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A generalization of Pardue's formula

by
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Abstract

In this paper, we introduce a new class of monomial ideals, called **d**-fixed ideals, which generalize the class of p-Borel ideals and show how some results for p-Borel ideals can be transfered to this new class. In particular, we give the form of a principal **d**-fixed ideal and we compute the socle of factors of this ideals, using methods similar as in [3]. This allowed us to give a generalization of Pardue's formula, i.e. a formula of the regularity for a principal **d**-fixed ideal.

Key Words: p-Borel ideals, Betti numbers, Mumford-Castelnuovo regularity.
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Introduction

A p-Borel ideal is a monomial ideal which satisfy certain combinatorial condition, where $p > 0$ is a prime number. It is well known that any positive integer a has an unique p-adic decomposition $a = \sum_{i\geq 0} a_i p^i$. If a, b are two positive integers, we write $a \leq_p b$ iff $a_i \leq b_i$ for any i, where $a = \sum_{i\geq 0} a_i p^i$ and $b = \sum_{i\geq 0} b_i p^i$. We say that a monomial ideal $I \subset S = k[x_1, \ldots, x_n]$ is p-Borel if for any monomial $u \in I$ and for any indices $j < i$, if $t \leq_p \nu_i(u)$ then $x_j^t u/x_i^t \in I$, where $\nu_i(u) = max\{k : x_i^k | u\}$.

This definition suggest a natural generalization. The idea is to consider a strictly increasing sequence of positive integers $\mathbf{d} : 1 = d_0|d_1|\cdots|d_s$, which we called a **d**-sequence. Lemma 1.1 states that for any positive integer a, there exists an unique decomposition $a = \sum_{t=0}^{s} a_t d_t$. If a, b are two positive integers, we write $a \leq_{\mathbf{d}} b$ iff $a_i \leq b_i$ for any i, where $a = \sum_{i\geq 0} a_i d_i$ and $b = \sum_{i\geq 0} b_i d_i$. We say that a monomial ideal I is **d**-fixed if for any monomial $u \in I$ and for any indices $j < i$, if $t \leq_{\mathbf{d}} \nu_i(u)$ then $x_j^t u/x_i^t \in I$. Obvious, the p-Borel ideals are a special case of **d**-fixed ideals for $\mathbf{d} : 1|p|p^2|\cdots$.

A principal $\mathbf{d}$-fixed ideal, is the smallest $\mathbf{d}$-fixed ideal which contain a given monomial. 1.6 and 1.8 gives the explicit form of a principal $\mathbf{d}$-fixed ideal. In the second section we compute the socle of factors for a principal $\mathbf{d}$-fixed ideal (2.1 and 2.4). The proofs are similar as in [3] but we consider that is necessary to present them in this context. In the third section we give a formula (3.1) for the regularity of a $\mathbf{d}$-fixed ideal, which generalize the Pardue's formula for the regularity of principal p-Borel ideals, proved by Aramova-Herzog [1] and Herzog-Popescu [4]. Using a theorem of Popescu [6] we compute the extremal Betti numbers of S/I (3.3). Also, we show that if I is a principal $\mathbf{d}$-fixed ideal generated by the power of a variable, then $I_{\geq e}$ is stable for any $e \geq reg(I)$ (3.6), so $reg(I) = min\{e \geq deg(I) : I_{\geq e} \ is \ stable\}$ (3.9). Thus a result of Eisenbud-Reeves-Totaro [2, Proposition 12] holds also in this frame.

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1 d-fixed ideals.

In the following $\mathbf{d} : 1 = d_0|d_1|\cdots|d_s$ is a strictly increasing sequence of positive integers. We say that $\mathbf{d}$ is a $\mathbf{d}$-sequence.

Lemma 1.1. *Let $\mathbf{d}$ be a $\mathbf{d}$-sequence. Then, for any $a \in \mathbb{N}$, there exists an unique sequence of positive integers $a_0, a_1, \ldots, a_s$ such that:*

1. $a = \sum_{t=0}^{s} a_t d_t$ *and*
2. $0 \leq a_t < \frac{d_{t+1}}{d_t}$, *for any* $0 \leq t < s$.

Conversely, if $\mathbf{d} : 1 = d_0 < d_1 < \cdots < d_s$ is a sequence of positive integers such that for any $a \in \mathbb{N}$ there exists an unique sequence of positive integers $a_0, a_1, \ldots, a_s$ as before, then $\mathbf{d}$ is a $\mathbf{d}$-sequence.

Proof: Let a_s be the quotient of a divided by d_s. For $0 \leq t < s$ let a_t be the quotient of $(a - q_{t+1})$ divided by d_t, where $q_{t+1} = \sum_{j=t+1}^{s} a_j d_j$. We will prove that $a_0, a_1, \ldots, a_s$ fulfill the required conditions. Indeed, it is obvious that $a = \sum_{t=0}^{s} a_t d_t$. On the other hand, $a - q_{t+1} < d_{t+1}$, since $a - q_{t+1}$ is $a - q_{t+2}$ modulo d_{t+1}. Therefore, since a_t is the quotient of $(a - q_{t+1})$ divided by d_t, it follows that $a_t < \frac{d_{t+1}}{d_t}$.

Suppose there exists another decomposition $a = \sum_{j=0}^{s} b_j d_j$ which also fulfill the conditions 1 and 2. Then, we may assume that there exists an integer $0 \leq t \leq s$ such that $b_s = a_s, \cdots, b_{t+1} = a_{t+1}$ and $b_t > a_t$. Notice that $d_t > \sum_{j=0}^{t-1} a_j d_j$. Indeed,

$$\sum_{j=0}^{t-1} a_j d_j \leq \sum_{j=0}^{t-1} (\frac{d_{j+1}}{d_j} - 1) d_j = (d_1 - d_0) + (d_2 - d_1) + \cdots + (d_t - d_{t-1}) = d_t - 1 < d_t.$$

We have $0 = \sum_{j=0}^{s}(b_j - a_j)d_j = \sum_{j=0}^{t}(b_j - a_j)d_j$, but on the other hand:

$$(b_t - a_t)d_t \geq d_t > \sum_{j=0}^{t-1} a_j d_j \geq \sum_{j=0}^{t-1}(a_j - b_j)d_j$$

and therefore $(b_t - a_t)d_t - \sum_{j=0}^{t-1}(a_j - b_j)d_j = \sum_{j=0}^{t}(b_j - a_j)d_j > 0$, which is a contradiction.

For the converse, we use induction on $0 \leq t < s$, the assertion being obvious for $t = 0$. Suppose $t > 0$ and $d_0|d_1|\cdots|d_t$ and consider the decomposition of $d_{t+1} - 1$. Since $d_{t+1} - 1 < d_{t+1}$, it follows that $d_{t+1} - 1 = \sum_{j=0}^{t} a_t d_t$. On the other hand, since $d_{t+1} - 1$ is the largest integer less than d_{t+1}, each a_j is maximal between the integers $< d_{j+1}/d_j$, for $j < t$. Therefore $a_j = d_{j+1}/d_j - 1$ for $0 \leq j < t$. Thus:

$$d_{t+1} = 1 + d_{t+1} - 1 = 1 + a_0 d_0 + a_1 d_1 + \cdots + a_t d_t = d_1 + a_1 d_1 + a_2 d_2 + \cdots + a_t d_t =$$

$$= d_2 + a_2 d_2 + \cdots + a_t d_t = \cdots = (a_t + 1)d_t, \ \ so \ \ d_t | d_{t+1}.$$

□

Definition 1.2. *Let a, b be two positive integers and consider the $\mathbf{d}$-decompositions $a = \sum_{j=0}^{s} a_j d_j$ and $b = \sum_{j=0}^{s} b_j d_j$. We say that $a \leq_{\mathbf{d}} b$ if $a_i \leq b_i$ for any $0 \leq i \leq s$.*

Lemma 1.3. *Let a, b be two positive integers with $a \leq_{\mathbf{d}} b$. Suppose $b = b' + b''$, where b' and b'' are positive integers. Then, there exists some positive integers $a' \leq_{\mathbf{d}} b'$ and $a'' \leq_{\mathbf{d}} b''$ such that $a = a' + a''$.*

Proof: Let $a = \sum_{t=0}^{s} a_t d_t$, $b = \sum_{t=0}^{s} b_t d_t$, $b' = \sum_{t=0}^{s} b'_t d_t$, $b'' = \sum_{t=0}^{s} b''_t d_t$. The hypothesis implies $a_t \leq b_t < d_{t+1}/d_t$ and $b'_t, b''_t < d_{t+1}/d_t$ for any $0 \leq t < s$. We construct the sequences a'_t, a''_t using decreasing induction on t. Suppose we have already defined a'_j, a''_j for $j > t$ such that $\sum_{i=j}^{s}(a'_i + a''_i)d_i = \sum_{i=j}^{s} a_i d_i$ and $b_{t+1} = b'_{t+1} + b''_{t+1}$. This is obvious for $t = s$.

We consider two cases. If $b_t = b'_t + b''_t$, then we choose $a'_t \leq b'_t$ and $a''_t \leq b''_t$ such that $a'_t + a''_t = a_t$. We can do this, because $a_t \leq b_t$. Also, it is obvious from the induction hypothesis that $\sum_{i=t}^{s}(a'_i + a''_i)d_i = \sum_{i=t}^{s} a_i d_i$, so we can pass from t to $t-1$.

If $b_t \neq b'_t + b''_t$ we claim that $b'_t + b''_t = b_t - 1$. Indeed, $\sum_{j=0}^{t-1}(b'_j + b''_j)d_j < 2d_t$ and therefore it is impossible to have $b'_t + b''_t \leq b_t - 2$, otherwise $\sum_{j=0}^{t}(b'_j + b''_j)d_j < b_t d_t$ and we contradict the equality $b = b' + b''$. Also, since $b'_{t+1} + b''_{t+1} = b_{t+1}$, we cannot have $b'_t + b''_t > b_t$. Similarly we get $b'_{t-1} + b''_{t-1} > b_{t-1}$. By recurrence, we conclude that there exists an integer $u < t$ such that: $b'_{u-1} + b''_{u-1} = b_{u-1}, \ , b'_u + b''_u = b_u + d_{u+1}/d_u$, $b'_{u+1} + b''_{u+1} = b_{u+1} + d_{u+2}/d_{u+1} - 1, \ \ldots, \ b'_{t-1} + b''_{t-1} = b_{t-1} + d_t/d_{t-1} - 1$.

If $a_j = b_j$ for any $j \in \{u,\dots,t\}$, we simply choose $a'_j = b'_j$ and $a''_j = b''_j$ for any $j \in \{u,\dots,t\}$ and the required conditions are fulfilled, so we can pass from t to $u-1$. If this is not the case, then there exists an integer $u \leq q \leq t$ such that $a_t = b_t,\dots,a_{q+1} = b_{q+1}$ and $a_q < b_q$. If $q = t$ then for any $j \in \{u,\dots,t\}$ we can choose $a'_j \leq b'_j$ and $a''_j \leq b''_j$ such that $a'_j + a''_j = a_j$. For $j < t$ the previous assertion is obvious because $b'_j + b''_j \geq b_j$, and for $j = t$, since $a_t < b_t$ we have in fact $a_t \leq b'_t + b''_t = b_t - 1$ and therefore we can choose again a'_t and a''_t. The conditions are satisfied so we can pass from t to $u-1$.

Suppose $q < t$. For $j \in \{u,\dots,q-1\}$ we choose $a'_j \leq b'_j$ and $a''_j \leq b''_j$ such that $a'_j + a''_j = a_j$. We can do this because $b'_j + b''_j \geq b_j \geq a_j$. We choose a'_q and a''_q such that $a'_q + a''_q = a_q + d_{q+1}/d_q$. We can make this choice, because $a_q \leq b_q - 1$ and $b'_q + b''_q \geq b_q + d_{q+1}/d_q - 1$. For $j > q$, we simply put $a'_j = b'_j$ and $a''_j = b''_j$. To pass from t to $u-1$ is enough to see that $\sum_{j=u}^{t} a_j d_j = \sum_{j=u}^{t}(a'_j + a''_j)d_j$. Indeed,

$$\sum_{j=u}^{t}(a'_j + a''_j)d_j = \sum_{j=u}^{q-1}(a'_j + a''_j)d_j + (a'_q + a''_q)d_q + \sum_{j=q+1}^{t}(a'_j + a''_j)d_j =$$

$$= \sum_{j=u}^{q-1} a_j d_j + (a_q + d_{q+1}/d_q)d_q + \sum_{j=q+1}^{t-1}(a_j + d_{j+1}/d_j - 1)d_j + (a_t - 1)d_t =$$

$$= \sum_{j=u}^{t} a_j d_j + d_{q+1} + \sum_{j=q+1}^{t-1}(d_{j+1} - d_j) - d_t = \sum_{j=u}^{t} a_j d_j,$$

The induction ends when $t = -1$. Finally, we obtain a' and a'' such that $a' + a'' = a$, $a'_t \leq b'_t$ and $a''_t \leq b''_t$, as required. □

Definition 1.4. *We say that a monomial ideal $I \subset S = k[x_1,\dots,x_n]$ is $\mathbf{d}$-fixed, if for any monomial $u \in I$ and for any indices $1 \leq j < i \leq n$, if $t \leq_{\mathbf{d}} \nu_i(u)$ (where $\nu_i(u)$ denotes the exponent of the variable x_i in u) then $u \cdot x_j^t / x_i^t \in I$.*

Notice that if $\mathbf{d} : 1|p|p^2|p^3|\cdots$ then I is a p-Borel ideal.

Definition 1.5. *A $\mathbf{d}$-fixed ideal I is called principal if it is generated, as a $\mathbf{d}$-fixed ideal by one monomial u, i.e. I is the smallest $\mathbf{d}$-fixed ideal which contain u. We write $I = < u >_{\mathbf{d}}$. More generally, if $u_1,\dots,u_r \in S$ are monomials, the $\mathbf{d}$-fixed ideal generated by $u_1,\dots,u_r$ is the smallest $\mathbf{d}$-fixed ideal I which contains $u_1,\dots,u_r$. We write $I = < u_1,\dots,u_r >_{\mathbf{d}}$.*

Our next goal is to describe the principal $\mathbf{d}$-fixed ideals. The easiest case is when we have a $\mathbf{d}$-fixed ideal generated by the power of a variable. Denote $\mathbf{m} = (x_1,\dots,x_n)$ and $\mathbf{m}^{[d]} = (x_1^d,\dots,x_n^d)$ for some nonnegative integer d. We have the following proposition.

Proposition 1.6. *If $u = x_n^{\alpha}$, then $I = < u >_{\mathbf{d}} = \prod_{t=0}^{s}(\mathbf{m}^{[d_t]})^{\alpha_t}$, where $\alpha = \sum_{t=0}^{s} \alpha_t d_t$.*

Proof: Let $I' = \prod_{t=0}^{s}(\mathbf{m}^{[d_t]})^{\alpha_t}$. The minimal generators of I' are monomials of the type $w = \prod_{t=0}^{s}\prod_{j=1}^{n} x_j^{\lambda_{tj}\cdot d_t}$, where $0 \le \lambda_{tj}$ and $\sum_{j=1}^{n}\lambda_{tj} = \alpha_t$. First, let us show that $I' \subset I$. In order to do this, we choose w a minimal generator of I' (the one bellow). We write x_n^α like this: $x_n^\alpha = x_n^{\alpha_0 d_0 + \alpha_1 d_1 + \cdots + \alpha_s d_s} = x_n^{\alpha_0 d_0} \cdot x_n^{\alpha_1 d_1} \cdots x_n^{\alpha_s d_s}$. Since $\lambda_{01} d_0 \le_{\mathbf{d}} \alpha_0 d_0 + \alpha_1 d_1 + \cdots + \alpha_s d_s$ and I is $\mathbf{d}$-fixed it follows that $x_1^{\lambda_{01}d_0} x_n^{\alpha - \lambda_{01} d_0} \in I$. Also, $\lambda_{02} d_0 < \alpha - \lambda_{01} d_0 = (\alpha_0 - \lambda_{01}) d_0 + \alpha_1 d_1 + \cdots + \alpha_s d_s$, and since I is $\mathbf{d}$-fixed it follows that $x_1^{\lambda_{01}d_0} x_2^{\lambda_{02}d_0} x_n^{\alpha - \lambda_{01}d_0 - \lambda_{02}d_0} \in I$. Using iteratively this argument, one can easily see that $x_1^{\lambda_{01}d_0} \cdots x_n^{\lambda_{0n}d_0} x_n^{\alpha - \alpha_0 d_0} \in I$. Also $\alpha - \alpha_0 d_0 = \alpha_1 d_1 + \cdots + \alpha_n d_n$. Again, using an inductive argument, we get:

$$(x_1^{\lambda_{01}d_0} \cdots x_n^{\lambda_{0n}d_0}) \cdot (x_1^{\lambda_{11}d_1} \cdots x_n^{\lambda_{1n}d_1}) \cdots (x_1^{\lambda_{s1}d_s} \cdots x_n^{\lambda_{sn}d_s}) = w \in I.$$

For the converse, i.e. $I \subset I'$, is enough to verify that I' is $\mathbf{d}$-fixed. In order to do this, is enough to prove that the minimal generators of I' fulfill the definition of a $\mathbf{d}$-fixed ideal. Let w be a minimal generator of I'. Let $2 \le i \le n$. Then $\nu_i(w) = \sum_{t=0}^{s}\lambda_{ti} d_t$. If $\beta \le_{\mathbf{d}} \nu_i(w)$ then $\beta = \sum_{t=0}^{s}\beta_t d_t$ with $\beta_t \le \lambda_{ti}$. Let $1 \le k < i$. We have

$$w \cdot x_k^\beta / x_i^\beta = \prod_{t=0}^{s}(\prod_{j\ne i,k} x_j^{\lambda_{tj}d_t}) \cdot x_i^{(\lambda_{ti}-\beta_t)d_t} \cdot x_k^{(\lambda_{tk}+\beta_t)d_t}.$$

Thus $w \cdot x_k^\beta / x_i^\beta \in I'$ and therefore I' is $\mathbf{d}$-fixed. Since I is the smallest $\mathbf{d}$-fixed ideal which contains x_n^α it follows that $I \subset I'$. □

Proposition 1.7. *If* $\alpha \le \beta$ *then* $< x_n^\beta >_{\mathbf{d}} \subseteq < x_n^\alpha >_{\mathbf{d}}$.

Proof: The case $\alpha = \beta$ is obvious, so we may assume $\alpha < \beta$. We denote $I = < x_n^\alpha >_{\mathbf{d}}$ and $I' = < x_n^\beta >_{\mathbf{d}}$. We write $\alpha = \sum_{t=0}^{s}\alpha_t d_t$ and $\beta = \sum_{t=0}^{s}\beta_t d_t$. If w is a minimal generator of I' then $w = \prod_{t=0}^{s}\prod_{i=1}^{n} x_i^{\lambda_{ti}d_t}$, where $0 \le \lambda_{ti}$ and $\sum_{i=1}^{n}\lambda_{ti} = \beta_t$. We claim that $w \in I$ and therefore $I' \subset I$ as required.

Since $\alpha < \beta$ there exists $t \in \{0, \ldots, s\}$ such that $\alpha_s = \beta_s, \ldots, \alpha_{t+1} = \beta_{t+1}$ and $\alpha_t < \beta_t$. We may assume $\lambda_{t1} > 0$. We have

$$w = \prod_{j=0}^{s}\prod_{i=1}^{n} x_i^{\lambda_{ji}d_j} = \prod_{j=0}^{t-1} x_1^{\alpha_j d_j} x_1^{(\lambda_{t1}-1)d_t} x_1^{d_t - \sum_{j=0}^{t-1}\alpha_j d_j} \prod_{i=2}^{n} x_i^{\lambda_{ti}d_t} \prod_{j>t}\prod_{i=1}^{n} x_i^{\lambda_{ji}d_j}$$

and now it is obvious that $w \in I$. □

We have the general description of a principal $\mathbf{d}$-fixed ideal given by the following proposition. In the proof, we will apply Lemma 1.3.

Proposition 1.8. *Let $1 \leq i_1 < i_2 < \cdots < i_r = n$ and let $\alpha_1, \ldots, \alpha_r$ be some positive integers. If $u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_r}^{\alpha_r}$ then:*

$$I = < u >_{\mathbf{d}} = < x_{i_1}^{\alpha_1} >_{\mathbf{d}} \cdot < x_{i_2}^{\alpha_2} >_{\mathbf{d}} \cdots < x_{i_r}^{\alpha_r} >_{\mathbf{d}} = \prod_{q=1}^{r} \prod_{t=0}^{s} (\mathbf{m}_q^{[d_t]})^{\alpha_{qt}},$$

where $\mathbf{m}_q = (x_1, x_2, \ldots, x_{i_q})$ and $\alpha_q = \sum_{t=0}^{s} \alpha_{qt} d_t$.

Proof: Let $I' = \prod_{q=1}^{r} \prod_{t=0}^{s} (\mathbf{m}_q^{[d_t]})^{\alpha_{qt}}$. The minimal generators of I' are monomials of the type $w = \prod_{q=1}^{r} \prod_{t=0}^{s} \prod_{j=1}^{i_q} x_j^{\lambda_{qtj} \cdot d_t}$, where $0 \leq \lambda_{qtj}$ and $\sum_{j=1}^{n} \lambda_{qtj} = \alpha_{qt}$. First, we show that $I' \subset I$. In order to do this, it is enough to prove that by iterative transformations we can modify u such that we obtain w.

The idea of this transformations is the same as in the proof of 1.6. Without given all the details, one can see that if we rewrite u as

$$(x_{i_1}^{\alpha_{10} d_0} x_{i_1}^{\alpha_{11} d_1} \cdots x_{i_1}^{\alpha_{1s} d_s}) \cdots (x_{i_r}^{\alpha_{r0} d_0} x_{i_r}^{\alpha_{r1} d_1} \cdots x_{i_r}^{\alpha_{rs} d_s}),$$

where $\alpha_q = \sum_{t=0}^{s} \alpha_{qt} d_t$, we can pass to w, using the transformations

$$x_{i_1}^{\alpha_{10} d_0} \mapsto \prod_{j=1}^{i_1} x_j^{\lambda_{10j} d_0}, \ldots, x_{i_1}^{\alpha_{1s} d_s} \mapsto \prod_{j=1}^{i_1} x_j^{\lambda_{1sj} d_s}, \ldots, x_{i_r}^{\alpha_{r0} d_0} \mapsto$$

$$\mapsto \prod_{j=1}^{i_r} x_j^{\lambda_{r0j} d_0}, \ldots, x_{i_r}^{\alpha_{rs} d_s} \mapsto \prod_{j=1}^{i_r} x_j^{\lambda_{rsj} d_s}.$$

Therefore $w \in I$, and thus $I' \subset I$. For the converse, it is enough to see that I' is a $\mathbf{d}$-fixed ideal. Let w be a minimal generator of I'. We choose an index $2 \leq i \leq n$. Then $\nu_i(w) = \sum_{q=1}^{r} \sum_{t=0}^{s} \lambda_{qti} d_t$. Let $\beta \leq \nu_i(w)$. Using Lemma 3.1, we can choose some positive integers $\beta_1, \ldots, \beta_r$ such that:

$$(a) \beta = \sum_{q=1, i_q \geq i}^{r} \beta_q \text{ and } (b) \beta_q \leq_{\mathbf{d}} \sum_{t=0}^{s} \lambda_{qti} d_t,$$

i.e. $\beta_{qt} \leq \lambda_{qti}$, where $\beta_q = \sum_{t=0}^{s} \beta_{qt} d_t$. Let $k < i$. Then,

$$w \cdot x_k^{\beta} / x_i^{\beta} = \prod_{q=1}^{r} \prod_{t=0}^{s} \left(\prod_{j=1, j \neq k, i}^{i_q} x_j^{\lambda_{qtj} \cdot d_t} \right) x_i^{(\lambda_{qti} - \beta_{qt}) d_t} x_k^{(\lambda_{qtk} + \beta_{qt}) d_t}.$$

Now, it is easy to see that $w \cdot x_k^{\beta} / x_i^{\beta} \in I'$, and therefore I' is $\mathbf{d}$-fixed. $\square$

Example 1.9. *Let* $\mathbf{d} : 1|2|4|12$.

1. *Let* $u = x_3^{21}$. *We have* $21 = 1 \cdot 1 + 0 \cdot 2 + 2 \cdot 4 + 1 \cdot 12$. *From* 1.6*, we get:*

$$< u >_{\mathbf{d}} = (x_1, x_2, x_3)(x_1^4, x_2^4, x_3^4)^2(x_1^{12}, x_2^{12}, x_3^{12}).$$

2. *Let* $u = x_1^2 x_2^9 x_3^{16}$. *We have* $9 = 1 \cdot 1 + 2 \cdot 4$ *and* $16 = 1 \cdot 4 + 1 \cdot 12$. *From* 1.8*, we get*

$$< u >_{\mathbf{d}} = x_1^2 < x_2^9 >_{\mathbf{d}} < x_3^{16} >_{\mathbf{d}} =$$
$$= x_1^2(x_1, x_2)(x_1^4, x_2^4)^2(x_1^4, x_2^4, x_3^4)(x_1^{12}, x_2^{12}, x_3^{12}).$$

Definition 1.10. *We say that a monomial ideal* $I \subset k[x_1, \dots, x_n]$ *is a Borel type ideal if*

$$I : x_j^{\infty} = I : (x_1, \dots, x_j)^{\infty}, \text{ for any } j = 1, \dots, n.$$

Proposition 1.11. *Any* $\mathbf{d}$*-fixed ideal* I *is a Borel type ideal.*

Proof: Indeed, [3, *Proposition* 2.2] says that an ideal I is of Borel type if and only if for any $1 \leq j < i \leq n$, there exists an positive integer t such that $x_j^t(u/x_i^{\nu_i(u)}) \in I$. Choosing $t = \nu_i(u)$, is easy to see that the definition of a $\mathbf{d}$-fixed ideal implies the condition above. □

Definition 1.12. *Let* $S = k[x_1, \dots, x_n]$ *and let* M *be a finitely generated graded* S*-module. The module* M *is sequentially Cohen-Macaulay if there exists a finite filtration* $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ *of* M *by graded submodules of* M *such that:*

- M_i/M_{i-1} *are Cohen-Macaulay for any* $i = 1, \dots, r$ *and*
- $dim(M_1/M_0) < dim(M_2/M_1) < \cdots < dim(M_r/M_{r-1})$.

In particular, if $I \subset S$ *is a graded ideal then* $R = S/I$ *is sequentially Cohen-Macaulay if there exists a chain of ideals* $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ *such that* I_j/I_{j-1} *are Cohen-Macaulay and* $dim(I_j/I_{j-1}) < dim(I_{j+1}/I_j)$ *for any* $j = 1, \dots, r-1$.

Remark 1.13. *Let* $I \subset S$ *be a monomial ideal. Recursively we define an ascending chain of monomial ideals as follows: We let* $I_0 := I$. *Suppose* I_ℓ *is already defined. If* $I_\ell = S$ *then the chain ends. Otherwise, let* $n_\ell = max\{i : x_i | u \text{ for an } u \in G(I_\ell)\}$. *We set* $I_{\ell+1} := (I_\ell : x_{n_\ell}^{\infty})$. *It is obvious that* $n_\ell > n_{\ell-1}$, *and therefore the chain* $I_0 \subset I_1 \subset \cdots \subset I_r = S$ *is finite and has length* $r \leq n$. *We call this chain of ideals, the sequential chain of* I.

If I *is a Borel type ideal,* [3, *Lemma* 2.4] *says that*

$$I_{\ell+1} := I_\ell : (x_1, x_2, \dots, x_{n_\ell})^{\infty}.$$

From [3, Corollary 2.5], it follows that $R = S/I$ is sequentially Cohen-Macaulay with the sequential chain $I_0 \subset I_1 \subset \cdots \subset I_r = S$ defined above. Moreover $I_{\ell+1}/I_\ell \cong J_\ell^{sat}/J_\ell[x_{n_\ell+1},\dots,x_n]$, where $J_\ell = I_\ell \cap k[x_1,\dots,x_{n_\ell}]$ and $J_\ell^{sat} = J_\ell : (x_1,\dots,x_{n_\ell})^\infty$.

Let $u = x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_r}^{\alpha_r}$ and $I = <u>_{\mathbf{d}} = \prod_{q=1}^r\prod_{t=0}^s(\mathbf{m}_q^{[d_t]})^{\alpha_{qt}}$, where $\mathbf{m}_q = (x_1,\dots,x_{i_q})$ and $\alpha_q = \sum_{t=0}^s \alpha_{qt}d_t$. Let $I_{r-e} = \prod_{q=1}^e\prod_{t=0}^s(\mathbf{m}_q^{[d_t]})^{\alpha_{qt}}$. Then $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ is the sequential chain of I. Let $n_\ell = i_{q_{r-\ell}}$. Indeed, since $x_{n_\ell}^{\alpha_{r-e}}I_{\ell+1} \subset I_\ell \Rightarrow I_{\ell+1} \subset (I_\ell : x_{n_\ell}^\infty)$. For the converse, let $w \in (I_\ell : x_{n_\ell}^\infty)$ be any minimal generator. Then there exists an integer b such that $w \cdot x_{n_\ell}^b \in I_\ell$. We may assume that w is a minimal generator of I_ℓ. Then $w \cdot x_{n_\ell}^b = w' \cdot y$ for a $w' \in I_{\ell+1}$ and $y \in \prod_{j=0}^t(\mathbf{m}_{r-\ell}^{[d_j]})^{\alpha_{r-\ell,j}}$ with $x_{n_\ell}^b|y$. Thus $w'|w$, and therefore $w \in I_{\ell+1}$.

Let $S = k[x_1,\dots,x_n]$ and let M be a finitely graded generated S-module with the minimal graded free resolution $0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to M \to 0$. Let $Syz_t(M) = Ker(F_t \to F_{t-1})$. The module M is called (r,t)-regular if $Syz_t(M)$ is $(r+t)$-regular in the sense that all generators of F_j for $t \leq j \leq s$ have degrees $\leq j+r$. The t-regularity of M is by definition $(t-reg)(M) = min\{r|\ M\ is\ (r,t)-regular\}$.

Obvious $(t-reg)(M) \leq ((t-1)-reg)(M)$. If the equality is strict and $r = (t-reg)(M)$ then (r,t) is called a corner of M and $\beta_{t,r+t}(M)$ is an extremal Betti number of M, where $\beta_{ij} = dim_k Tor_i(k,M)_j$ denotes the ij-th graded Betti number of M. Later, we will use the following result:

Theorem 1.14. *[6, Theorem 3.2] If $I \subset S$ is a Borel type ideal, then S/I has at most $r+1$-corners among $(n_\ell, s(J_\ell^{sat}/J_\ell))$ and the corresponding extremal Betti numbers are*

$$\beta_{n_\ell, s(J_\ell^{sat}/J_\ell)+n_\ell}(S/I) = dim_k(J_\ell^{sat}/J_\ell)_{s(J_\ell^{sat}/J_\ell)}.$$

2 Socle of factors by principal d-fixed ideals.

In the following, we suppose $n \geq 2$.

Lemma 2.1. *Let $\mathbf{d} : 1 = d_0|d_1|\cdots|d_s$, $\alpha \in \mathbb{N}$ and $I = <x_n^\alpha>_{\mathbf{d}} = \prod_{t=0}^s(\mathbf{m}^{[d_t]})^{\alpha_t}$. Let $q_t = \sum_{j=t}^s \alpha_j d_j$. Let*

$$J = \sum_{t=0,\alpha_t>0}^{s}(x_1\cdots x_n)^{d_t-1}(\mathbf{m}^{[d_t]})^{\alpha_t-1}\prod_{j>t}(\mathbf{m}^{[d_j]})^{\alpha_j}.$$

Then:

1. *$Soc(S/I) = \frac{J+I}{I}$*
2. *Let e be a positive integer. Then $(\frac{J+I}{I})_e \neq 0 \Leftrightarrow e = q_t + (n-1)(d_t-1) - 1$, for some $0 \leq t \leq s$ with $\alpha_t > 0$.*

3. $max\{e|(\frac{J+I}{I})_e \neq 0\} = \alpha_s d_s + (n-1)(d_s - 1) - 1.$

Proof: 1. First we prove that $\frac{J+I}{I} \subset Soc(S/I)$. Since $Soc(S/I) = (O :_{S/I} \mathbf{m})$, it is enough to show that $\mathbf{m}J \subset I$.

We have $J = \sum_{t=0,\ \alpha_t>0}^{s} J_t$, where

$$J_t = (x_1 \cdots x_n)^{d_t - 1}(\mathbf{m}^{[d_t]})^{\alpha_t - 1} \prod_{j>t}(\mathbf{m}^{[d_j]})^{\alpha_j}.$$

It is enough to prove that $x_i J_t \subset I$ for any i and any t. Suppose $i = 1$:

$$x_1 J_t = x_1^{d_t}(x_2 \cdots x_n)^{d_t-1}(\mathbf{m}^{[d_t]})^{\alpha_t - 1}\prod_{j>t}(\mathbf{m}^{[d_j]})^{\alpha_j} \subset (x_2 \cdots x_n)^{d_t - 1}\prod_{j\geq t}(\mathbf{m}^{[d_j]})^{\alpha_j}.$$

On the other hand, $(x_2 \cdots x_n)^{d_t-1} \in \prod_{j<t}(\mathbf{m}^{[d_j]})^{\alpha_j}$, because $d_t - 1 \geq \sum_{j<t} \alpha_j d_j$. Thus $x_1 J_t \subset I$.

For the converse, we apply induction on α. If $\alpha = 1$ then $s = 0$ and $I = (x_1, \ldots, x_n) = \mathbf{m}$. $J = (x_1, \ldots, x_n)^{d_0 - 1} = S$, and obvious $Soc(S/I) = Soc(S/\mathbf{m}) = S/\mathbf{m}$. Let us suppose that $\alpha > 1$. We prove that if $w \in S \setminus I$ is a monomial such that $\mathbf{m}w \subset I$, then $w \in J$. Let $t_e = max\{t :\ x_e^{d_t - 1}|w\}$. Renumbering $x_1, \ldots, x_n$ which does not affect either I or J, we may suppose that $t_1 \geq t_2 \geq \cdots \geq t_n$. We have two cases: (i)$t_1 > t_n$ and (ii)$t_1 = t_n$. But first, let's make the following remark: $(*)$ If $u = x_1^{\beta_1} \cdots x_n^{\beta_n} \in \prod_{j \geq t} \mathbf{m}^{d_j}$ and $\beta_i < d_t$ for certain i then $u/x_i^{\beta_i} \in \prod_{j\geq t} \mathbf{m}^{[d_j]}$ (the proof is similarly to [3, Lemma 3.5]).

In the case (i), there exists an index e such that $t_e > t_{e+1} = \cdots = t_n$. Then we have $w = (x_n \cdots x_{e+1})^{d_{t_n} - 1} \cdot x_e^{d_{t_e} - 1} \cdot y$, for a monomial $y \in S$. We consider two cases (a) x_e does not divide y and (b) x_e divide y. (a) From $x_n w = x_n^{d_{t_n}} \cdot (x_{n-1} \cdots x_{e+1})^{d_{t_n} - 1} x_e^{d_{t_e} - 1} \cdot y \in I$ we see that $y \in \prod_{j \geq t_e}(\mathbf{m}^{[d_j]})^{\alpha_j}$, by $(*)$. Therefore $w \in I$, because $x_e^{d_{t_e} - 1} \in \prod_{j<t_e}(\mathbf{m}^{[d_j]})^{\alpha_j}$, which is an contradiction.

(b) In this case, $w = (x_n \cdots x_{e+1})^{d_{t_n} - 1} x_e^{d_{t_e}} y'$, where $y' = y/x_e$. We claim that there exist $\lambda \leq t_e$ such that $\alpha_\lambda \neq 0$. Indeed, if all $\alpha_\lambda = 0$ for $\lambda \leq t_e$, then $I = \prod_{j=t_e+1}^{s}(\mathbf{m}^{[d_j]})^{\alpha_j}$ and $x_n w \in I$ implies $y' \in I$ because of the maximality of t_n and $(*)$. It follows $w \in I$, which is false. Choose $\lambda \leq t_e$ maximal possible with $\alpha_\lambda \neq 0$. Set $w' = w/x_e^{d_\lambda}$. Note that $\mathbf{m}w \subset I$ implies

$$\mathbf{m}w \subset I' = (\mathbf{m}^{[d_{t_\lambda}]})^{\alpha_\lambda - 1}\prod_{j \neq \lambda}(\mathbf{m}^{[d_{t_j}]})^{\alpha_j}.$$

It is obvious that $x_q w' \in I'$ for $q \neq e$. Also, since $x_e^{d_{t_e}+1}$ does not divide $x_e w$ implies $x_e w' \in I'$. Choosing $\alpha' = \alpha - d_\lambda$, we get $\alpha'_j = \alpha_j$ for $j \neq \lambda$ and $\alpha'_\lambda = \alpha_\lambda - 1$ and therefore we can apply our induction hypothesis for I' (because $\alpha' < \alpha$) and for the ideal J' associated to I', which has the form:

$$J' = \sum_{q=0, \alpha'_q \neq 0} (x_1 \cdots x_n)^{d_q - 1}(\mathbf{m}^{[d_q]})^{\alpha'_q - 1}\prod_{j>q}(\mathbf{m}^{[d_j]})^{\alpha'_j},$$

and so $w = x_e^{d_\lambda} w' \in x_e^{d_\lambda} J' \subset J$.

It remains to consider the case (ii) in which we have in fact $t_1 = t_2 = \cdots = t_n$. If $y = w/(x_1 \cdots x_n)^{d_{t_n}-1} \in \mathbf{m}$, then there exists e such that $x_e|y$, and we apply our induction hypothesis as in the case (b) above. Thus we may suppose $y = 1$, i.e. $w = (x_1 \cdots x_n)^{d_{t_n}-1}$. Since $\mathbf{m}w \subset I$, we see that $\alpha_j = 0$ for $j > t_n$ and $\alpha_{t_n} = 1$ (otherwise $w \in I$, which is absurd). Thus $w \in J$.

2. Let $v = x_1^{q_t-1}(x_2 \cdots x_n)^{d_t-1}$. Then $deg(v) = q_t + (n-1)(d_t-1) - 1$. But $v \in J$ and $v \notin I$, therefore $v \neq 0$ in $Soc(S/I) = \frac{J+I}{I}$.

3. Let $e_t = q_t + (n-1)(d_t-1) - 1$ for $0 \le t \le s$. Let $t < s$. Then

$$e_{t+1} - e_t = q_{t+1} - q_t + (n-1)(d_{t+1} - d_t) =$$

$$= -\alpha_t d_t + (n-1)(d_{t+1} - d_t) \ge d_{t+1} - (\alpha_t + 1)d_t \ge 0, \ so$$

$$max\{e | ((J+I)/I)_e \neq 0\} = e_s = \alpha_s d_s + (n-1)(d_s - 1) - 1.$$

□

Remark 2.2. *From the proof of the above lemma, we may easily conclude that for* $n \ge 3$, $e_t = e_{t'}$ *if and only if* $t = t'$, *and if* $n = 2$, *then* $e_t = e_{t'}$ $(t < t')$ *if and only if*
$\alpha_{t'-1} = d_{t'}/d_{t'-1}, \ldots, \alpha_t = d_{t+1}/d_t$.

Corollary 2.3. *With the notations of previous lemma and remark, let* $0 \le t \le s$ *be an integer such that* $\alpha_t \neq 0$. *Let* $h_t = dim_K((I + J_t)/I)$. *Then:*

1. $G(J_t) \cap (I + J_{t'}) = 0$ *for* $0 \le t' \le s$, $t' \neq t$.
2. $h_t = \binom{n+\alpha_t-2}{n-1} \prod_{j>t} \binom{n+\alpha_j-1}{n-1}$.
3. $$dim_K(Soc(S/I)_e) = \begin{cases} h_q, & if\ n \ge 3\ and\ e = e_q\ for\ a\ q \le s \\ & with\ \alpha_q \neq 0. \\ \sum_q h_q, & if\ n = 2\ and\ q \in \{\epsilon | e = e_\epsilon\ for\ \epsilon \le s\ . \\ & with\ \alpha_\epsilon \neq 0\}. \\ 0, & otherwise. \end{cases}$$

Proof: 1. First suppose $t' < t$. A minimal generator $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ of J_t has the form

$$(x_1 \cdots x_n)^{d_t-1} \prod_{j \ge t} (x_1^{\lambda_{1j} d_j} \cdots x_n^{\lambda_{nj} d_j}),\ where\ \sum_{\nu=1}^{n} \lambda_{\nu j} = \begin{cases} \alpha_j, & if\ j > t, \\ \alpha_t - 1, & if\ j = t. \end{cases}.$$

Thus, $\beta_i = d_t - 1 + \sum_{j=t}^{s} \lambda_{ij} d_j$. On the other hand, $d_t - 1 = \sum_{j=0}^{t-1} (d_{j+1}/d_j - 1) d_j$, so β_i has the writing $\sum_{j=0}^{s} \beta_{ij} d_j$, where $\beta_{ij} = d_{j+1}/d_j - 1$ for $j < t$ and $\beta_{ij} = \lambda_{ij}$ for $j \ge t$.

Assume that $x^\beta \in I + J_{t'}$ for a certain $t' < t$. Then there exists $\gamma \in \mathbb{N}^n$ such that $x^\gamma \in G(I)$ (or $x^\gamma \in G(J_{t'})$) and $x^\gamma | x^\beta$, that is $\gamma_i \leq \beta_i$ for all $1 \leq i \leq n$. Let $\gamma_i = \sum_{j=0}^s \gamma_{ij} d_j$, the $\mathbf{d}-$ decomposition of γ_i. We notice that $(\beta_{is}, \dots, \beta_{i0}) \geq (\gamma_{is}, \dots, \gamma_{i0})$ in the lexicographic order.

Note that all minimal generators x^γ of I have the same degree $\alpha < e_t$ and $\sum_{i=1}^n \gamma_{iq} = \alpha_q$ for each $0 \leq q \leq s$. Also all minimal generators x^γ of $J_{t'}$ have the same degree $e_{t'} < e_t$ and $\sum_{i=1}^n \gamma_{iq} = \alpha_q$ for each $t \leq q \leq s$. It follows $deg(x^\beta) > deg(x^\gamma)$ and so $\beta_i > \gamma_i$ for some i. Choose a maximal $q < s$ such that $\beta_{iq} > \gamma_{iq}$ for some i. Thus $\beta_{ij} = \gamma_{ij}$ for $j > q$. It follows $\beta_{iq} \geq \gamma_{iq}$ since $(\beta_{is}, \dots, \beta_{i0}) \geq_{lex} (\gamma_{is}, \dots, \gamma_{i0})$. If $q \leq t$ then we have

$$\alpha_q = \sum_{i=1}^n \gamma_{iq} < \sum_{i=1}^n \beta_{iq} = \sum_{i=1}^n \lambda_{iq} \leq \alpha_q,$$

which is not possible. It follows $q < t$ and so $\beta_{it} = \gamma_{it}$ for each i. But this is not possible because we get $\alpha_t = \sum_{i=1}^n \gamma_{it} = \sum_{i=1}^n \lambda_{it} = \alpha_t - 1$. Hence $x^\beta \notin I + J_{t'}$.

Suppose now $t' > t$. If $e_{t'} > e_t$, then $G(J_t) \cap G(J_{t'}) = \emptyset$ by degree reason. Assume $e_t = e_{t'}$. If follows $n = 2$ by the previous remark. If $x_1^{\beta_1} x_2^{\beta_2} \in G(J_t) \cap J_{t'}$ we necessarily get $x_1^{\beta_1} x_2^{\beta_2} \in G(J_t) \cap G(J_{t'})$ again by degree reason. But this is not possible since it implies that $\alpha_{t'} - 1 = \beta_{1t'} + \beta_{2t'} = \alpha_{t'}$.

2. and 3. follows from 1. □

Theorem 2.4. *Let* $u = \prod_{q=1}^r x_{i_1}^{\alpha_q}$, *where* $2 \leq i_1 < i_2 < \dots < i_r \leq n$. *Let*

$$I = < u >_{\mathbf{d}} = \prod_{q=1}^r \prod_{j=0}^s (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}},$$

where $\alpha_q = \sum_{j=0}^s \alpha_{qj} d_j$. *Suppose* $i_r = n$. *Let* $1 \leq a \leq r$ *be an integer and*

$$P_a(I) := \{(\lambda, t) \in \mathbb{N}^a \times \mathbb{N}^a \,|\, 1 \leq \lambda_1 < \dots < \lambda_a = r, t_a > \dots > t_1, \alpha_{\lambda_\nu t_\nu} \neq 0,$$

$$for\ 1 \leq \nu \leq a\}.$$

Let $J = \sum_{a=1}^r \sum_{(\lambda,t) \in P_a(I)} J_{(\lambda,t)}$, *where* $J_{(\lambda,t)}$ *is the ideal*

$$\prod_{e=1}^a (x_{i_{\lambda_e}} \cdots x_{i_{\lambda_{e-1}}+1})^{d_{t_e}-1} \prod_{\nu=1}^a \mathbf{m}_{\lambda_\nu}^{[d_{t_\nu}+1]} \prod_{j>t_\nu} (\mathbf{m}_{\lambda_\nu}^{[d_j]})^{\alpha_{\lambda_\nu j}} (\mathbf{m}_{\lambda_\nu}^{[d_{t_\nu}]})^{\alpha_{\lambda_\nu, t_\nu} - 1} \cdot$$

$$\cdot \prod_{q=\lambda_{\nu-1}+1}^{\lambda_\nu - 1} \prod_{j \geq t_\nu} (m_q^{[d_j]})^{\alpha_{qj}},$$

where we denote $\mathbf{m}^{[d_{t_a+1}]} = S$. *Then* $Soc(S/I) = (J + I)/I$.

Proof: The proof will be given by induction on r, the case $r=1$ being done in Lemma 2.1. Suppose that $r>1$. For $1\leq q\leq r$, let: $I_q=\prod_{e=1}^{q}\prod_{j=0}^{s}(\mathbf{m}_e^{[d_j]})^{\alpha_{ej}}$ and $S_q=k[x_1,x_2,\ldots,x_{i_q}]$ For t with $\alpha_{rt}\neq 0$, denote:

$$I^{(t)}=\mathbf{m}_{r-1}^{[d_t]}\prod_{j<t}(\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{r-1,j}}I_{r-2}.$$

Let $J^{(t)}$ be an ideal in S_{r-1} such that $Soc(S_{r-1}/I^{(t)})=(J^{(t)}+I^{(t)})/I^{(t)}$. The induction step is given in the following lemma:

Lemma 2.5. *Suppose $i_r=n$ and let*

$$J=\sum_{t=0,\alpha_{rt}\neq 0}(x_n\cdots x_{i_{r-1}+1})^{d_t-1}\prod_{j>t}(\mathbf{m}_r^{[d_j]})^{\alpha_{rj}}\prod_{j\geq t}(\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{r-1,j}}(\mathbf{m}_r^{[d_t]})^{\alpha_{rt}-1}J^{(t)}.$$

Then $Soc(S/I)=(J+I)/I$.

Proof: Let $w\in S\setminus I$ be a monomial such that $\mathbf{m}_r w\subset I$. As in the proof of lemma 2.1, we choose for each $1\leq\rho\leq n$, $e_\rho=max\{e:\ x_\rho^{d_e-1}|w\}$. Renumbering variables $\{x_n,\ldots,x_{i_{r-1}+1}\}$ (it does not affect I, J and $I^{(t)}$), we may suppose $e_n\leq e_{n-1}\leq\cdots\leq e_{i_{r-1}+1}$. Set $t=e_n$. We claim that $\alpha_{rt}\neq 0$. Indeed, if $\alpha_{rt}=0$ then from $x_nw\in I$ we get $x_nw/x_n^{d_t-1}\in\widetilde{I}=\prod_{j>t}(\mathbf{m}_r^{[d_j]})^{\alpha_{rj}}I_{r-1}$ because $x_n^{d_t-1}\in\prod_{j<t}(\mathbf{m}_r^{[d_j]})^{\alpha_{rj}}$. Since $t=e_n$ is maximal chosen, we get $w/x_n^{d_t-1}\in\widetilde{I}$ and so $w\in I$ a contradiction.

Reduction to the case that $x_n^{d_t}$ does not divide w. Suppose that $w=x_n^{d_t}\widetilde{w}$ and set

$$\widetilde{I}=(\mathbf{m}_r^{[d_t]})^{\alpha_{rt}-1}\prod_{\epsilon\leq 0,\epsilon\neq t}(\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{r\epsilon}}I_{r-1}.$$

We see that $\mathbf{m}w\in I\Leftrightarrow\mathbf{m}\widetilde{w}\in\widetilde{I}$. Replacing w and I with $\widetilde{w}$ and $\widetilde{I}$, we reduce our problem to a new $\widetilde{t}<t$. The above argument implies that $\widetilde{\alpha}_{r\widetilde{t}}\neq 0$, where $\widetilde{\alpha}$ is the 'new' α of $\widetilde{I}$.

Reduction to the case when $\alpha_{rj}=\alpha_{r-1,j}=0$ for $j>t$, $\alpha_{rt}=1$ and $\alpha_{r-1,t}=0$. From $x_nw\in I$, we see that there exists $\rho<n$ such that $x_\rho^{d_j}|w$ for $j>t$ if $\alpha_{rj}\neq 0$, or $j=t$ if $\alpha_{rt}>1$. Choose such maximal possible ρ. Set $w'=w/x_\rho^{d_j}$,

$$I'=(\mathbf{m}_r^{[d_j]})^{\alpha_{rj}-1}\prod_{\epsilon\geq 0,\epsilon\neq j}(\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{r\epsilon}}I_{r-1}.$$

We see that $\mathbf{m}w\subset I\Leftrightarrow\mathbf{m}w'\subset I'$, because from $x_nw\in I$, we get $x_nw'\in I'$ from the maximality of ρ.

Let $\alpha'_{rj} = \alpha_{rj} - 1$ and $\alpha'_{q\epsilon} = \alpha_{q\epsilon}$ for $(q,\epsilon) \neq (r,j)$. α' is the 'new' α for I'. If we show that

$$w' \in J' = \sum_{e\geq 0, \alpha'_{re}\neq 0} (x_n \cdots x_{i_{r-1}+1})^{d_e-1} \cdot$$

$$\cdot \prod_{\epsilon > e} (\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{r\epsilon}} \prod_{j\geq e} (\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{r-1,j}} (\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{re}-1} J^{(t)},$$

then $w = x_\rho^{d_j} w' \in \mathbf{m}_r^{[d_j]} J' \subset J$. Using this procedure, by recurrence we arrive to the case $\alpha_{rj} = 0$ for $j > t$ and $\alpha_{rt} = 1$. Again from $x_n w \in I$, we note that there exists $\rho < i_{r-1}$ such that $x_\rho^{d_j} | w$ for $j \geq t$ with $\alpha_{r-1,j} \neq 0$. Choose such maximal possible ρ and note that $\mathbf{m}w \subset I$ if and only if $\mathbf{m}w'' \in I''$ for $w'' = w/x_\rho^{d_j}$, where

$$I'' = (\mathbf{m}_{r-1}^{d_j})^{\alpha_{r-1,j-1}} \prod_{\epsilon\geq 0,\ \epsilon\neq j} (\mathbf{m}_{r-1}^{[d_\epsilon]})^{\alpha_{r-1,\epsilon}} \prod_{\epsilon \geq 0} (\mathbf{m}_r^{[d_\epsilon]})^{\alpha_{r\epsilon}} I_{r-2}.$$

As above, we reduce our problem to I'' and the α'', which is the new α of I'', is given by $\alpha''_{r-1,j} = \alpha_{r-1,j-1}$, $\alpha''_{q\epsilon} = \alpha_{q\epsilon}$ for $(q,\epsilon) \neq (r-1,j)$. Using this procedure, by recurrence we end our reduction.

Case $\alpha_{rj} = \alpha_{r-1j} = 0$ for $j > t$, $\alpha_{rt} = 1$, $\alpha_{r-1t} = 0$ and $x_n^{d_t}$ does not divide w. Let express $w = (x_n \cdots x_{i_{r-1}+1})^{d_t-1} y$. We will show that y does not depend on $\{x_n, \ldots, x_{i_{r-1}+1}\}$. Indeed, if $n = i_{r-1}+1$ then there is nothing to show since $x_n^{d_t}$ does not divide w. Suppose that $n > i_{r-1}+1$, then from $x_n w \in I$ we get $y \in I_{r-1}$ because $x_{n-1}^{d_t-1} \in \prod_{j<t} (\mathbf{m}_r^{d_j})^{\alpha_{rj}}$ and the variables $x_n, \ldots, x_{i_{r-1}+1}$ are regular on $S/I_{r-1}S$. If $y = x_\eta y'$ for $\eta > i_{r-1}$, then as above $y' \in I_{r-1}$. Thus $w \in x_\eta^{d_t} x_\rho^{d_t-1} y' \subset I$ for any $\rho \neq \eta, i_{r-1} < \rho \leq n$, a contradiction.

Note that $\mathbf{m}_r w \in I \Rightarrow \mathbf{m}_{r-1} y \in I^{(t)}$ and so $w \in (x_n \cdots x_{i_{r-1}+1})^{d_t-1} J^{(t)}$. Since $\alpha_{rj} = \alpha_{r-1j} = 0$ for $j > t$ and $\alpha_{rt} = 1$ and $\alpha_{r-1,t} = 0$, we get $w \in J$. Conversely, if $y \in J^{(t)}$, then it is clear that $w \in J$. □

We see by the above lemma that:

$$(*)\ J = \sum_{e\geq 0, \alpha_{re}\neq 0} (x_n \cdots x_{i_{r-1}+1})^{d_e-1} \prod_{j>e} (\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{rj}}$$

$$\prod_{j\geq e} (\mathbf{m}_{r-1}^{[d_j]})^{\alpha_{r-1,j}} (\mathbf{m}_r^{[d_e]})^{\alpha_{re}-1} J^{(e)}.$$

Since $\lambda_a = r-1$, by the induction hypothesis applied to $I^{(e)}$ we get:

$$J^{(e)} = \sum_{a=1}^{r-1} [\sum_{(\lambda,t)\in P_a(I^{(e)}), t_a = e} \prod_{s=1}^{a} (x_{i_{\lambda_s}} \cdots x_{i_{\lambda_{s-1}+1}})^{d_{t_s}-1} \cdot J'_{(\lambda,t)} +$$

$$+ \sum_{(\lambda,t)\in P_a(I^{(e)}), t_a < e} \prod_{s=1}^{a} (x_{i_{\lambda_s}} \cdots x_{i_{\lambda_{s-1}+1}})^{d_{t_s}-1} \cdot J''_{(\lambda,t)}],\ \textit{where}$$

$$J'_{(\lambda,t)} = \prod_{q=\lambda_{a-1}+1}^{\lambda_a-1} \prod_{j\geq e} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}} \widetilde{J}_{(\lambda,t)} \textit{ and}$$

$$J''_{(\lambda,t)} = \mathbf{m}_{r-1}^{[d_e]} \prod_{j>t_a}^{e-1} (\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{\lambda_a,j}} (\mathbf{m}_{\lambda_a}^{[d_{t_a}]})^{\alpha_{\lambda_a,t_a-1}} \cdot \prod_{q=\lambda_{a-1}+1}^{\lambda_a-1} \prod_{j\geq t_a} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}} \widetilde{J}_{(\lambda,t)}, \textit{ and}$$

$$\widetilde{J}_{(\lambda,t)} = \prod_{\nu=1}^{a-1} \mathbf{m}_{\lambda_\nu}^{[d_{t_\nu+1}]} \prod_{j>t_\nu} (\mathbf{m}_{\lambda_\nu}^{[d_j]})^{\alpha_{\lambda_\nu,j}} (\mathbf{m}_{\lambda_\nu}^{[d_{t_\nu}]})^{\alpha_{\lambda_\nu,t_\nu-1}} \cdot \prod_{q=\lambda_{\nu-1}+1}^{\lambda_\nu-1} \prod_{j\geq t_\nu} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}.$$

If $t_a = e$, set $\lambda'_\nu = \lambda_\nu$ for $\nu < a$, $\lambda'_a = r$ and see that $(\lambda', t) \in P_a(I)$. If $t_a < e$, then put $\lambda''_\nu = \lambda_\nu$ for $\nu \leq a$, $\lambda''_{a+1} = r$, $t''_\nu = t_\nu$ for $\nu \leq a$ and $t''_{a+1} = e$ and then $(\lambda'', t) \in P_{a+1}(I)$. Substituting $J^{(e)}$ in $(*)$, we get the following expression for J:

$$\sum_{a=1}^{r-1} \sum_{(\lambda',t)\in P_a(I)} \prod_{\nu=1}^{a} (x_{i_{\lambda'_\nu}} \cdots x_{i_{\lambda'_{\nu-1}+1}})^{d_{t_\nu}-1} \cdot [\prod_{j>e} (\mathbf{m}_{\lambda'_a}^{[d_j]})^{\alpha_{\lambda'_a j}} (\mathbf{m}_{\lambda'_a}^{[d_e]})^{\alpha_{\lambda'_a e}-1} \cdot$$

$$\cdot \prod_{q=\lambda'_{a-1}+1}^{\lambda'_a-1} \prod_{j\geq e} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}] \cdot \widetilde{J}_{(\lambda,t)} + \sum_{a=1}^{r-1} \sum_{(\lambda'',t'')\in P_{a+1}(I)} \prod_{\nu=1}^{a+1} (x_{i_{\lambda''_\nu}} \cdots x_{i_{\lambda''_{\nu-1}+1}})^{d_{t_\nu}-1} \cdot$$

$$\cdot [\prod_{j>e} (\mathbf{m}_{\lambda''_{a+1}}^{[d_j]})^{\alpha_{\lambda''_{a+1},j}} (\mathbf{m}_{\lambda''_{a+1}}^{[d_{t''_a+1}]})^{\alpha_{\lambda''_{a+1} t''_{a+1}}-1}]$$

$$[\mathbf{m}_{\lambda''_a}^{[d_{t''_{a+1}}]} \prod_{j\geq t''_a} (\mathbf{m}_{\lambda''_a}^{[d_j]})^{\alpha_{\lambda''_{a+1},j}} (\mathbf{m}_{\lambda''_a}^{[d_{t''_a}]})^{\alpha_{\lambda''_a t''_a}-1} \prod_{q=\lambda''_{a-1}+1}^{\lambda''_a-1} \prod_{j\geq t''_a} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}] \cdot \widetilde{J}_{(\lambda,t)}.$$

Since all the pairs of $P_b(I)$ have the form (λ', t) or (λ'', t'') for a pair $(\lambda, t) \in P_b(I)$ or $(\lambda, t) \in P_{b-1}(I)$ respectively, it is not hard to see that the expression above is the formula of J as stated. □

Let $s_q = max\{j|\alpha_{qj} \neq 0\}$, $d_{qt} = \sum_{e=1}^{q} \sum_{j\geq t}^{s_q} \alpha_{ej} d_j$, $D_q = d_{q,s_q} + (i_q - 1)(d_{s_q} - 1)$ for $1 \leq q \leq r$.

Corollary 2.6. *With the notation and hypothesis of above theorem, for $(\lambda, t) \in P_a(I)$ let:*

$$d_{(\lambda,t)} = \sum_{\nu=1}^{a} \sum_{q=\lambda_{\nu-1}+1}^{\lambda_\nu} \sum_{j\geq t_\nu} \alpha_{qj} d_j. \ Then:$$

1. $Soc(I_{r-1}S/I) = Soc(S/I)$.
2. $((J+I)/I)_e \neq 0$, *if and only if* $e = d_{(\lambda,t)} + \sum_{\nu=1}^{a} (i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu} - 1) - d_{t_1}$, *for some* $1 \leq a \leq r$ *and* $(\lambda, t) \in P_a(I)$.

3. $c = max\{e|((J+I)/I)_e \neq 0\} = d_{r,s_r} + (n-1)(d_{s_r}-1) - 1 = D_r - 1.$

Proof: 1.Note that $J_{(\lambda,t)}$ is contained in

$$\prod_{q=1, q\notin\{\lambda_1,\ldots,\lambda_q\}}^{r} (\mathbf{m}_q^{[d_j]})^{\alpha_{qj}} \prod_{\nu=1}^{a}[\prod_{j\neq t_\nu}(\mathbf{m}_{\lambda_\nu}^{d_j})^{\alpha_{\lambda_\nu,j}}(\mathbf{m}_{\lambda_\nu}^{t_\nu})^{\alpha_{\lambda_\nu,t_\nu}-1}]\prod_{\epsilon=1}^{a-1}\mathbf{m}_{\lambda_{\epsilon+1}}^{d_{t_\epsilon}+1}.$$

Since $\mathbf{m}_{\lambda_\epsilon}^{d_{t_\epsilon}+1} \subset \mathbf{m}_{\lambda_\epsilon}^{d_{t_\epsilon}}$ for $t_{\epsilon+1} > t_\epsilon$ and $\lambda_a = r$ if follows that

$$J \subset \prod_{j\neq t_a}(\mathbf{m}_r^{[d_j]})^{\alpha_{rj}}(\mathbf{m}_r^{[d_{t_a}]})^{\alpha_{rt_a}-1} I_{r-1},$$

as desired.

2.If $((J+I)/I)_e \neq 0$ then there exists a monomial $u \in J\setminus I$ of degree e. But $u \in J$, implies that there exists $a \in \{1,\ldots,r\}$ and $(\lambda,t) \in P_a(I)$ such that $u \in J_{(\lambda,t)}$. Thus the degree of u is $e = d_{(\lambda,t)} + \sum_{\nu=1}^{a}(i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu}-1) - d_{t_1}$, as required.

Conversely, let $e = d_{(\lambda,t)} + \sum_{\nu=1}^{a}(i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu}-1) - d_{t_1}$ for some $a \in \{1,\ldots,r\}$ and $(\lambda,t) \in P_a(I)$. We show that the monomial

$$w = \prod_{\nu=1}^{a}(x_{i_{\lambda_\nu}}\cdots x_{i_{\lambda_{\nu-1}+1}})^{d_{t_\nu}-1}\cdot x_1^{d_{(\lambda,t)}-d_{t_1}} \in J\setminus I.$$

Obvious $w \in J$. Let us assume that $w \notin I$. Then

$$w/x_{i_{\lambda_a}}^{d_{t_a}-1} \in \prod_{j\geq t_a}(\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{\lambda_a j}} I_{\lambda_a-1}$$

because $x_{i_{\lambda_a}}^{d_{t_a}-1} \in \prod_{j<t_a}(\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{\lambda_a j}}$ and $x_{i_{\lambda_a}} \notin \mathbf{m}_j$ for $j < \lambda_a$. Inductively we get that:

$$w/(x_{i_{\lambda_a}}\cdots x_{i_{\lambda_{a-1}+1}})^{d_{t_a}-1} \in \prod_{q=\lambda_{a-1}+1}^{\lambda_a}\prod_{j\geq t_a}(\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{qj}} I_{\lambda_a-1}.$$

Following the same reduction and using that $t_a > \cdots > t_1$ we obtain that:

$$x_1^{d_{(\lambda,t)}-d_{t_1}} \in \prod_{\nu=1}^{a}\prod_{q=\lambda_{\nu-1}+1}^{\lambda_\nu}\prod_{j\geq t_\nu}(\mathbf{m}_{\lambda_a}^{[d_j]})^{\alpha_{qj}}.$$

So $d_{(\lambda,t)} - d_{t_1} \geq d_{(\lambda,t)}$, a contradiction.

3.Note that $c = d_{(\lambda',t')}$ for $(\lambda',t') \in P_1(I)$ with $\lambda' = \lambda_1 = r$ and $t' = t_1 = s_r$. We have to show that:

$$c = d_{r,s_r} + (n-1)(d_{s_r}-1) - 1 \leq d_{(\lambda,t)} + \sum_{\nu=1}^{a}(i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu}-1) - d_{t_1},$$

for any $1 \leq a \leq r$ and $(\lambda, t) \in P_a(I)$. Since $d_{s_r} - 1 \leq (d_{t_\nu} - 1) + \sum_{j \leq t_\nu}^{s_r - 1} \alpha_{qj} d_j$ for all q with $i_{\nu-1} < q \leq i_\nu$, we see that:

$$D_r - 1 \geq d_{(\lambda,t)} + \sum_{\nu=2}^{a} (i_{\lambda_\nu} - i_{\lambda_{\nu-1}})(d_{t_\nu} - 1) + (i_{\lambda_1} - 1)(d_{t_1} - 1) - 1.$$

On the other hand, $(i_{\lambda_1} - i_{\lambda_0})(d_{t_1} - 1) = (i_{\lambda_1} - 1)(d_{t_1} - 1) + d_{t_1} - 1$, and replacing that in the above relation we obtained what is required. □

Example 2.7. *Let* $\mathbf{d} : 1|2|4|12$.

1. *Let* $u = x_3^{21}$. *We have* $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = 2$ *and* $\alpha_3 = 1$ *so:*

$$I = < u >_{\mathbf{d}} = (x_1, x_2, x_3)(x_1^4, x_2^4, x_3^4)^2 (x_1^{12}, x_2^{12}, x_3^{12}).$$

Let $J = \sum_{t=0, \alpha_t > 0} J_t$, *where*

$$J_t = (x_1 x_2 x_3)^{d_t - 1} (x_1^{d_t}, x_2^{d_t}, x_3^{d_t})^{\alpha_t - 1} \prod_{j > t} (x_1^{d_j}, x_2^{d_j}, x_3^{d_j})^{\alpha_j}.$$

$$J_0 = (x_1 x_2 x_3)^{1-1} \cdot (x_1, x_2, x_3)^{1-1} \cdot \prod_{j>t} (x_1^{d_j}, x_2^{d_j}, x_3^{d_j})^{\alpha_j} =$$
$$= (x_1^4, x_2^4, x_3^4)^2 (x_1^{12}, x_2^{12}, x_3^{12}).$$

$$J_2 = (x_1 x_2 x_3)^{4-1} (x_1^4, x_2^4, x_3^4)^{2-1} (x_1^{12}, x_2^{12}, x_3^{12}) =$$
$$= (x_1 x_2 x_3)^3 (x_1^4, x_2^4, x_3^4)(x_1^4, x_2^4, x_3^{12})$$

and

$J_3 = (x_1 x_2 x_3)^{12-1} = (x_1 x_2 x_3)^{11}$. *From* 2.1 , *Soc*$(S/I) = (J + I)/I$.

2. *Let* $u = x_2^9 x_3^{16}$. *We have* $r = 2$, $i_1 = 2$ *and* $i_2 = 3$. *Also* $\alpha_{10} = 1$, $\alpha_{12} = 2$, $\alpha_{22} = 1$, $\alpha_{23} = 1$ *and the other components of* α *are zero. Then*

$$I = < u >_{\mathbf{d}} = < x_2^9 >_{\mathbf{d}} < x_3^{16} >_{\mathbf{d}} = (x_1, x_2)(x_1^4, x_2^4)^2 (x_1^4, x_2^4, x_3^4)(x_1^{12}, x_2^{12}, x_3^{12}).$$

We have two possible partitions: (a) (2) *and (b)* $(1 < 2)$.

(a)$\lambda = \lambda_1 = 2$, $t = t_1$ *such that* $\alpha_{2t} \neq 0$. *We have two possible* t*:* $t = 2$ *or* $t = 3$.

(i)For $t = 2$ *we obtain (according to the Theorem* 2.4*) the following part of the socle:*

$$J_{(2,2)} = (x_1 x_2 x_3)^3 (x_1^{12}, x_2^{12}, x_3^{12})(x_1^4, x_2^4)^4$$

(ii)For $t = 3$ we obtain:

$$J_{(2,3)} = (x_1x_2x_3)^{11}$$

(b)$1 = \lambda_1 < \lambda_2 = 2$*, $t = (t_1, t_2)$ such that $\alpha_{\lambda_e,t_e} \neq 0$ for $1 \leq e \leq 2$ and $t_1 < t_2$. According to our expressions for α_i we have three possible cases: $t_1 = 0, t_2 = 2$ or $t_1 = 0, t_2 = 3$ or $t_1 = 2, t_2 = 3$.*

(i)For $t_1 = 0$ and $t_2 = 2$ we obtain:

$$J_{(1,2),(0,2)} = x_3^3(x_1^4, x_2^4)(x_1^4, x_2^4)^2(x_1^{12}, x_2^{12}, x_3^{12}).$$

(ii)For $t_1 = 0$ and $t_2 = 3$ we obtain:

$$J_{(1,2),(0,3)} = x_3^{11}(x_1^{12}, x_2^{12})(x_1^4, x_2^4)^2$$

(iii)For $t_1 = 2$ and $t_2 = 3$ we obtain:

$$J_{(1,2),(2,3)} = x_1^3x_2^3x_3^{11}(x_1^{12}, x_2^{12})(x_1^4, x_2^4)$$

From 2.4 it follows that if $J = J_{(2,2)} + J_{(2,3)} + J_{(1,2),(0,2)} + J_{(1,2),(0,3)} + J_{(1,2),(2,3)}$ then $Soc(S/I) = (I + J)/J$.

3 A generalization of Pardue's formula.

In this section, we give a generalization of a theorem proved by Aramova-Herzog [1] and Herzog-Popescu [4] which is known as "Pardue's formula".

Let $1 \leq i_1 < i_2 < \cdots < i_r = n$ and let $\alpha_1, \ldots, \alpha_r$ some positive integers. Let $u = \prod_{i=1}^{r} x_{i_q}^{\alpha_q} \in S = K[x_1, \ldots, x_n]$. Our goal is to give a formula for the regularity of the ideal

$$I =< u >_{\mathbf{d}} = \prod_{r=1}^{q}\prod_{j=0}^{s}(\mathbf{m}_q^{[d_j]})^{\alpha_{qj}},$$

where $\alpha_q = \sum_{j=0}^{s} \alpha_{qj}d_j$. If $i_1 = 1$, it follows that $I = x_1^{\alpha_1}I'$, where $I' = \prod_{r=2}^{q}\prod_{j=0}^{s}(\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}$, and therefore $reg(I) = \alpha_1 + reg(I')$. Thus, we may assume $i_1 \geq 2$.

If N is a graded S-module of finite length, we denote $s(N) = max\{i | N_i \neq 0\}$. Let $s_q = max\{j|\alpha_{qj} \neq 0\}$ and $d_{qt} = \sum_{e=1}^{q}\sum_{j\geq t}^{s_e} \alpha_{ej}d_j$. Let $D_q = d_{qs_q} + (i_q - 1)(d_{s_q} - 1)$, for $1 \leq q \leq r$. With this notations we have:

Theorem 3.1. *$reg(I) = max_{1\leq q\leq r}D_q$. In particular, if $I =< x_n^{\alpha} >_{\mathbf{d}}$ and $\alpha = \sum_{t=0}^{s} \alpha_t d_t$ with $\alpha_s \neq 0$ then $reg(I) = \alpha_s d_s + (n - 1)(d_s - 1)$.*

Proof: Let $I_\ell = \prod_{q=1}^{r-\ell}\prod_{j=0}^{s}(\mathbf{m}_q^{[d_j]})^{\alpha_{qj}}$, for $0 \leq \ell \leq r$. Then $I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ is the sequential chain of ideals of I, i.e. $I_{\ell+1} = (I_\ell : x_{n_\ell}^\infty)$, where $n_\ell = i_{r-\ell}$. Moreover, from the Remark 1.13, we see that this chain is in fact the chain from the definition of a sequentially Cohen-Macauly module for S/I. Let $S_\ell = k[x_1, \ldots, x_{n_\ell}]$ and $m_\ell = (x_1, \ldots, x_{n_\ell})$.

The corollary 2.6 implies that $c_e = D_e - 1$ is the maximal degree for a nonzero element of $Soc(S_\ell/J_\ell)$. [3, Corollary 2.7] implies

$$reg(I) = max\{s(I_\ell S_\ell^{sat}/I_\ell S_\ell) \mid \ell = 0, \ldots, r-1)\} + 1.$$

Also, from the corollary 2.6, we get

$$Soc(S_\ell/I_\ell S_\ell) = Soc(I_{\ell+1}S_\ell/I_\ell S_\ell) = (I_{\ell+1} : m_\ell)S_\ell/I_\ell S_\ell = I_\ell S_\ell^{sat}/I_\ell S_\ell,$$

which complete the proof. □

Corollary 3.2. *$reg(I) \leq n\cdot deg(u) = n\cdot deg(I)$, where $deg(I) = max\{deg(w) | w \in G(I)\}$.*

Corollary 3.3. *S/I has at most r-corners among $(i_q, D_q - 1)$ for $1 \leq q \leq r$. If $i_1 = 1$ we replace $(i_1, D_1 - 1)$ with $(1, \alpha_1)$. The corresponding extremal Betti numbers are β_{i_q, D_q+i_q-1}.*

Proof: By Theorem 1.14 combined with the proof of Theorem 3.1, S/I has at most r-corners among $(n_\ell, s(I_{\ell+1}S_\ell/I_\ell S_\ell))$ and is enough to apply Corollary 2.6. □

Example 3.4. *Let $\mathbf{d} : 1|2|4|12$.*

1. *Let $u = x_3^{21} \in k[x_1, x_2, x_3]$. We have $21 = 1\cdot 1 + 0 \cdot 2 + 2 \cdot 4 + 1 \cdot 12$. From 3.1, we get:*

$$reg(< u >_{\mathbf{d}}) = 1 \cdot 12 + (3-1)\cdot(12-1) = 34.$$

2. *Let $u = x_1^2 x_2^{16} x_3^9$. Then $reg(< u >_{\mathbf{d}}) = 2 + reg(< u' >_{\mathbf{d}})$, where $u' = u/x_1^2$. We compute $reg(< u' >_{\mathbf{d}})$. With the notations above, we have $i_1 = 2$, $i_2 = 3$, $r = 2$, $\alpha_1 = 16$ and $\alpha_2 = 9$. We have $\alpha_1 = 1\cdot 4 + 1\cdot 12$ and $\alpha_2 = 1 \cdot 1 + 2 \cdot 4$, thus $s_1 = 3$ and $s_2 = 2$. $D_1 = d_{13} + (2-1)(d_3 - 1) = 12 + 11 = 23$ and $D_2 = d_{22} + (3-1)(d_2 - 1) = 24 + 6 = 30$. In conclusion, $reg(< u >_{\mathbf{d}}) = 2 + max\{23, 30\} = 32$.*

In the following, we show that if I is a principal $\mathbf{d}$-fixed ideal generated by the power of a variable, then $I_{\geq e}$ is stable for any $e \geq reg(I)$.

Lemma 3.5. *Let $I =< x_n^\alpha >_{\mathbf{d}}$ and $\alpha = \sum_{t=0}^{s}\alpha_t d_t$ with $\alpha_s \neq 0$. If $e \geq reg(I) + 1$ then for every monomial $v \in I_{\geq e}$ there exists $w \in G(I)$ and a monomial $y \in S$ such that $v = w \cdot y$ and $m(v) = m(y)$.*

Proof: We may assume $e = reg(I)+1$ and $v \in I_e$. Then $v = w'\cdot y'$ for some $w' \in G(I)$ and a monomial $y' \in S$. Suppose $w' = \prod_{t=0}^{s}\prod_{j=1}^{n} x_j^{\lambda_{tj}\cdot d_t}$, where $0 \leq \lambda_{tj}$ and $\sum_{j=1}^{n}\lambda_{tj} = \alpha_t$. Suppose $n = m(v) = m(w') > m(y')$. Then $y' = x_1^{\beta_1}\cdots x_{n-1}^{\beta_{n-1}}$. Let $m = min\{t|\lambda_{tn} \neq 0\}$.

We claim that there exists some $1 \leq i \leq n$ such that $d_m - \sum_{t=0}^{m-1}\lambda_{ti}d_t \leq \beta_i$. Otherwise, it follows that $d_m - \sum_{t=0}^{m-1}\lambda_{ti}d_t \geq \beta_i + 1$ for any $i = 1,\dots,n-1$. So,

$$(n-1)d_m - \sum_{i=1}^{n-1}\sum_{t=0}^{m-1}\lambda_{ti}d_t \geq \beta_1 + \cdots + \beta_{n-1} + n - 1 = reg(I) + 1 - \alpha + n - 1 \Leftrightarrow$$

$$(n-1)(d_m - 1) - \sum_{t=0}^{m-1}\alpha_t d_t \geq (n-1)(d_s - 1) - \sum_{t=0}^{s-1}\alpha_t d_t + 1 \Leftrightarrow$$

$$\sum_{t=m}^{s-1}\alpha_t d_t \geq (n-1)(d_s - d_m) + 1,$$

because $reg(I) = \alpha_s d_s + (n-1)(d_s - 1)$ from Theorem 3.1. But on the other hand, $d_s - d_m = \sum_{t=m}^{s-1}(d_{t+1}/d_t - 1)d_t \geq \sum_{t=m}^{s-1}\alpha_t d_t$ and this contradict the above inequality.

Thus, we may choose $i < n$ such that $\gamma = d_m - \sum_{t=0}^{m-1}\lambda_{ti}d_t \leq \beta_i$. Therefore, we can write: $v = w'\cdot y' = w\cdot y$, where $w = w'\cdot x_i^{\gamma}/x_n^{\gamma}$ and $y = w'\cdot x_n^{\gamma}/x_i^{\gamma}$. It is easy to see that $w \in G(I)$ and $m(v) = m(y) = n$. $\square$

Corollary 3.6. *If* $I =< x_n^{\alpha} >_{\mathbf{d}}$ *and* $e \geq reg(I)$ *then* $I_{\geq e}$ *is stable.*

Proof: Let $v \in I_{\geq e}$. Let $i < m(v)$. Since $x_i \cdot v \in I_{\geq e+1}$ it follows from the above lemma that $x_i v = w\cdot y$ for some $w \in G(I)$ and $y \in S$ such that $m(x_i v) = m(y)$. But $m(v) = m(x_i v)$ and thus $x_i v/x_{m(v)} = w\cdot y/x_{m(v)} \in I$. $\square$

The converse is also true. Indeed we have the following more general result of Eisenbud-Reeves-Totaro:

Proposition 3.7. *[2, Proposition 12] Let I be a monomial ideal with* $deg(I) = d$ *and let* $e \geq d$ *such that* $I_{\geq e}$ *is stable. Then* $reg(I) \leq e$.

Remark 3.8. 3.6 *gives another proof for the "*$\leq$*" inequality of the generalised Pardue's formula in the case when* $I =< x_n^{\alpha} >_{\mathbf{d}}$. *Indeed, considering* $e = \alpha_s d_s + (n-1)(d_s - 1)$ *from* 3.6 *it follows that* $I_{\geq e}$ *is stable and thus* 3.7 *implies* $reg(I) \leq e$.

Corollary 3.9. *If* $I =< x_n^{\alpha} >_{\mathbf{d}}$ *then* $reg(I) = min\{e|\ I_{\geq e}$ *is stable* $\}$.

References

[1] ANNETTA ARAMOVA, JÜRGEN HERZOG, "p-Borel principal ideals", Illinois J.Math.41,no 1.(1997),103-121.

[2] D.EISENBUD, A.REEVES, B.TOTARO, "Initial ideals, veronese subrings and rates of algebras", Adv.Math. 109 (1994), 168-187.

[3] JÜRGEN HERZOG, DORIN POPESCU, MARIUS VLADOIU, "On the Ext-Modules of ideals of Borel type", Contemporary Math. 331 (2003), 171-186.

[4] JÜRGEN HERZOG, DORIN POPESCU, "On the regularity of p-Borel ideals", Proceed.of AMS, Volume 129, no.9, 2563-2570.

[5] KEITH PARDUE, "Non standard Borel fixed ideals", Dissertation, Brandeis University, 1994.

[6] DORIN POPESCU, "Extremal Betti numbers and regularity of Borel type ideals", Bull. Math. Soc. Sc. Math. Roum. 48(96), no 1, (2005), 65-72.

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