (4) for the development of the permanent of the matrix Z on the rows $\{k, t\}$ and we obtain:

$$\begin{aligned} \operatorname{per}(Z) &= \operatorname{per}\begin{pmatrix} x_{1,1} a_{1,1} & \cdots & x_{1,n} a_{1,n} \\ \vdots & & \vdots \\ x_{p,1} a_{p,1} & \cdots & x_{p,n} a_{p,n} \end{pmatrix} = \\ &= \sum_{1 \leq i < j \leq n} \operatorname{per}\begin{pmatrix} x_{k,i} a_{k,i} & x_{k,j} a_{k,j} \\ x_{t,i} a_{t,i} & x_{t,j} a_{t,j} \end{pmatrix} \operatorname{per}\left(Z_{\overline{\{k,t\}}, \overline{\{i,j\}}} \right) = \\ &= \sum_{1 \leq i < j \leq n} \operatorname{per}\begin{pmatrix} x_{k,i} & x_{k,j} \\ x_{t,i} a_{k,i} a_{t,i} & x_{t,j} a_{k,j} a_{t,j} \end{pmatrix} \operatorname{per}\left(Z_{\overline{\{k,t\}}, \overline{\{i,j\}}} \right) - \\ &- \sum_{1 \leq i < j \leq n} (a_{k,i} - a_{k,j}) \left(\frac{x_{t,i}}{x_{k,i}} a_{t,i} - \frac{x_{t,j}}{x_{k,j}} a_{t,j} \right) x_{k,i} x_{k,j} \ \operatorname{per}\left(Z_{\overline{\{k,t\}}, \overline{\{i,j\}}} \right) = \end{aligned}$$

$$= \text{per}\begin{pmatrix} x_{1,1}a_{1,1} & \cdots & x_{1,n}a_{1,n} \\ \vdots & & \vdots \\ x_{k,1} & \cdots & x_{k,n} \\ \vdots & & \vdots \\ x_{t,1}a_{k,1}a_{t,1} & \cdots & x_{t,n}a_{k,n}x_{t,n} \\ \vdots & & \vdots \\ x_{p,1}a_{p,1} & \cdots & x_{p,n}a_{p,n} \end{pmatrix} -$$

$$- \sum_{1\le i<j\le n} (a_{k,i} - a_{k,j}) \left(\frac{x_{t,i}}{x_{k,i}} a_{t,i} - \frac{x_{t,j}}{x_{k,j}} a_{t,j} \right) x_{k,i} x_{k,j} \text{ per}\left(Z_{\overline{\{k,t\}},\overline{\{i,j\}}} \right).$$

Now, we successively apply the formula above for the development of the permanent of the matrix Z on the rows $\{1,2\}, \{2,3\}, \{3,4\}, \ldots, \{p-2, p-1\}$ and $\{p-1, p\}$ and we obtain the formula from the statement. □

3 Applications

3.1. <u>The case $x_{1,j} = x_{2,j} = x_{3,j} = \cdots = x_{p,j} = x_j$ for $j \in N$.</u>

Theorem 3.1.1. *Let $A = (a_{i,j})_{i\in P j\in N}$ be a real positive matrix where $P = \{1, 2, 3, \ldots, p\}$, $N = \{1, 2, \ldots, n\}$ and $p, n \in \mathbb{N}_{\ge 2}$, $p \le n$ and for each $i \in P$*

$$a_{i,1} \le a_{i,2} \le a_{i,3} \le \cdots \le a_{i,n}.$$

For a real positive numbers $x_1, \ldots, x_n$ we denote by $Z = (z_{i,j})_{i\in P j\in N}$ the matrix defined by $z_{i,j} = x_j a_{i,j}$ for $i \in P$ and $j \in N$.

We have:

$$\text{per}(Z) \le (p-1)! \sum_{k=1}^{n} x_k a_{1,k} \cdots a_{p,k} s_{p-1}(x_1, x_2, x_3, \ldots, \widehat{x}_k, \cdots, x_n), \qquad (7)$$

where by convention $\widehat{x}_k = 0$.

Proof: We substitute in (6) $x_{i,j} = x_j$ for $(i,j) \in P \times N$. Each sum from the right hand side of (6) is a nonnegative number. We develop the first permanent from the right hadn side of (6) on the row p and we obtain:

$$\text{per}(Z) \le \text{per}\begin{pmatrix} x_1 & x & x_p \\ \vdots & \vdots & \vdots \\ x_1 & x_2 & x_p \\ x_1 a_{1,1} \cdots a_{p,1} & x_2 a_{1,2} \cdots a_{p,2} \cdots & x_p a_{1,p} \cdots a_{p,n} \end{pmatrix} =$$

$$= \sum_{k=1}^{n} x_k a_{1,k} \cdots a_{p,k} \sum_{\overline{i_1 \dots i_{p-1} \in (N-\{k\})^{(p-1)}}} x_{i_1} \cdots x_{i_{p-1}} =$$

$$= (p-1)! \sum_{k=1}^{n} x_k a_{1,k} \cdots a_{p,k} \sum_{\{i_1,\dots,i_{p-1}\} \in (N-\{k\})^{(p-1)}} x_{i_1} \cdots x_{i_{p-1}} =$$

$$= (p-1)! \sum_{k=1}^{n} x_k a_{1,k} \cdots a_{p,k} s_{p-1} (x_1, x_2, x_3, \cdots, \widehat{x}_k, \cdots, x_n) .$$

□

3.2. The case $x_{i,j} = 1$ for $(i,j) \in P \times N$:

Theorem 3.2.1. *Let* $A=(a_{i,j})_{i\in P j\in N}$ *be a real matrix, where* $P=\{1,2,3,\dots,p\}$, $N = \{1, 2, 3, \dots, n\}$ *and* $p, n \in \mathbb{N}_{\geq 2}$, $p \leq n$. We have:

$$\operatorname{per}(A) = (n-1)^{\underline{p-1}} \sum_{k\in N} a_{1,k} \cdots a_{p,k} - \tag{8}$$

$$(n-2)^{\underline{p-2}} \sum_{1\leq i<j\leq n} (a_{1,i} \cdots a_{p-1,i} - a_{1,j} \cdots a_{p-1,j})(a_{p,i} - a_{p,j}) \operatorname{per}(A_{\phi,\overline{\{i,j\}}}) -$$

$$(n-3)^{\underline{p-3}} \sum_{1\leq i<j\leq n} (a_{1,i} \cdots a_{p-2,i} - a_{1,j} \cdots a_{p-2,j})(a_{p-1,i} - a_{p-1,j}) \operatorname{per}(A_{\{p\},\overline{\{i,j\}}}) -$$

$$(n-4)^{\underline{p-4}} \sum_{1\leq i<j\leq n} (a_{1,i} \cdots a_{p-3,i} - a_{1,j} \cdots a_{p-3,j})(a_{p-2,i} - a_{p-2,j}) \operatorname{per}(A_{\{p-1,p\},\overline{\{i,j\}}}) -$$

$$\cdots \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$(n-p)^{\underline{0}} \sum_{1\leq i<j\leq n} (a_{1,i} - a_{1,j})(a_{2,i} - a_{2,j}) \operatorname{per}(A_{\{3,4,\dots,p\},\overline{\{i,j\}}}).$$

Proof: For the beginning we observe that for a real matrix $W = (w_{i,j})_{i\in L j\in C}$ where $L = \{1, 2, 2, \dots, \ell\}$, $C = \{1, 2, 3, \dots, c\}$, $\ell \leq c$ in which the first ℓ' row consist of $1'$s we have

$$\operatorname{per}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \\ w_{\ell'+1,1} & \cdots & w_{\ell'+1,c} \\ \vdots & & \vdots \\ w_{\ell'+\ell",1} & \cdots & w_{\ell'+\ell",c} \end{pmatrix} = (c-\ell'')^{\underline{\ell'}} \operatorname{per}\begin{pmatrix} w_{\ell'+1,1} & \cdots \cdots & w_{\ell'+1,c} \\ \vdots & & \vdots \\ w_{\ell'+\ell",1} & \cdots \cdots & w_{\ell'+\ell",c} \end{pmatrix} \tag{9}$$

where $\ell' + \ell" = \ell = |L|$.

Now, the formula (8) follows immediately from theorem 2.2. in which we consider $x_{i,j} = 1$ for any $i \in P, j \in N$ and we apply the formula (6). □

Corollary 3.2.2. *Let* $A = (a_{i,j})_{i\in P j\in N}$ *be a real positive matrix where* $P = \{1,2,3,\dots,p\}$, $N = \{1,2,3,\dots,n\}$ *and* $p,n \in \mathbb{N}_{\geq 2}, p \leq n$ *and for each* $i \in P$

$$a_{i,1} \leq a_{i,2} \leq a_{i,3} \leq \cdots \leq a_{i,n}.$$

We have

$$\operatorname{per}(A) \leq (n-1)^{\underline{p-1}} \sum_{k\in N} a_{1,k} \cdots a_{p,k}. \tag{10}$$

Proof: The inequality follows from theorem 3.2.1. because, in the equality (8) each sum from the right hand side of (8) is a nonnegative number. □

Corollary 3.2.3. *Let* $A = (a_{i,j})_{i\in P j\in N}$ *be a real positive matrix, where* $P = \{1,2,3,\dots,p\}$, $V = \{1,2,3,\dots,v\}$ *and* $p,v \in \mathbb{N}_{\geq 2}, p \leq v$ *and for each* $i \in P$

$$a_{i,1} \leq a_{i,2} \leq a_{i,3} \leq \cdots \leq a_{i,v}.$$

For any n*-uniform hypergraph* $G = (V,E)$ *where* $E \subseteq V^{(n)}$ *and* $n \in \mathbb{N}_{\geq 2}, p \leq n \leq v$, *we have*

$$\sum_{N\in E} \operatorname{per}(A_{P,N}) \leq (n-1)^{\underline{p-1}} \sum_{k\in V} d_G(k)\, a_{1,k} \cdots a_{p,k}. \tag{11}$$

3.3. The case $x_{i,j} = 1$ for $(i,j) \in P \times N$ and $a_{1,j} = a_{2,j} = a_{3,j} = \cdots a_{p,j} = a_j$ for $j \in N$.

Theorem 3.3.1. *For any two natural numbers* $p,n \in \mathbb{N}_{\geq 2},\ p \leq n$ *and* $a_1,\dots,a_n \in \mathbb{R}$ *we have the following identity:*

$$p!s_p(a_1,\dots,a_n) = (n-1)^{\underline{p-1}}(a_1^p + \cdots + a_n^p) -$$

$$-(n-2)^{\underline{p-2}}0! \sum_{1\leq i<j\leq n} (a_i^{p-1} - a_j^{p-1})(a_i - a_j)$$

$$-(n-3)^{\underline{p-3}}1! \sum_{1\leq i<j\leq n} (a_i^{p-2} - a_j^{p-2})(a_i - a_j)s_1(a_1,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_n)$$

$$-(n-4)^{\underline{p-4}}2! \sum_{1\leq i<j\leq n} (a_i^{p-3} - a_j^{p-3})(a_i - a_j)s_2(a_1,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_n)$$

$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \tag{12}$$

$$-(n-k-2)^{\underline{p-k-2}}k! \sum_{1\leq i<j\leq n} (a_i^{p-k-1} - a_j^{p-k-1})(a_i - a_j)s_k(a_1,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_n)$$

$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$$

$$-(n-p)^{\underline{0}}(p-2)! \sum_{1\leq i<j\leq n} (a_i^1 - a_j^1)(a_i - a_j)s_{p-2}(a_1,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_n)$$

where by convention $\widehat{a}_i = \widehat{a}_j = 0$.

• We remark that for $p = n$ we obtain from (12) a form of an identity of Hurwitz and Muirhead ([3], p.50 and p.55):

$$n!a_1 \dots a_n = (n-1)!\,(a_1^n + \dots + a_n^n) -$$
$$-(n-2)!0! \sum_{1\le i<j\le n} \left(a_i^{n-1} - a_j^{n-1}\right)(a_i - a_j)$$
$$-(n-3)!1! \sum_{1\le i<j\le n} \left(a_i^{n-2} - a_j^{n-2}\right)(a_i - a_j)\, s_1\left(a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n\right)$$
$$-(n-4)!2! \sum_{1\le i<j\le n} \left(a_i^{n-3} - a_j^{n-3}\right)(a_i - a_j)\, s_2\left(a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n\right)$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \qquad (13)$$
$$-(n-k-2)!k! \sum_{1\le i<j\le n} \left(a_i^{n-k-1} - a_j^{n-k-1}\right)(a_i - a_j) s_k\left(a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n\right)$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$
$$-0!\,(n-2)!1! \sum_{1\le i<j\le n} \left(a_i^1 - a_j^1\right)(a_i - a_j)\, s_{n-2}\left(a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n\right).$$

• For $a_1, \dots, a_n \in \mathbb{R}_{\ge 0}$, we obtain from (13) the mean′s inequality:

$$na_1 \dots a_n \le a_1^n + \dots + a_n^n. \qquad (14)$$

• We can also obtain this inequality from the Corollary 3.2.3.

Indeed, we consider the case $v = n = p$ and $a_{1,k} = a_{2,k} = a_{3,k} = \dots = a_{n,k} = a_k$. Then the hypergraph G contains a single n-edge, $E = \{V\}$. From (11) we obtain the mean′s inequality (14):

$$\operatorname{per}\begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ a_1 & \cdots & a_n \end{pmatrix} \le (n-1)!\,(a_1^n + \dots + a_n^n).$$

3.4. <u>The case $p = 2$ and $n \in \mathbb{N}_{\ge 2}$</u>

From Theorem 2.2 we can deduce the following

Proposition 3.4.1. *For* $x_1, \dots, x_n,\ a_1, \dots, a_n,\ b_1, \dots, b_n \in \mathbb{R}$ *we have*

$$\operatorname{per}\begin{pmatrix} x_1a_1 & x_2a_2 \cdots & x_na_n \\ x_1b_1 & x_2b_2 \cdots & x_nb_n \end{pmatrix} =$$

$$\operatorname{per}\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1a_1b_1 & x_2a_2b_2 \cdots & x_na_nb_n \end{pmatrix} - \sum_{1\le i<j\le n}(a_i-a_j)(b_i-b_j)x_ix_j \tag{15}$$

$$\left(\sum_{1\le i\le n} x_ia_i\right)\left(\sum_{1\le i\le n} x_ib_i\right) =$$

$$= \left(\sum_{1\le i\le n} x_i\right)\left(\sum_{1\le i\le n} x_ia_ib_i\right) - \sum_{1\le i<j\le n}(a_i-a_j)(b_i-b_j)x_ix_j. \tag{16}$$

• For $a_1 \le \ldots \le a_n, b_1 \le \ldots \le b_n$ and $x_1 \le \ldots \le x_n \in \mathbb{R}_{\ge 0}$ we obtain from (16) the weighted Chebyshev inequality [1, p.175]:

$$\left(\sum_{1\le i\le n} x_ia_i\right)\left(\sum_{1\le i\le n} x_ib_i\right) \le \left(\sum_{1\le i\le n} x_i\right)\left(\sum_{1\le i\le n} x_ia_ib_i\right). \tag{17}$$

• For $x_1 = \ldots = x_n = 1$ we obtain from (17) the Chebyshev inequality

$$\left(\frac{1}{n}\sum_{1\le i\le n} a_i\right)\left(\frac{1}{n}\sum_{1\le i\le n} b_i\right) \le \frac{1}{n}\sum_{1\le i\le n} x_ia_ib_i. \tag{18}$$

• We can obtain the inequality (17) from the Theorem 3.1.1. by putting $p = 2$ and replaceing $a_{1,k}$ by a_k and $b_{1,k}$ by b_k for $k \in \{1,2,3,\ldots,n\}$,

$$\operatorname{per}\begin{pmatrix} x_1a_1 & x_2a_2 & x_3a_3 & \ldots & x_na_n \\ x_1b_1 & x_2b_2 & x_3b_3 & \ldots & x_nb_n \end{pmatrix} \le$$

$$\le \operatorname{per}\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_1a_1b_1 & x_2a_2b_2 & x_3a_3b_3 & \cdots & x_na_nb_n \end{pmatrix} =$$

$$= (2-1)!\sum_{1\le k\le n} x_ka_kb_ks_1(x_1,\ldots,\widehat{x}_k,\ldots,x_n)$$

and summing up $\sum\limits_{1\le k\le n} x_k^2a_kb_k$ in both members.

• We can also obtain the inequality (18) from the Corollary 3.2.3.

Indeed, we consider the case $n = p = 2$. Then G is the complete graph $K_n\left(E = V^{(2)}\right)$ and the degree of every vertex $k \in V$ is $v-1$.

The inequality (11) becomes:

$$\sum_{\{i,j\}\in V^{(2)}} \operatorname{per}\begin{pmatrix} a_{1,i} & a_{1,j} \\ a_{2,i} & a_{2,j} \end{pmatrix} \le (2-1)!\sum_{k\in V}(v-1)\,a_{1,k}b_{1,k}.$$

We sum up $\sum\limits_{k\in V} a_{1,k}b_{1,k}$ in both members and we deduce (18) with a_k replaced by $a_{1,k}$ and $a_{2,k}$ by b_k and n by v.

From lemma 2.1, (5), we can deduce the following

Proposition 3.4.2. *For $x_1,\ldots,x_n,\ a_1,\ldots,a_n,\ b_1,\ldots,b_n \in \mathbb{R}$ we have:*

$$\operatorname{per}\begin{pmatrix} x_1a_1 & x_2a_2 & \cdots & x_na_n \\ y_1b_1 & y_2b_2 & \cdots & y_nb_n \end{pmatrix} = \operatorname{per}\begin{pmatrix} x_1b_1 & x_2b_2 & \cdots & x_nb_n \\ y_1a_1 & y_2a_2 & \cdots & y_na_n \end{pmatrix} + \\ + \sum_{1\le i<j\le n} \det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} \tag{19}$$

$$\left(\sum_{1\le i\le n} x_ia_i\right)\left(\sum_{1\le i\le n} y_ib_i\right) = \left(\sum_{1\le i\le n} x_ib_i\right)\left(\sum_{1\le i\le n} y_ia_i\right) + \\ + \sum_{1\le i<j\le n} \det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}. \tag{20}$$

(20) is the identity of Binet-Cauchy [1, p.174].

For $x_i = a_i$ and $y_i = b_i$ we obtain from (20) the Lagrange's identity:

$$\sum_{1\le i\le n} a_i^2 \sum_{1\le i\le n} b_i^2 = \left(\sum_{1\le i\le n} a_ib_i\right)^2 + \sum_{1\le i\le n} \det{}^2\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}. \tag{21}$$

From Lemma 2.1., (5), we also derive the following identity, (23), of Tiberiu Popoviciu ([1], p.179 and [2], p.38).

Proposition 3.4.3. *For $p_{i,j}, x_i, a_i, b_i \in \mathbb{R}$ where $i,j \in \{1,2,3,\ldots,n\}$ we have*

$$\operatorname{per}\begin{pmatrix} p_{i,j}x_ia_j \\ p_{i,j}y_ib_jb_k \end{pmatrix}_{1\le i\le n 1\le j\le n} = \operatorname{per}\begin{pmatrix} p_{i,j}x_ib_j \\ p_{i,j}y_ia_j \end{pmatrix}_{1\le i\le n 1\le j\le n} + \\ \sum_{1\le i<h\le n 1\le j<k\le n} \det\begin{pmatrix} p_{i,j} & p_{i,k} \\ p_{h,j} & b_{h,k} \end{pmatrix} \det\begin{pmatrix} x_i & x_h \\ y_i & y_h \end{pmatrix} \det\begin{pmatrix} a_k & a_j \\ b_k & b_j \end{pmatrix} \tag{22}$$

$$\left(\sum_{1\le i,j\le n} p_{i,j}x_ia_j\right)\left(\sum_{1\le i,j\le n} p_{i,j}y_ib_j\right) = \left(\sum_{1\le i,j\le n} p_{i,j}x_ib_j\right)\left(\sum_{1\le i,j\le n} p_{i,j}y_ia_j\right) + \\ + \sum_{1\le i<h\le n 1\le j<k\le n} \det\begin{pmatrix} p_{i,j} & p_{i,k} \\ p_{h,j} & p_{h,k} \end{pmatrix} \det\begin{pmatrix} x_i & x_h \\ y_i & y_h \end{pmatrix} \det\begin{pmatrix} a_k & a_j \\ b_k & b_j \end{pmatrix}. \tag{23}$$

Proof: We get applying (5):

$$\operatorname{per}\begin{pmatrix} p_{i,j}x_ia_j \\ p_{i,j}y_ib_j \end{pmatrix}_{1\le i\le n 1\le j\le n} = \operatorname{per}\begin{pmatrix} p_{i,j}x_ib_j \\ p_{i,j}y_ia_j \end{pmatrix}_{1\le i\le n 1\le j\le n} +$$

$$+\sum_{1\le i<h\le n 1\le j<k\le n}\left[\det\begin{pmatrix} p_{i,j}x_i & p_{h,k}x_h \\ p_{i,j}y_i & p_{h,k}y_h \end{pmatrix}\det\begin{pmatrix} a_j & a_k \\ b_j & b_k \end{pmatrix}+\right.$$

$$\left.+\det\begin{pmatrix} p_{i,k}x_i & p_{h,j}x_k \\ p_{i,k}y_i & p_{h,j}y_k \end{pmatrix}\det\begin{pmatrix} a_k & a_j \\ b_k & b_j \end{pmatrix}\right] = \operatorname{per}\begin{pmatrix} p_{i,j}x_ib_j \\ p_{i,j}y_ia_j \end{pmatrix}_{1\le i\le n 1\le j\le n}$$

$$+\sum_{1\le i<h\le n 1\le j<k\le n}\left[p_{i,j}p_{h,k}\det\begin{pmatrix} x_i & x_h \\ y_i & y_h \end{pmatrix}\det\begin{pmatrix} a_j & a_k \\ b_j & b_k \end{pmatrix}\right.$$

$$\left.+p_{i,k}p_{h,j}\det\begin{pmatrix} x_i & x_h \\ y_i & y_h \end{pmatrix}\det\begin{pmatrix} a_k & a_j \\ b_k & b_j \end{pmatrix}\right] =$$

$$= \operatorname{per}\begin{pmatrix} p_{i,j}x_ib_j \\ p_{i,j}y_ia_j \end{pmatrix}_{1\le i\le n 1\le j\le n} +$$

$$+\sum_{1\le i<h\le n 1\le j<k\le n}\det\begin{pmatrix} p_{i,j} & p_{i,k} \\ p_{h,,j} & p_{h,k} \end{pmatrix}\det\begin{pmatrix} x_i & x_h \\ y_i & y_h \end{pmatrix}\det\begin{pmatrix} a_k & a_j \\ b_k & b_j \end{pmatrix}.$$

where we have used the equality

$$\det\begin{pmatrix} x_i & x_h \\ y_i & y_h \end{pmatrix}\det\begin{pmatrix} a_j & a_k \\ b_j & b_k \end{pmatrix} = 0 \text{ for } i=h \text{ or } j=k.$$

Finally, we obtain (23) from (22) by summing up $\sum_{1\le i,j\le n} p_{i,j}x_iy_ia_jb_j$ in both members. □

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