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Quasiconformality and compatibility for direct products of homeomorphisms

by
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Abstract

Replacing limsup by liminf we establish the characterization of quasiconformality for direct products of homeomorphisms by a compatibility condition in Karmazin's meaning.

We use regular families of neighborhoods and distance (0.1) on direct products of Euclidean domains. We prove that direct products of two quasiconformal mappings are quasiconformal under certain conditions of compatibility.

Key Words: Direct product, quasiconformal map, linear liminf-dilatation, regular families of neighborhoods.
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Introduction

This paper develops the foundations of the theory of quasiconformal maps on direct product spaces with respect to linear liminf-dilatation.

In [4] and [5] the quasiconformality is introduced on the basis of Markushevich-Pesin's definition in connection with linear limsup-dilatation, whose equivalence to Väisälä's metric definition is given in [6].

Let D and D' be domains in $\mathbb{R}^n$, $F : D \to D'$ a homeomorphism and

$$d(z, z_0) = |z - z_0| = (|x - x_0|^2 + |y - y_0|^2)^{1/2} \tag{0.1}$$

the Euclidean distance. For any point $z_0 \in D$ and $t > 0$ such that the ball $\bar{B}(z_0, t) = \{z : d(z, z_0) = |z - z_0| \leq t\}$ be included in D, denote

$$L(z_0, F, t) = \max_{d(z,z_0)=t} d(F(z), F(z_0)) \tag{0.2}$$

and

$$l(z_0, F, t) = \min_{d(z,z_0)=t} d(F(z), F(z_0)). \tag{0.3}$$

It is known that if the *linear limsup-dilatation* of F at z_0

$$H(z_0, F) = \limsup_{t\to 0} \frac{L(z_0, F, t)}{l(z_0, F, t)} \tag{0.4}$$

is bounded in D, i.e. there exists a constant $H < \infty$ such that $H(z_0, F) \le H$ for every $z_0 \in D$, F is a quasiconformal mapping after the metric definition [6]. At the end of 80's, M. Cristea [1] used also in the study of the quasiconformal mappings the *linear liminf-dilatation*

$$h(z_0, F) = \liminf_{t\to 0} \frac{L(z_0, F, t)}{l(z_0, F, t)} \tag{0.5}$$

and unexpected in 1995, J. Heinonen and P. Koskela [3] proved that $h(z_0, F) \le h$ for every $z_0 \in D$, h being a finit constant, implies that F is quasiconformal, what increased the importance of $h(z_0, F)$. The present paper deals with M. Cristea's following problem [2]: the study of the quasiconformality for direct product of homeomorphisms by means of Karmazin's compatibility, however working with liminf.

Let U and V be domains in $\mathbb{R}^k$ and $\mathbb{R}^m$; x and y arbitrary points in U and V respectively; $f : U \to U' \subset \mathbb{R}^k$, $g : V \to V' \subset \mathbb{R}^m$ homeomorphisms; $F = f \times g : U \times V \to U' \times V'$ the direct product of f and g, $U \times V$ and $U' \times V'$ being domains in $R^k \times R^m$ identified with R^n, $n = k + m$; $z = (x, y)$ a point in $U \times V$ and $F(z) = (f(x), g(y)) \in U' \times V'$.

The homeomorphisms F, f, g are quasiconformal by Karmazin's Definition if $H(z_0, F)$ and respectively $H(x_0, f)$, $H(y_0, g)$ are bounded in $U \times V$, U or V respectively, and by Definition 2 from below if $h(z_0, F)$, $h(x_0, f)$, $h(y_0, g)$ are bounded in $U \times V$, U and V.

Starting from Karmazin's compatibility condition, we say that f and g are compatible if there is a constant C, respectively c, such that:

Condition 1: $\limsup_{t\to 0} \frac{L(x_0, f, t)}{l(y_0, g, t)} \le C$ and $\limsup_{t\to 0} \frac{L(y_0, g, t)}{l(x_0, f, t)} \le C$, for any $x_0 \in U$ and $y_0 \in V$, and

Condition 2: there exists a sequence $t_p \to 0$, $p \in \mathbb{N}$ such that

$$\frac{L(x_0, f, t_p)}{l(y_0, g, t_p)} \le c \text{ and } \frac{L(y_0, g, t_p)}{l(x_0, f, t_p)} \le c, \text{ for any } x_0 \in U, y_0 \in V.$$

Remark: It is clear that Condition 2 is fulfilled e.g. if

$$\liminf_{t\to 0} \frac{L(x_0,f,t)}{l(y_0,g,t)} \le c \text{ and } \limsup_{t\to 0} \frac{L(y_0,g,t)}{l(x_0,f,t)} \le c$$

or vice versa, and for case when

$$\liminf_{t\to 0} \frac{L(x_0,f,t)}{l(y_0,g,t)} \le c \text{ and } \liminf_{t\to 0} \frac{L(y_0,g,t)}{l(x_0,f,t)} \le c$$

but there exists a sequence t_p as above. Clearly, Condition 2 implies

$$\liminf_{t\to 0} \frac{L(x_0,f,t)}{l(y_0,g,t)} \le c \text{ and } \liminf_{t\to 0} \frac{L(y_0,g,t)}{l(x_0,f,t)} \le c.$$

With these compatibility conditions we succeeded to characterize the quasiconformality of F extending Karmazin's Theorem 1 in [4].

1 Regular families of neighborhoods

Definition1 *Let be x_0 an arbitrary point in $\mathbb{R}^n$. A family of open sets $\{U_t| \quad t \in (0,1]\}$ is called a regular family of neighborhoods at the point x_0, if*

1^0) *$U_t \subset U_1$ for all t;*

2^0) *there exists a homeomorphism $\tau : U_1 \to B^n$ such that $\tau(U_t) = B^n(t)$ for every $t \in (0,1)$;*

3^0) $\tau(x_0) = 0$

and a h-regular family of neighborhoods at x_0 if in addition

$$\liminf_{t\to 0} \frac{\max\limits_{x\in\partial U_t} |x-x_0|}{\min\limits_{x\in\partial U_t} |x-x_0|} \le h = h(x_0), (1 \le h(x_0) < \infty). \tag{1.1}$$

Definition 2 *We say that homeomorphism $F : D \to D'(D, D^{'} \subset \mathbb{R}^n)$, $n \ge 2$, is quasiconformal at the point $x_0 \in D$, if there is a h- regular family of neighborhoods $\{U_t|t \in (0,1]\}$ of the point x_0 such that family of neighborhoods $U_t' = F(U_t)$ of the point $x_0' = F(x_0)$ forms a regular family of parameter $h^{'}(x_0)(1 \le h^{'} < \infty)$*

$$\liminf_{t\to 0} \frac{\max_{y\in\partial V_t} |y-y_0|}{\min\limits_{y\in\partial V_t} |y-y_0|} \le h' = h^{'}(x_0), 1 \le h^{'} < \infty. \tag{1.2}$$

Definition 3 *We call $q(x_0,F) = \inf h(x_0)h^{'}(x_0)$ a quasiconformality characteristic of F at x_0, where the infimum is taken over all regular families of neighborhoods of x_0.*

Definition 4 *A homeomorphism F is called K - quasiconformal in D if there is a constant K, $1 \le K < \infty$, such that the characteristic $q(x,F)$ is bounded in D and if $q(x,F) \le K$ a.e. in D.*

The following Lemma 1 is essentially known, see [4], but we represent the proof in detail since it is needed later.

Lemma 1 *Let $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^m$ be open sets and $D = U \times V$. Then, by ρ we denote the distance,*

$$\rho(z_0, \partial D) = \min\{\rho(x_0, \partial U), \rho(y_0, \partial V)\}. \tag{1.3}$$

Proof: We use the general property of direct product

$$\partial(U \times V) = (\partial U \times V) \cup (U \times \partial V) \cup (\partial U \times \partial V)). \tag{1.4}$$

Denote $\rho_0 = \min\{\rho(x_0, \partial U), \rho(y_0, \partial V)\}$ and we show that ρ_0 is the distance from z_0 at ∂D.

Step I. If $U = \mathbb{R}^k$ and $V = \mathbb{R}^m$, then $\rho(x_0, \partial U) = \infty$ and $\rho(y_0, \partial V) = \infty$ what implies $\rho_0 = \rho(z_0, \partial D) = \infty$, where $D = \mathbb{R}^k \times \mathbb{R}^m$.

In the rest consider $\rho_0 < \infty$.

Step II. For example, if $U = \mathbb{R}^k$ and $V \neq \mathbb{R}^m$, then $\rho_0 = \rho(y_0, \partial V)$

$$\begin{aligned} \rho(z_0, \partial D) &= \rho(z_0, U \times \partial V) = \inf_{z \in U \times \partial V} |(x_0, y_0) - (x, y)| = \\ &= \inf_{z \in U \times \partial V} \sqrt{|x - x_0|^2 + |y - y_0|^2} \\ &= \inf_{y \in \partial V} |y - y_0| = \rho(y_0, \partial V) = \rho_0. \end{aligned}$$

Step III. Similarly ,we show lemma for $U \neq \mathbb{R}^k$ and $V \neq \mathbb{R}^m$.

a) We prove that $\rho(z_0, \partial D) \geq \rho_0$. Let $x^{'} \in \partial U$ be to fulfilling distance from x_0 at ∂U, that is

$$\rho(x_0, \partial U) = \inf_{x \in \partial U} \{\rho(x_0, x) | x \in \partial U\} = \rho(x_0, x^{'})$$

and let be $y' \in \partial V$ fulfilling distance from y_0 at ∂V, that is

$$\rho(y_0, \partial V) = \inf_{y \in \partial V} \{\rho(y_0, y) | y \in \partial V\} = \rho(y_0, y^{'})$$

Let be $\rho(z^{''}, z_0) = \rho(z_0, \partial D)$ where $z^{''} \in \partial D$.

Assuming that $\rho(z^{''}, z_0) < \rho_0$, then

$$\begin{aligned} |z_0 - z^{''}|^2 &= \sum_{i=1}^{k} (x_i^0 - x_i^{''})^2 + \sum_{j=1}^{m} (y_i^0 - y_i^{''})^2 < \rho_0^2 \leq \\ &\leq \sum_{i=1}^{k} (x_i^0 - x_i^{'})^2 = \rho^2(x_0, \partial U). \end{aligned}$$

Furthermore,

$$\sum_{i=1}^{k}(x_i^0 - x_i^{''})^2 < \sum_{i=1}^{k}(x_i^0 - x_i^{'})^2$$

Hence $\rho(x_0, x^{''}) < \rho(x_0, x^{'}) = \rho(x_0, \partial U) \Rightarrow x^{''} \in \mathrm{IntU}$, analogously $y^{''} \in \mathrm{IntV}$. Thus, $z^{''} = (x^{''}, y^{''}) \in \mathrm{IntD}$, what leads to a contradition.

Hence $\rho(z^{''}, z_0) \geq \rho_0$.

b) Suppose that $x^{'} \in \partial U, y^{'} \in \partial V$ as in case (a) and consider the points $z_1 = (x^{'}, y_0)$ and $z_2 = (x_0, y^{'})$, which belong to ∂D.

Then

$$|z_1 - z_0| = |(x^{'}, y_0) - (x_0, y_0)| = |x^{'} - x_0| = \rho(x_0, \partial U),$$

$$|z_2 - z_0| = |(x_0, y^{'}) - (x_0, y_0)| = |y^{'} - y_0| = \rho(y_0, \partial V).$$

We obtain $\rho(z_0, \partial D) \leq \rho(z_0, z_1) = \rho(x_0, \partial U)$ and $\rho(z_0, \partial D) \leq \rho(z_0, z_2) = \rho(y_0, \partial V)$.

Hence

$$\rho(z_0, \partial D) \leq \min(\rho(x_0, \partial U), \rho(y_0, \partial V) = \rho_0.$$

The cases (a) and (b) imply (1.3). □

2 Relation between regularity and compatibility of the families of neighborhoods

Let $\{U_t | t \in (0,1]\}$ and $\{V_t | t \in (0,1]\}$ be regular families of neighborhoods of x_0 and y_0 respectively.

Definition 5 *We say that families of neighborhoods $\{U_t\}$ and $\{V_t\}$ are compatible at $z_0 = (x_0, y_0)$, if there exist a sequence $t_p \to 0$, $p \in \mathbb{N}$, and a constant $c < \infty$ such that the following conditions are satisfied:*

$$(\alpha') \qquad \frac{\max\limits_{x \in \partial U_{t_p}} |x - x_0|}{\min\limits_{y \in \partial V_{t_p}} |y - y_0|} \leq c = c(z_0)$$

and

$$(\alpha'') \qquad \frac{\max\limits_{y \in \partial V_{t_p}} |y - y_0|}{\min\limits_{x \in \partial U_{t_p}} |x - x_0|} \leq c = c(z_0).$$

Remark: Definition 5 expresses Condition 2 from Introduction.

Lemma 2 *Family of neighborhoods* $\{W_t | t \in (0,1]\} = \{U_t \times V_t\}$ *of the point* z_0 *is regular, if and only if families* $\{U_t\}$ *and* $\{V_t\}$ *are compatible at the point* $z_0 = (x_0, y_0)$.

More exactly if $\{W_t\}$ *is h-regular at the* z_0*, then* $\{U_t\}$*,* $\{V_t\}$ *are compatible at* z_0 *with the same constant* h*. Conversely if* $\{U_t\}$ *and* $\{V_t\}$ *are c-compatible at the point* z_0*, then* $\{W_t\}$ *is* (c^2+c)*-regular at* z_0 *point.*

Proof: **Necessity**

Let

$$R(x_0,t) = \max_{x\in\partial U_t} |x-x_0|, r(x_0,t) = \min_{x\in\partial U_t} |x-x_0|. \tag{2.1}$$

Similarly, we can define the functions $R(y_0,t), r(y_0,t), R(z_0,t), r(z_0,t)$.

Suppose that a family of neighborhoods $\{W_t\}$ of the point z_0 is referred to as regular of parameter h, in accordance with (1.1)

$$\liminf_{t\to 0} \frac{\max\limits_{z\in\partial W_t} |z-z_0|}{\min\limits_{z\in\partial W_t} |z-z_0|} \le h. \tag{2.2}$$

We choose $z' = (x', y') \in \partial W_t$ such that $x' \in \partial U_t$, $y' \in \partial V_t$ and

$$\max_{x\in\partial U_t} |x-x_0| = |x'-x_0|, \quad \max_{y\in\partial V_t} |y-y_0| = |y'-y_0|,$$

hence

$$|z'-z_0| \ge |x'-x_0|, |z'-z_0| \ge |y'-y_0| \text{ and}$$

$$\max_{z\in\partial W_t} |z-z_0| \ge |z'-z_0| \ge \max\{|x'-x_0|, |y'-y_0|\}.$$

On the basis of (2.1) and Lemma 1 we obtain

$$R(z_0,t) \ge \max\{R(x_0,t), R(y_0,t)\}, r(z_0,t) = \min\{r(x_0,t), r(y_0,t)\} \tag{2.3.}$$

Obviously,

$$\frac{R(z_0,t)}{r(z_0,t)} \ge \max\left\{\frac{R(x_0,t)}{r(x_0,t)}; \frac{R(x_0,t)}{r(y_0,t)}; \frac{R(y_0,t)}{r(x_0,t)}; \frac{R(y_0,t)}{r(y_0,t)}\right\}. \tag{2.4}$$

By Definition 1

$$\liminf_{t\to 0} \frac{R(z_0,t)}{r(z_0,t)} \le h$$

therefore for any $\varepsilon > 0$ there exists sequence $t_p \to 0$, $p \in \mathbb{N}$, with

$$\frac{R(z_0,t_p)}{r(z_0,t_p)} \le h + \varepsilon.$$

By (2.4)

$$\frac{R(x_0,t_p)}{r(y_0,t_p)} \le h+\varepsilon \quad \text{and} \quad \frac{R(y_0,t_p)}{r(x_0,t_p)} \le h+\varepsilon.$$

Hence, families of neighborhoods $\{U_t\}$ and $\{V_t\}$ satisfy the compatibility conditions $(\alpha^{'})$ and $(\alpha^{''})$ of Definition 5 and are h-compatible. Thus the necessity is proved.

Sufficiency

Consider that the families of neighborhoods $\{U_t\}$ and $\{V_t\}$ are c-compatible at z_0. We have to show that family $\{W_t\}$ is regular at the point z_0.

Let $d(z,z_0) = |z-z_0| = (|x-x_0|^2 + |y-y_0|^2)^{\frac{1}{2}}$ be the distance in the direct product space $U \times V$. Then $|z-z_0| \le |(x,y)-(x_0,y)| + |(x_0,y)-(x_0,y_0)|$, hence $R(z_0,t) \le R(x_0,t) + R(y_0,t)$ and by Lemma 1, $r(z_0,t) = \min\{r(x_0,t), r(y_0,t)\}$.

Obviously,

$$\begin{aligned} \frac{R(z_0,t)}{r(z_0,t)} &\le \frac{R(x_0,t)+R(y_0,t)}{\min(r(x_0,t),r(y_0,t))} \\ &= \max\left\{\frac{R(x_0,t)+R(y_0,t)}{r(x_0,t)}, \frac{R(x_0,t)+R(y_0,t)}{r(y_0,t)}\right\}. \end{aligned} \tag{2.5}$$

By Definition 5, there exists a sequence $t_p \to 0$, $p \in \mathbb{N}$, such that

$$\frac{R(x_0,t_p)}{r(y_0,t_p)} \le c.$$

and

$$\frac{R(y_0,t_p)}{r(x_0,t_p)} \le c$$

Consequently,

$$\begin{aligned} \frac{R(x_0,t_p)}{r(x_0,t_p)} &\le \frac{R(x_0,t_p)}{r(y_0,t_p)} \cdot \frac{R(y_0,t_p)}{r(x_0,t_p)} \le c^2 \Rightarrow \liminf_{t\to 0} \frac{R(x_0,t)}{r(x_0,t)} \le c^2, \\ \frac{R(y_0,t_p)}{r(y_0,t_p)} &\le \frac{R(y_0,t_p)}{r(x_0,t_p)} \cdot \frac{R(x_0,t_p)}{r(y_0,t_p)} \le c^2 \Rightarrow \liminf_{t\to 0} \frac{R(y_0,t)}{r(y_0,t)} \le c^2, \end{aligned} \tag{2.6}$$

by using $r(y_0,t_p) \le R(y_0,t_p)$, $r(x_0,t_p) \le R(x_0,t_p)$.

This means that families $\{U_t\}$ and $\{V_t\}$ are regular at the points x_0 and y_0 respectively, of parameter c^2.

Obviously, by (2.5)

$$\frac{R(z_0,t_p)}{r(z_0,t_p)} \le c^2 + c \Rightarrow \liminf_{t\to 0} \frac{R(z_0,t)}{r(z_0,t)} \le c^2 + c. \tag{2.7}$$

It follows that family $\{W_t\}$ is regular at the point z_0 of parameter c^2+c.

□

Remarks:

(i) The regularity condition of family $\{W_t\}$ implies $\{U_t\}$ and $\{V_t\}$ regular.

(ii) The compatibility condition of families $\{U_t\}$ and $\{V_t\}$ also implies $\{U_t\}$ and $\{V_t\}$ regular.

3 Compatible mappings and quasiconformality of the direct product

Definition 6 *We shall say that the homeomorphisms $f : U \to U^{'}$ and $g : V \to V^{'}$ are compatible, if there exist the c-compatible families of neighborhoods $\{U_t\}$ and $\{V_t\}$ at the point $z_0 = (x_0, y_0) \in U \times V$, such that $\{f(U_t)\}$ and $\{g(V_t)\}$ are the c'-compatible at the point $z_0' = (f(x_0), g(y_0))$.*

This means that there is a sequence $t_p \to 0$, $p \in \mathbb{N}$, and a constant $c^{'} < \infty$ such that the following conditions are satisfied:

$$(\beta^{'}) \qquad \frac{\max\limits_{x \in \partial U_{t_p}} |f(x) - f(x_0)|}{\min\limits_{y \in \partial V_{t_p}} |g(y) - g(y_0)|} \leq c^{'} = c'(x_0, f, g),$$

$$(\beta^{''}) \qquad \frac{\max\limits_{y \in \partial V_{t_p}} |g(y) - g(y_0)|}{\min\limits_{x \in \partial U_{t_p}} |f(x) - f(x_0)|} \leq c^{'} = c'(z_0, f, g).$$

Definition 7 *We call $p(z_0) = p(z_0, f, g) = \inf c\, c^{'}$ the compatibility characteristic of the homeomorphisms f and g at z_0, where $c, c^{'}$ are compatibility parameters of the families of neighborhoods $\{U_t\}$ and $\{V_t\}$, $\{f(U_t)\}$ and $\{g(V_t)\}$ respectively, and infimum is taken for all families $\{U_t\}$ and $\{V_t\}$ (Definition 5 and 6).*

Definition 8 *We say that the homeomorphisms f and g are K-compatible if they are compatible at every point $z_0 \in U \times V$, the compatible characteristic $p(x_0)$ is bounded in D and there is a constant K, $1 \leq K < \infty$, such that $p(z_0) \leq K$ a.e. in D.*

Theorem 1 *Suppose $f : U \to U^{'}$ and $g : V \to V^{'}$ are homeomorphisms. Then $F(z) = f(x) \times g(y)$ is quasiconformal in the domain $D = U \times V$, if and only if the homeomorphisms f and g are compatible in D.*

Proof: Let $F = f \times g$ be a quasiconformal homeomorphism. By Definition 2, we have that there is a h-regular family of neighborhoods $\{W_t\}$ of the point x_0 such that family of neighborhoods $\{F(W_t)\}$ is regular of parameter h'.

This means that there is a constant $h' < \infty$, such that linear liminf-dilatation is

$$(3.1) \qquad h(z_0, F) = \liminf_{t \to 0} \frac{\max\limits_{z \in \partial W_t} |F(z) - F(z_0)|}{\min\limits_{z \in \partial W_t} |F(z) - F(z_0)|} \leq h'$$

for almost all $z_0 \in D$ and bounded in D.

Necessity

Step 1

We shall show that if homeomorphism F is quasiconformal then f and g are compatible.

Denote by

$$D_0 = \{z_0 \in D | h(z_0, F) \leq h'\}$$

for which $\text{mes}D = \text{mes}D_0$.

Fix $z_0 = (x_0, y_0) \in D_0$, where $x_0 \in U$ and $y_0 \in V$ with $U_t \subset U$, $V_t \subset V$. Evidently that families of neighborhoods $\{U_t\}$, $\{V_t\}$ at the points x_0, y_0 respectively, are h-compatible at the point z_0, satisfying conditions $(\alpha^{'})$ and $(\alpha^{''})$.

By (3.1) $h(z_0, F) \leq h'$, what implies that family of neighborhoods $\{F(U_t \times V_t)\} = \{f(U_t) \times g(V_t)\}$ is h'-regular at point $F(z_0) = (f(x_0), g(y_0))$.

By virtue of Lemma 2, if family of neighborhoods $F(U_t \times V_t)$ is h'-regular then families of neighborhoods $\{f(U_t)\}, \{g(V_t)\}$ are h'-compatible at point $F(z_0)$ satisfying conditions $(\beta^{'}), (\beta^{''})$, what implies that f and g are compatible in $z_0 \in D$.

Step 2

Let $\tilde{z}_0 \in D \backslash D_0$, to repeat reasoning used above families of neighborhoods $\{U_t\}, \{V_t\}$ are $\tilde{h}$-compatible at the point $\tilde{z}_0 = (\tilde{x}_0, \tilde{y}_0)$. Family of neighborhoods $\{f(U_t) \times g(V_t)\}$ is regular at the point $F(\tilde{z}_0)$ with parameter $\tilde{h}'$. Similarly, as in Lemma 2, families of neighborhoods $\{f(U_t)\}$, $\{g(V_t)\}$ are compatible of parameter $\tilde{h}'$ at the point $F(\tilde{z}_0)$.

It follows that the homeomorphisms f and g are K-compatible, because are compatible at each point $z \in U_t \times V_t$ and their compatibility characteristic $p(z_0) = \inf h(z_0)h'(z_0) \leq K$, $(1 \leq K < \infty)$ in D_0 and $p(z)$ is bounded in D.

Sufficiency

Let f and g be K-compatible homeomorphisms in domain D.

Denote by $D_1 = \{z_0 \in D | p(z_0) \leq K\}$.

Since $p(z_0) \leq K$ for almost all $z_0 \in D$, $\text{mesD}_1 = \text{mesD}$.

We can choose $z_0 \in D_1$ and for all $\varepsilon > 0$ arbitrary number there exist families of neighborhoods $\{U_t\}, \{V_t\}$ c-compatible at the point $z_0 = (x_0, y_0)$ such that families of neighborhoods $\{f(U_t)\}, \{g(V_t)\}$ are $c^{'}$-compatible at the point $\{f(x_0), g(y_0)\}$ and $c(z_0) \cdot c^{'}(z_0) \leq K + \varepsilon$.

By Lemma 2, (2.7) family $\{W_t\} = \{U_t \times V_t\}$ is $(c^2 + c)$-regular at the point z_0 and family $\{F(W_t)\} = \{f(U_t) \times g(V_t)\}$ is $(c'^2 + c')$-regular at the point $F(z_0)$.

Obviously, homeomorphism F is quasiconformal at the point z_0 with quasiconformality characteristic.

$$q(z_0) \leq (c^2 + c)(c^{'2} + c^{'}).$$

Because $c, c^{'} \geq 1$, we have that

$$c^2 + c < 2c^2, \quad c^{'2} + c^{'} < 2c^{'2}$$

hence $q(z_0) \leq 4(c \cdot c^{'})^2 \leq 4(K+\varepsilon)^2$ and homeomorphism F is $4K^2$-quasiconformal in D.

This completes the proof. □

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