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A Unified Theory of Closed Functions

by
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Abstract

We obtain some characterizations and several properties of M-closed functions defined between sets satisfying some minimal conditions. The functions enable us to formulate a unified theory of modifications of closedness: α-closedness [20], semi-closedness [25], preclosedness [19] and β-closedness [1].

Key Words: m-structure, m-open set, M-closed, closed, α-closed, semi-closed, preclosed, β-closed, m-regular, m-normal.
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1 Introduction

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researches of generalizations of closed functions in topological spaces. By using these sets, many authors introduced and studied various types of modifications of closed functions. The analogy in their definitions and results suggests the need of formulating a unified theory.

In this paper, in order to unify several characterizations and properties of some kind of modifications of closed functions, we introduce a new class of functions called M-closed functions as functions defined between sets satisfying some conditions. We obtain several characterizations and properties of such functions. In Section 3, we obtain several characterizations of M-closed functions. In Section 4, we obtain some preservation theorems of modifications of regular spaces and normal spaces. In the last section, we recall some modifications of open sets and point out the posibility of new forms of M-closed functions.

2 Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$). A subset A is said to be *δ-open* [37] if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$. A point $x \in X$ is called a *δ-cluster* point of A if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by $\mathrm{Cl}_\delta(A)$. The set $\{x \in X : x \in U \subset A$ for some regular open set U of $X\}$ is called the *δ-interior* of A and is denoted by $\mathrm{Int}_\delta(A)$.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be *semi-open* [15] (resp. *preopen* [19], *α-open* [24], *β-open* [1] or *semi-preopen* [3]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))))$.

The family of all semi-open (resp. preopen, α-open, β-open, δ-open, regular open) sets in (X, τ) is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$ or τ^α, $\beta(X)$, $\delta(X)$, $\mathrm{RO}(X)$).

Definition 2.2. The complement of a semi-open (resp. preopen, α-open, β-open) set is said to be *semi-closed* [5] (resp. *preclosed* [19], *α-closed* [20], *β-closed* [1] or *semi-preclosed* [3]).

Definition 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed) sets of X containing A is called the *semi-closure* [5] (resp. *preclosure* [10], *α-closure* [20], *β-closure* [2] or *semi-preclosure* [3]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$).

Definition 2.4. The union of all semi-open (resp. preopen, α-open, β-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior* or *semi-preinterior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$).

A subset A is said to be *semi-regular* [6] if it is semi-open and semi-closed. A point $x \in X$ is called a semi-θ-adherent point of A if $U \cap A \neq \emptyset$ for every semi-regular set U of X containing x. The set of all semi-θ-adherent points of A is denoted by $\mathrm{sCl}_\theta(A)$. If $A = \mathrm{sCl}_\theta(A)$, then A is said to be *semi-θ-closed* [6]. The family of all semi-regular sets of X is denoted by $\mathrm{SR}(X)$.

Definition 2.5. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *semi-closed* [25] (resp. *preclosed* [10], *α-closed* [20], *β-closed* [1], *star-closed* [13]) if $f(F)$ is semi-closed (resp. preclosed, α-closed, β-closed, δ-closed) for each closed set F of X,

(2) *presemiclosed* [11] (resp. *M-preclosed* [21], *strongly α-closed* [23], *pre-β-closed* [17], *semi-θ-closed* [14], *δ-closed* [27]) if $f(A)$ is semi-closed (resp. preclosed, α-closed, β-closed, semi-θ-closed, δ-closed) in Y for every semi-closed (resp. preclosed, α-closed, β-closed, semi-θ-closed, δ-closed) set A of X,

(3) *quasi α-closed* [23] (resp. *almost closed* [36] or *regular closed* [16]) if $f(B)$ is closed in Y for every α-closed (resp. regular closed) set B of X.

3 Minimal structures and M-closed functions

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X [33] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Remark 3.1. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\delta(X)$, $\mathrm{RO}(X)$ and $\mathrm{SR}(X)$ are all m-structures on X.

Definition 3.2. Let (X, m_X) be an m-space. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [18] as follows:
(1) $m_X\text{-Cl}(A) = \bigcap\{F : A \subset F, X - F \in m_X\}$,
(2) $m_X\text{-Int}(A) = \bigcup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$), then we have
(1) $m_X\text{-Cl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$),
(2) $m_X\text{-Int}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$).
In the sequel, $m_X\text{-Cl}(A)$ and $m_X\text{-Int}(A)$ are briefly denoted by $\mathrm{mCl}(A)$ and $\mathrm{mInt}(A)$, respectively.

Lemma 3.1. (Maki et al. [18]) *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) $m_X\text{-Cl}(X - A) = X - (m_X\text{-Int}(A))$ *and* $m_X\text{-Int}(X - A) = X - (m_X\text{-Cl}(A))$,
(2) If $(X - A) \in m_X$*, then* $m_X\text{-Cl}(A) = A$ *and if* $A \in m_X$*, then* $m_X\text{-Int}(A) = A$,
(3) $m_X\text{-Cl}(\emptyset) = \emptyset$, $m_X\text{-Cl}(X) = X$, $m_X\text{-Int}(\emptyset) = \emptyset$ *and* $m_X\text{-Int}(X) = X$,
(4) If $A \subset B$*, then* $m_X\text{-Cl}(A) \subset m_X\text{-Cl}(B)$ *and* $m_X\text{-Int}(A) \subset m_X\text{-Int}(B)$,
(5) $A \subset m_X\text{-Cl}(A)$ *and* $m_X\text{-Int}(A) \subset A$,
(6) $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$ *and* $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$.

Lemma 3.2. (Popa and Noiri [33]) *Let (X, m_X) be an m-space and A a subset of X. Then $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $(\mathcal{B})$ [18] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 3.3. (Popa and Noiri [34]) *Let (X, m_X) an m-space, where m_X satisfies the property $(\mathcal{B})$. For a subset A of X, the following properties hold:*
(1) $A \in m_X$ *if and only if* $m_X\text{-Int}(A) = A$,
(2) A is m_X-closed if and only if $m_X\text{-Cl}(A) = A$,
(3) $m_X\text{-Int}(A) \in m_X$ *and* $m_X\text{-Cl}(A)$ *is m_X-closed.*

Definition 3.4. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-closed* if for each m-closed set F of (X, m_X), $f(F)$ is m-closed in (X, m_Y).

Remark 3.3. Let (X, τ) and (Y, σ) be topological spaces and $f : (X, m_X) \to (Y, m_Y)$ be an M-closed function.

(1) If $m_X = \tau$ and $m_Y = \sigma$ (resp. SO(Y), PO(Y), $\alpha(Y)$, $\beta(Y)$, $\delta(X)$), then f is closed (resp. semi-closed, preclosed, α-closed, β-closed, star-closed).

(2) If $m_X =$ SO(X) (resp. PO(X), $\alpha(X)$, $\beta(X)$, SR(X), $\delta(X)$) and $m_Y =$ SO(Y) (resp. PO(Y), $\alpha(Y)$, $\beta(Y)$, SR(Y), $\delta(Y)$), then f is presemiclosed (resp. M-preclosed, strongly α-closed, pre-β-closed, semi-θ-closed, δ-closed),

(3) If $m_X = \alpha(X)$ (resp. RO(X)) and $m_Y = \sigma$ and f is M-closed, then f is quasi α-closed (resp. almost closed or regular closed).

Theorem 3.1. *For a function $f : (X, m_X) \to (Y, m_Y)$, where m_Y has property $(\mathcal{B})$, the following properties are equivalent:*

(1) f is M-closed;

(2) for each subset F of Y and each $U \in m_X$ with $f^{-1}(F) \subset U$, there exists $V \in m_Y$ such that $F \subset V$ and $f^{-1}(V) \subset U$;

(3) for each $y \in Y$ and each $U \in m_X$ with $f^{-1}(y) \subset U$, there exists $V \in m_Y$ containing y such that $f^{-1}(V) \subset U$.

Proof: (1) $\Rightarrow$ (2): Let f be M-closed. Let F be any subset of Y and $U \in m_X$ with $f^{-1}(F) \subset U$. Put $V = Y - f(X - U)$. Then $f(X - U)$ is m-closed and hence V is m-open in (Y, m_Y), $F \subset V$ and $f^{-1}(V) \subset U$.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Let F be an m-closed set of X and $y \in Y - f(F)$. Since $f^{-1}(y) \subset X - F \in m_X$, there exists $V \in m_Y$ with $y \in V$ and $f^{-1}(V) \subset X - F$. Therefore, $V \cap f(F) = \emptyset$. By Lemma 3.2, $y \in Y - \mathrm{mCl}(f(F))$. Hence $\mathrm{mCl}(f(F)) = f(F)$ and by Lemma 3.3 $f(F)$ is m-closed. This implies that f is M-closed. □

Theorem 3.2. *A function $f : (X, m_X) \to (Y, m_Y)$, where m_X and m_Y have property $(\mathcal{B})$, is M-closed if and only if m_Y-$\mathrm{Cl}(f(A)) \subset f(m_X$-$\mathrm{Cl}(A))$ for every subset A of X.*

Proof: *Necessity.* Suppose that f is M-closed and let A be any subset of X. Since m_X has property $(\mathcal{B})$, by Lemma 3.3 $\mathrm{mCl}(A)$ is m-closed. Since f is M-closed, $f(\mathrm{mCl}(A))$ is m-closed. By Lemma 3.1, $f(A) \subset f(\mathrm{mCl}(A))$ and hence $\mathrm{mCl}(f(A)) \subset \mathrm{mCl}(f(\mathrm{mCl}(A))) = f(\mathrm{mCl}(A))$. Therefore, we obtain $\mathrm{mCl}(f(A)) \subset f(\mathrm{mCl}(A))$.

Sufficiency. Let F be any m-closed set of X. Then by the hypothesis, $\mathrm{mCl}(f(F)) \subset f(\mathrm{mCl}(F)) = f(F)$. Therefore, we have $\mathrm{mCl}(f(F)) = f(F)$ and by Lemma 3.3 $f(F)$ is m-closed. Hence f is M-closed. □

Remark 3.4. Let (X, τ) and (Y, σ) be topological spaces and $f : (X, m_X) \to (Y, m_Y)$ be an M-closed function. If $m_X = \alpha(X)$ (resp. $\beta(X)$, $\mathrm{SR}(X)$, $\delta(X)$) and $m_Y =$ (resp. $\alpha(Y)$, $\beta(Y)$, $\mathrm{SR}(Y)$, $\delta(Y)$), then by Theorems 3.1 and 3.2 we obtain the results established in Lemma 5.4 and Theorem 5.4 of [23] (resp. Theorem 3.3 of [17], Theorem 4.1 of [6], Theorem 2.3 of [27]).

Theorem 3.3. *For a function $f : (X, m_X) \to (Y, m_Y)$, where m_Y has property $(\mathcal{B})$, the following properties are equivalent:*

(1) f is M-closed;

(2) if $U \in m_X$, then the set $G = \{y \in Y : f^{-1}(y) \subset U\}$ is m_Y-open;

(3) if F is m_X-closed, then the set $M = \{y \in Y : f^{-1}(y) \cap F \neq \emptyset\}$ is m_Y-closed.

Proof: (1) $\Rightarrow$ (2): Let $U \in m_X$ and $y \in G$. By Theorem 3.1, there exists $V \in m_Y$ containing y such that $f^{-1}(V) \subset U$. Thus $V \subset G$ and $G = \mathrm{mInt}(G)$. By Lemma 3.3, G is m_Y-open.

(2) $\Rightarrow$ (1): Let F be any m_X-closed set. Then $X - F$ is m_X-open and by (2), $G = \{y \in Y : f^{-1}(y) \subset X - F\}$ is m_Y-open. Since $f(F) = Y - G$, it follows that $f(F)$ is m_Y-closed. Therefore, f is M-closed.

(2) $\Leftrightarrow$ (3): This is obvious. □

Remark 3.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $f : (X, m_X) \to (Y, \sigma)$, where $m_X = \mathrm{RO}(X)$, then by Theorem 3.3 we obtain the result established in Theorem 2.3 of [16].

Definition 3.5. A function $f : (X, \tau) \to (Y, m_Y)$ is said to be *m-closed* if $f(F)$ is m_Y-closed for every closed set F of X.

Corollary 3.1. *For a function $f : (X, \tau) \to (Y, m_Y)$, where m_Y has property $(\mathcal{B})$, the following properties are equivalent:*

(1) f is m-closed;

(2) for each subset F of Y and each open set U of X with $f^{-1}(F) \subset U$, there exists $V \in m_Y$ such that $F \subset V$ and $f^{-1}(V) \subset U$;

(3) for each $y \in Y$ and each open set U of X such that $f^{-1}(y) \subset U$, there exists $V \in m_Y$ containing y such that $f^{-1}(V) \subset U$;

(4) $m_Y\text{-}\mathrm{Cl}(f(A)) \subset f(\mathrm{Cl}(A))$ for every subset A of X.

Proof: This follows immediately from Theorems 3.1 and 3.2. □

Remark 3.6. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $f : (X, \tau) \to (Y, m_Y)$, where $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\alpha(Y)$, $\beta(Y)$), then by Corollary 3.1 we obtain the results established in Theorems 4 and 5 of [25] (resp. Lemma 6.3 of [10], Theorems 2.2 and 2.3 of [20], Theorem 3.2 of [1]).

Definition 3.6. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *quasi m-closed* if $f(F)$ is closed in (Y, σ) for every m_X-closed set F of X.

Corollary 3.2. *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is quasi m-closed;

(2) for each subset F of Y and each $U \in m_X$ with $f^{-1}(F) \subset U$, there exists an open set V of Y such that $F \subset V$ and $f^{-1}(V) \subset U$;

(3) for each $y \in Y$ and each $U \in m_X$ with $f^{-1}(y) \subset U$, there exists an open set V of Y containing y such that $f^{-1}(V) \subset U$.

Proof: This follows immediately from Theorem 3.1. □

Remark 3.7. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $f : (X, m_X) \to (Y, \sigma)$, where $m_X = \alpha(X)$ (resp. $\mathrm{RO}(X)$)), then by Corollary 3.2 we obtain the results established in Theorem 4.3 of [23] (resp. Lemma 1 of [26]).

Corollary 3.3. *A function $f : (X, m_X) \to (Y, \sigma)$, where m_X has property ($\mathcal{B}$), is quasi m-closed if and only if $\mathrm{Cl}(f(A)) \subset f(m_X\text{-}\mathrm{Cl}(A))$ for every subset A of X.*

Proof: This follows immediately from Theorem 3.2. □

4 Preservations of m-regularity and m-normality

Definition 4.1. An m-space (X, m_X) is said to be *m-regular* [34] if for each m-closed set F and each $x \notin F$, there exist disjoint m-open sets U and V such that $x \in U$ and $F \subset V$.

Remark 4.1. Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X)$). Then m-regularity coincides with regularity (resp. semi-regularity [7], pre-regularity [31], semi-preregularity [28]).

Lemma 4.1. (Noiri and Popa [29]). *Let (X, m_X) be an m-space, where m_X satisfies property ($\mathcal{B}$). Then (X, m_X) is m-regular if and only if for each $x \in X$ and each m_X-open set U containing x, there exists an m_X-open set V such that $x \in V \subset m_X\text{-}\mathrm{Cl}(V) \subset U$.*

Definition 4.2. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-continuous* [33] if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$.

Lemma 4.2. (Noiri and Popa [33]). *For a function $f : (X, m_X) \to (Y, m_Y)$, where m_X satisfies property ($\mathcal{B}$), the following properties are equivalent:*

(1) f is M-continuous;

(2) $f^{-1}(V)$ is m_X-open for every m_Y-open set V of Y;

(3) $f^{-1}(K)$ is m_X-closed for every m_Y-closed set K of Y.

Definition 4.3. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-open* [22] if for each $U \in m_X$, $f(U) \in m_Y$.

Theorem 4.1. *If $f : (X, m_X) \to (Y, m_Y)$, where m_X and m_Y have property $(\mathcal{B})$, is an M-continuous, M-open and M-closed surjection and (X, m_X) is m-regular, then (Y, m_Y) is m-regular.*

Proof: Let $y \in Y$ and V be any m_Y-open set of Y containing y. Let $x \in f^{-1}(y)$. By Lemma 4.2, we have $x \in f^{-1}(V) \in m_X$. By Lemma 4.1, there exists $U \in m_X$ such that $x \in U \subset \mathrm{mCl}(U) \subset f^{-1}(V)$. Then, we have $y \in f(U) \subset f(\mathrm{mCl}(U)) \subset V$. Since f is M-open, $f(U)$ is m_Y-open. Since m_X has property $(\mathcal{B})$, by Lemma 3.3, $\mathrm{mCl}(U)$ is m-closed and hence $f(\mathrm{mCl}(U))$ is m-closed because f is M-closed. Hence, we obtain $y \in f(U) \subset \mathrm{mCl}(f(U)) \subset V$. It follows from Lemma 4.1 that (Y, m_Y) is m-regular. □

Corollary 4.1. (Noiri [26]). *If $f : (X, \tau) \to (Y, \sigma)$ is a continuous, open and closed surjection and (X, τ) is regular, then (Y, σ) is regular.*

Theorem 4.2. *If $f : (X, m_X) \to (Y, m_Y)$, where m_X has property $(\mathcal{B})$, is an M-continuous and M-closed injection and (Y, m_Y) is m-regular, then (X, m_X) is m-regular.*

Proof: Let F be any m-closed set of X and $x \in X - F$. Since f is M-closed, $f(F)$ is m-closed and $f(x) \in Y - f(F)$. Since (Y, m_Y) is m-regular, there exist disjoint m-open sets U and V such that $f(x) \in U$ and $f(F) \subset V$. Since f is M-continuous, by Lemma 4.2 $f^{-1}(U)$ and $f^{-1}(V)$ are m_X-open. Thus we obtain $x \in f^{-1}(U), F \subset f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. This shows that (X, m_X) is m-regular. □

Definition 4.4. A function $f : (X, \tau) \to (Y, m_Y)$ is said to be *(τ, m)-continuous* [34] if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists an open set U containing x such that $f(U) \subset V$.

Lemma 4.3. *For a function $f : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f is (τ, m)-continuous;
(2) $f^{-1}(V)$ is open in X for every m_Y-open set V of Y;
(3) $f^{-1}(K)$ is closed in X for every m_Y-closed set K of Y.

Theorem 4.3. *If $f : (X, \tau) \to (Y, m_Y)$, where m_Y has property $(\mathcal{B})$, is a (τ, m)-continuous and m-closed surjection with compact point inverses and (X, τ) is regular, then (Y, m_Y) is m-regular.*

Proof: Let F be any m-closed set of Y and $y \in Y - F$. Since f is (τ, m)-continuous, by Lemma 4.3 $f^{-1}(F)$ is closed in X. By the hypothesis, $f^{-1}(y)$ is compact. Moreover, $f^{-1}(F)$ and $f^{-1}(y)$ are disjoint in the regular space (X, τ). Therefore, there exist disjoint open sets U_y and U_F such that $f^{-1}(y) \in U_y$ and $f^{-1}(F) \subset U_F$. Since f is m-closed, by Corollary 3.1 there exist m-open sets V_y and V_F such that $y \in V_y, F \subset V_F, f^{-1}(V_y) \subset U_y$ and $f^{-1}(V_F) \subset U_F$. Since f is surjective and $U_y \cap U_F = \emptyset$, we obtain $V_y \cap V_F = \emptyset$. This shows that (Y, m_Y) is m-regular. □

Definition 4.5. An m-space (X, m_X) is said to be *m-normal* if for each pair of disjoint m-closed sets F_1, F_2 of X, there exist $U_1, U_2 \in m_X$ such that $F_1 \subset U_1, F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Remark 4.2. Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X), \beta(X)$). Then m-normality coincides with normality (resp. semi-normality [8], pre-normality [30], β-normality [17]).

Theorem 4.4. *If $f : (X, m_X) \to (Y, m_Y)$, where m_X and m_Y have property ($\mathcal{B}$), is an M-closed and M-continuous surjection and (X, m_X) is m-normal, then (Y, m_Y) is m-normal.*

Proof: Let K_1, K_2 be any disjoint m-closed sets of (Y, m_Y). Since f is M-continuous, by Lemma 4.2 $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint m-closed sets of (X, m_X). Since (X, m_X) is m-normal, there exist $U_1, U_2 \in m_X$ such that $f^{-1}(K_1) \subset U_1, f^{-1}(K_2) \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Since f is M-closed, by Theorem 3.1 there exists $V_i \in m_Y$ such that $K_i \subset V_i$ and $f^{-1}(V_i) \subset U_i$ for $i = 1$, 2. Since $U_1 \cap U_2 = \emptyset$ and f is surjective, we obtain $V_1 \cap V_2 = \emptyset$. Therefore, (Y, m_Y) is m-normal. □

Remark 4.3. Let (X, τ) and (Y, σ) be topological spaces. If $f : (X, m_X) \to (Y, m_Y)$ is an M-closed and M-continuous surjection, $m_X = \tau$ (resp. $\mathrm{SO}(X)$) and $m_Y = \sigma$ (resp. $\mathrm{SO}(Y)$), then we obtain the result established in Theorem 3.3 of [9] (p.145) (resp. Theorem 25 of [4]).

Theorem 4.5. *If $f : (X, m_X) \to (Y, m_Y)$, where m_X has property ($\mathcal{B}$), is an M-continuous and M-closed injection and (Y, m_Y) is m-normal, then (X, m_X) is m-normal.*

Proof: Let A and B be disjoint m-closed sets of X. Since f is an M-closed injection, $f(A)$ and $f(B)$ are disjoint m-closed sets of Y. By the m-normality of (Y, m_Y), there exist disjoint m-open sets U and V such that $f(A) \subset U$ and $f(B) \subset V$. Since f is M-continuous, by Lemma 4.2 $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint m-open sets containing A and B, respectively. This shows that (X, m_X) is m-normal. □

Corollary 4.2. (Arya and Bhamini [4]). *The inverse images of semi-normal spaces under irresolute and presemiclosed injections are semi-normal.*

5 New forms of M-closed functions

First, we recall some modifications of open sets defined recently.

Definition 5.1. A subset A of a topological space (X, τ) is said to be *δ-semiopen* [32] (resp. *δ-preopen* [35], *δ-β-open* [12]) if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A)))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A))))$.

By $\delta\mathrm{SO}(X)$ (resp. $\delta\mathrm{PO}(X)$, $\delta\beta(X)$), we denote the collection of all δ-semiopen (resp. δ-preopen, δ-β-open) sets of a topological space (X, τ). These three collections are all m-structures with property $(\mathcal{B})$. The following implications hold:

DIAGRAM I

$$\begin{array}{ccccccc}
\delta\text{-open} & \Rightarrow & \text{open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\delta\text{-semiopen} & \Rightarrow & \text{semi-open} & \Rightarrow & \beta\text{-open} & \Rightarrow & \delta\text{-}\beta\text{-open}
\end{array}$$

Definition 5.2. A subset A of a topological space (X, τ) is said to be *δ-semiclosed* [32] (resp. *δ-preclosed* [35], *δ-β-closed* [12]) if the complement of A is δ-semiopen (resp. δ-preopen, δ-β-open).

Definition 5.3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be
(1) *δ-semi-closed* if $f(F)$ is δ-semi-closed in Y for every closed set F of X,
(2) *δ-preclosed* if $f(F)$ is δ-preclosed in Y for every closed set F of X,
(3) *δ-β-closed* if $f(F)$ is δ-β-closed in Y for every closed set F of X.

Remark 5.1. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $f : (X, m_X) \to (Y, m_Y)$ is M-closed, $m_X = \tau$ and $m_Y = \delta\mathrm{SO}(Y)$ (resp. $\delta\mathrm{PO}(Y)$, $\delta\beta(X)$), then $f : (X, \tau) \to (Y, \sigma)$ is δ-semi-closed (resp. δ-preclsoed, δ-β-closed).

Lemma 5.1. *Let m_X^1 and m_X^2 be two m-structures on a nonempty set X. If $m_X^1 \subset m_X^2$ and a function $f : (X, m_X^2) \to (Y, m_Y)$ is M-closed, then $f : (X, m_X^1) \to (Y, m_Y)$ is M-closed.*

Proof: Suppose that $f : (X, m_X^2) \to (Y, m_Y)$ is M-closed. Let F be m-closed in (X, m_X^1). Since $m_X^1 \subset m_X^2$, F is m-closed in (X, m_X^2) and $f(F)$ is m-closed in (Y, m_Y). This shows that $f : (X, m_X^1) \to (Y, m_Y)$ is M-closed. □

Lemma 5.2. *Let m_Y^1 and m_Y^2 be two m-structures on a nonempty set Y. If $m_Y^1 \subset m_Y^2$ and a function $f : (X, m_X) \to (Y, m_Y^1)$ is M-closed, then $f : (X, m_X) \to (Y, m_Y^2)$ is M-closed.*

Proof: Suppose that $f : (X, m_X) \to (Y, m_Y^1)$ is M-closed. Let F be m-closed in (X, m_X). Then $f(F)$ is m-closed in (Y, m_Y^1). Since $m_Y^1 \subset m_Y^2$, $f(F)$ is m-closed in (Y, m_Y^2). This shows that $f : (X, m_X) \to (Y, m_Y^2)$ is M-closed. □

By DIAGRAM I and Lemma 5.2, for functions defined above we obtain the following diagram:

DIAGRAM II

$$\begin{array}{ccccccc} \text{star-closed} & \Rightarrow & \text{closed} & \Rightarrow & \text{preclosed} & \Rightarrow & \delta\text{-preclosed} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \delta\text{-semi-closed} & \Rightarrow & \text{semi-closed} & \Rightarrow & \beta\text{-closed} & \Rightarrow & \delta\text{-}\beta\text{-closed} \end{array}$$

Remark 5.2. If, as m_X and m_Y, we take the collections of δ-open sets, δ-preopen sets or δ-β-open sets, we can define new kinds of functions and use the results established in Sections 3 and 4.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.

[2] M. E. Abd El-Monsef, R. A. Mahmoud and E. R. Lashin, *β-closure and β-interior*, J. Fac. Ed. Ain Shams Univ. **10** (1986), 235–245.

[3] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik **38** (1986), 24–32.

[4] S. P. Arya and M. P. Bhamini, *A generalization of normal spaces*, Mat. Vesnik **35** (1983), 1–10.

[5] S. G. Crossley and S. K. Hildebrand, *Semi-closure* Texas J. Sci. **22** (1971), 99–112.

[6] G. Di Maio and T. Noiri, *On s-closed spaces*, Indian J. Pure Appl. Math. **18** (1987), 226–233.

[7] C. Dorsett, *Semi-regular spaces*, Soochow J. Math. **8** (1982), 45–53.

[8] C. Dorsett, *Semi-normal spaces*, Kyungpook Math. J. **25** (1985), 173–180.

[9] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[10] S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, *On p-regular spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie **27(75)** (1983), 311–315.

[11] G. L. Garg and D. Sivaraj, *Presemiclosed mappings*, Period. Math. Hungar. **19** (1988), 97–106.

[12] E. Hatir and T. Noiri, *Decompositions of continuity and complete continuity*, Acta Math. Hungar. (to appear).

[13] L. L. Herrington, *Some properties preserved by the almost-continuous function*, Boll. Un. Mat. Ital. (4) **10** (1974), 556–568.

[14] M. Khan, B. Ahmad and T. Noiri, *On semi θ-perfect functions*, Bull. Malays. Math. Sci. Soc. (2) **26** (2003), 175–180.

[15] N. Levine, *Semi-open sets and semi-continuity in topological spaces,* Amer. Math. Monthly **70** (1963), 36–41.

[16] P. E. Long and L. L. Herrington, *Basic properties of regular closed functions*, Rend. Circ. Mat. Palermo (2), **27** (1978), 20–28.

[17] R. A. Mahmoud and M. E. Abd El-Monsef, *β-irresolute and topological β-invariant*, Proc. Pakistan Acad. Sci. **27** (1990), 285–296.

[18] H. Maki, K. Chandrasekhara Rao and A. Nagoor Gani, *On generalizing semi-open and preopen sets*, Pure Appl. Math. Sci. **49** (1999), 17–29.

[19] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, *On precontinuous and weak precontinuous mappings,* Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.

[20] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, *α-continuous and α-open mappings*, Acta Math. Hungar. **41** (1983), 213–218.

[21] A. S. Mashhour, M. E. Abd El-Monsef and I. A. Hasanein, *On pretopological spaces,* Bull. Math. Soc. Math. R. S. Roumanie **98**(**76**) (1984), 39–45.

[22] M. Mocanu, *Generalizations of open functions*, Stud. Cerc. St. Ser. Mat., Univ. Bacău, **13** (2003), 67–78.

[23] G. B. Navalagi, *Quasi α-closed, strongly α-closed, weakly α-irresolute mappings*, Topology Atlas, ♯421.

[24] O. Njåstad, *On some classes of nearly open sets,* Pacific J. Math. **15** (1965), 961–970.

[25] T. Noiri, *A generalization of closed mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fiz. Mat. Natur. (8) **54** (1973), 412–415.

[26] T. Noiri, *Une note sur les applications presque fermées*, Ann. Soc. Sci. Bruxelles **88** (1974), 301–304.

[27] T. Noiri, *A generalization of perfect functions*, J. London Math. Soc. (2) **17** (1978), 540–544.

[28] T. Noiri, *Weak and strong forms of β-irresolute functions*, Acta Math. Hungar. **99** (2003), 315–328.

[29] T. Noiri and V. Popa, *A unified theory of θ-continuity for functions*, Rend. Circ. Mat. Palermo (2) **52** (2003), 163–188.

[30] T. M. Nour, *Contributions to Theory of Bitopological Spaces*, Ph. D. Thesis, Univ. of Delhi, 1989.

[31] M. C. Pal and P. Bhattacharyya, *Feeble and strong forms of preirresolute functions*, Bull. Malaysian Math. Sci. Soc. (2) **19** (1996), 63–75.

[32] J. H. Park, B. Y. Lee and M. J. Son, *On δ-semiopen sets in topological spaces,* J. Indian Acad. Math. **19** (1997), 59–67.

[33] V. Popa and T. Noiri, *On M-continuous functions,* Anal. Univ. "Dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor., Fasc. II, **18** (**23**) (2000), 31–41.

[34] V. Popa and T. Noiri, *On weakly (τ, m)-continuous functions*, Rend. Circ. Mat. Palermo (2) **51** (2002), 295–316.

[35] S. Raychaudhuri and M. N. Mukherjee, *On δ-almost continuity and δ-preopen sets,* Bull. Inst. Math. Acad. Sinica **21** (1993), 357–366.

[36] M. K. Singal and A. R. Singal, *Almost continuous mappings*, Yokohama Math. J. **16** (1968), 63–73.

[37] N. V. Veličko, *H-closed topological spaces,* Amer. Math. Soc. Transl. (2) **78** (1968), 103–118.

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