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The geometry of curves of a complex Finsler space

by
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Abstract

In some previous papers ([Mu, Mu1, Mu2]), our attention focused on the general theory of holomorphic subspaces in a complex Finsler space. In the present paper two approaches in the study of complex curves of a complex Finsler space will be proposed.

In the first section we study curves on the holomorphic tangent bundle $T'M$ depending on the arc length parameter s. This study is in some sense through analogy with that made for curves in real Finsler spaces ([B-F]), by which an orthonormal moving frame of Frenet type is introduced.

In the second part of the paper we study the geometry of a complex curve (Riemannian surface) viewed as a particular one dimensional holomorphic subspace. The induced tangent and normal Chern-Finsler connections and the Gauss-Weingarden formulas will be obtained. A special attention is devoted to its geodesic curvature.

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The study of curves of a Riemannian space, or more generally of a real Finsler space, is mainly based on the fact that the tangent vector along the curve is unitary with respect to the arc length parameter. In the classical theory, by differentiating this tangent vector the first principal normal vector is obtained and afterwards successive differentiations of the principal normals finally get a moving Frenet frame along the curve.

When we pass to the study of curves in a complex Finsler space (M, F), the first remark is that the arc length parameter is real valued and hence an onset of the geometry of curves as in the classical way can not include the class of complex curves. And this class is indubitably one of interest for the geometry of holomorphic mappings, for example. Hereby, in the study of complex curves we are lead to the general theory of holomorphic subspaces of a complex Finsler space, with the particularities derived from the fact that they are one dimensional.

However, we appreciate that both approaches are of interest for the applications.

1 Curves depending on the arc length parameter

Let (M, F) be a $n-$dimensional complex Finsler space. Recall (see [A-P, Ai, Mu],...) it means that on the nonzero sections of the holomorphic tangent bundle $T'M$ of the complex manifold M, a metric function $F : T'M \to \mathbb{R}^+$ is homogeneous in η direction, i.e. $F(z, \lambda\eta) = \mid \lambda \mid F(z, \eta)$ for any $\lambda \in \mathbb{C}$, and F satisfies a strong pseudoconvexity condition that implies the Hermitian metric tensor $g_{i\bar{j}} = \partial^2 L/\partial\eta^i\partial\bar{\eta}^j$ is positively nondegenerate, where $L = F^2$.

Consider $c : t \to (z^i(t))$ a curve on the manifold M, where $(z^i)_{i=\overline{1,n}}$ are coordinates in a local chart, $t \in \mathbb{R}^+$. It follows immediately that $s(t) = \int_0^t F(z(t), \frac{dz}{dt})dt$ does not depend on changes of parameter t and also $\frac{ds}{dt} = F(z(t), \frac{dz}{dt})$. It follows that $z^i = z^i(s)$, $i = \overline{1,n}$, is a natural parametrization along the curve c for $s \in [a, b]$.

Since s is a real positive parameter, is obvious that c is not a Riemannian surface on M.

Let $\theta = \frac{ds}{dt}$ be the tangent vector at c and (s, θ) a coordinate system on the manifold $c^* = Tc$, tangent to the curve c.

Now, let us consider $T'M$ the vector bundle of $(1, 0)$ vectors (the holomorphic tangent bundle) and (z^i, η^i) local coordinates in a chart on $T'M$. Forasmuch $\eta^i = dz^i/dt = \theta dz^i/ds$, along the curve c the following parametric equations of the curve c^* on $T'M$ hold:

$$z^i = z^i(s) \ ; \ \eta^i = \theta\frac{dz^i}{ds} \quad , \ s \in [a, b]. \tag{1.1}$$

Subsequently, as a rule, by prime it will be denoted the derivative with respect to parameter s, that is to say $z'^i = \frac{dz^i}{ds}$.

From the homogeneity of Finsler function F and since ds/dt is positive, we easily check that $F(z(s), z'(s)) = 1$. At the same time, we know ([A-P, Mu]) that $L(z, \eta) = g_{i\bar{j}}\eta^i\bar{\eta}^j$, where $L = F^2$. So we conclude that $g_{i\bar{j}}z'^i\bar{z}'^j = 1$, that is $\| \frac{dz^i}{ds} \| = 1$.

Now, let us consider $VT'M$ the vertical distribution of $T_C T'M$, locally spanned by $\{\dot{\partial}_i := \frac{\partial}{\partial\eta^i}\}_{i=\overline{1,n}}$ and $HT'M$ the Chern-Finsler complex nonlinear connection (in brief $(c.n.c.)$) which is generated by the adapted frame

$$\delta_i := \frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^j\frac{\partial}{\partial\eta^j} \ , \ i = \overline{1,n}, \tag{1.2}$$

where $N_i^j = g^{\bar{m}j}\dfrac{\partial^2 L}{\partial z^i\partial\bar{\eta}^m}$ (see [A-P, Mu]). Hence $T_C T'M$ splits into $T_C T'M = HT'M \oplus \overline{HT'M} \oplus VT'M \oplus \overline{VT'M}$.

Let Tc^* be the real tangent space of the manifold c^* and let $\{\frac{d}{ds}, \frac{d}{d\theta}\}$ be a local frame in $(z^i(s), \eta^i(s,\theta))$. From (1.1) we deduce the link between the bases $\{\frac{d}{ds}; \frac{d}{d\theta}\}$ and $\{\delta_i, \delta_{\bar{\imath}}, \dot{\partial}_i, \dot{\partial}_{\bar{\imath}}\}$ (where $\delta_{\bar{\imath}}, \dot{\partial}_{\bar{\imath}}$ are obtained by conjugation everywhere in $\delta_i, \dot{\partial}_i$):

$$\frac{d}{ds} = \frac{dz^i}{ds}\delta_i + \frac{d\bar{z}^i}{ds}\delta_{\bar{\imath}} \ ; \ \frac{d}{d\theta} = \frac{dz^i}{ds}\dot{\partial}_i + \frac{d\bar{z}^i}{ds}\dot{\partial}_{\bar{\imath}}. \tag{1.3}$$

From this formula we remark that $\{\frac{d}{d\theta}\}$ spans a vertical distribution VTc^*, which is a subdistribution of $T_RT'M$.

Let $\{dz^i, \delta\eta^i\}$ be the dual adapted frame to (1.2) and $G = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j$ the Sasaki type lift of the Finsler metric tensor $g_{i\bar{j}}$.

G defines a Hermitian metric structure on fibres of $T_CT'M$ and hereby $Re\ G$ is a metric structure on $T_RT'M$.

Like in [A-P], p. 98, let us consider the following vertical and horizontal lifts along the curve c^*,

$$T^v = \frac{dz^k}{ds}\dot{\partial}_k \ \ , \ \ T^h = \frac{dz^k}{ds}\delta_k$$

and $\bar{T}^v$, $\bar{T}^h$ their conjugates.

Then (1.3) says that $\frac{d}{ds} = T^h + \bar{T}^h$ and $\frac{d}{d\theta} = T^v + \bar{T}^v$.

According to [B-F], p. 153, a Finsler field X is said to be *projectable* if $X = X^i(s)\dot{\partial}_i$ and hence, in view of the isomorphism $VT'M \simeq T'M$, it is derived from a vector $X^* = X^i(s)\partial/\partial z^i$.

It is well known ([A-P, Mu, Ai]) that on the sections of $T_CT'M$ a remarkable linear connection of (1,0)-type is acting, namely the Chern-Finsler complex linear connection, and let us denote it by $D\Gamma(N) = (L^i_{jk}, L^{\bar{\imath}}_{\bar{j}k} = 0, C^i_{jk}, C^{\bar{\imath}}_{\bar{j}k} = 0)$, where $L^i_{jk} = g^{\bar{m}i}\delta_k g_{j\bar{m}}$ and $C^i_{jk} = g^{\bar{m}i}\dot{\partial}_k g_{j\bar{m}}$. In our terminology from [Mu], this connection is a $N-(c.l.c.)$, i.e. $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$ and $D_{\delta_k}\dot{\partial}_j = L^i_{jk}\dot{\partial}_i$, etc.

Let X be projectable. An immediate computation gets that $D_{\dot{\partial}_k}X = X^jC^i_{j\bar{k}}$ and $D_{T^v}X = X^jz'^kC^i_{jk}$. Because $C^i_{j\bar{k}} = 0$ and $\eta^kC^i_{jk} = 0$ (the last identity results from the 0-homogeneity of $g_{i\bar{j}}$), it follows that $D_{T^v}X = D_{\bar{T}^v}X = 0$ along the curve c^*, and hence $D_{\frac{d}{d\theta}}X = 0$ for any projectable complex Finsler vector field X.

Since $\overline{D_XY} = D_{\bar{X}}\bar{Y}$, $\frac{d}{d\theta} = T^v + \bar{T}^v$ and $\frac{d}{ds}$ is projectable, it follows

Proposition 1.1. *We have:*

$$D_{\frac{d}{d\theta}}\frac{d}{d\theta} = 0 \ ; \ D_{\frac{d}{d\theta}}\frac{d}{ds} = 0. \tag{1.4}$$

By taking into account that $D_{\delta_k}\delta_{\bar{j}} = D_{\delta_k}\dot{\partial}_{\bar{j}} = 0$, direct calculus gives that

$$\begin{aligned} D_{\frac{d}{ds}}T^v &= (z''^i + L^i_{jk}z'^jz'^k)\dot{\partial}_i = (z''^i + N^i_0)\dot{\partial}_i \\ D_{\frac{d}{ds}}\bar{T}^v &= \bar{z}''^i\dot{\partial}_{\bar{\imath}} \ , \end{aligned}$$

which are the (1,0) and respectively (0,1) components of

$$D_{\frac{d}{ds}}\frac{d}{d\theta} = (z''^i + N_0^i)\dot{\partial}_i + \bar{z}''^i\dot{\partial}_{\bar{\imath}}. \tag{1.5}$$

Since D is a $N-(c.l.c)$ it results that $D_{\frac{d}{ds}}T^h = (z''^i + N_0^i)\delta_i$ and $D_{\frac{d}{ds}}\bar{T}^h = \bar{z}''^i\delta_{\bar{\imath}}$. Thus, we have also

$$D_{\frac{d}{ds}}\frac{d}{ds} = (z''^i + N_0^i)\delta_i + \bar{z}''^i\delta_{\bar{\imath}}. \tag{1.6}$$

Now, from (1.4), (1.5) and (1.6) we infer that the Chern-Finsler $N-(c.n.c.)$ induces a derivative law along the curve c^*, which is still denoted by D. From a geometrical point of view, since D is vertically vanishing, its vertical derivative is not of interest.

Because D is metrical with respect to G (i.e. $XG(Y,Z) = D(D_XY,Z) + G(Y,D_XZ)$ for $\forall X,Y,Z \in \Gamma T_C T'M$), and G is Hermitian, i.e. $\overline{G(X,Y)} = G(\bar{X},\bar{Y})$, it can be concluded that the induced connection D along the curve is also metrical with respect to $Re\ G = \frac{1}{2}(G+\bar{G})$. Now the claim $g_{i\bar{j}}z'^i\bar{z}'^j = 1$ implies that $G(\frac{d}{d\theta},\frac{d}{d\theta}) = G(\frac{d}{ds},\frac{d}{ds}) = 2$, and this formula suggests that a more suitable choice for a real metric along the curve c^* is $\tilde{G} = \frac{1}{2}Re\ G$. Now we have $\| \frac{d}{d\theta} \|_{\tilde{G}} = \| \frac{d}{ds} \|_{\tilde{G}} = 1$.

Let $\mathcal{R} = \{T^v, N_a\}_{a=\overline{2,n}}$ be an orthonormal frame in $VT'M$ with respect to G and let $VT'c^{*\perp}$ be the subdistribution of $VT'M$ spanned by $\{N_a\}_{a=\overline{2,n}}$. Its conjugate frame is denoted by $\overline{\mathcal{R}} = \{\bar{T}^v, \bar{N}_a\}_{a=\overline{2,n}}$.

Since the induced connection D is metrical with respect to $\tilde{G}$, we can easily check that $\tilde{G}(D_{\frac{d}{ds}}\frac{d}{d\theta},\frac{d}{d\theta}) = 0$, which says that $D_{\frac{d}{ds}}\frac{d}{d\theta}$ is orthogonal to $\frac{d}{d\theta}$. But $D_{\frac{d}{ds}}\frac{d}{d\theta} \in V_RT'M \simeq Re\ VT'M$, and hence we have

$$D_{\frac{d}{ds}}\frac{d}{d\theta} = k_1\mathcal{N}_1 \tag{1.7}$$

where $k_1 : c^* \to \mathbb{R}^+$ and $\mathcal{N}_1 \in Re\ VT'M$ is assumed an unitary vertical field, $\| \mathcal{N}_1 \|_{\tilde{G}} = 1$.

On the other hand, the $N-(c.l.c.)$ D is metrical on $T'M$ with respect to G and from $G(T^v,\bar{T}^v) = 1$ it follows $0 = G(D_{\frac{d}{ds}}T^v,\bar{T}^v) + G(T^v, D_{\frac{d}{ds}}\bar{T}^v)$, that is $2Re\ G(D_{\frac{d}{ds}}T^v,\bar{T}^v) = 0$. Thus we have $D_{\frac{d}{ds}}T^v \in VT'c^{*\perp}$, which means $D_{\frac{d}{ds}}T^v = k'_1\mathcal{N}'_1$ with $\mathcal{N}'_1 = \mathcal{N}'^a_1 N_a$ a normal vector that can be assumed unitary, $\| \mathcal{N}'_1 \|_G = 1$.

Completely analogously we prove that $D_{\frac{d}{ds}}\bar{T}^v \in \overline{VT'c^{*\perp}}$, and hence $D_{\frac{d}{ds}}\bar{T}^v = k''_1\mathcal{N}''_1$ with $\| \mathcal{N}''_1 \|_G = 1$. Again using the fact $\overline{D_XY} = D_{\bar{X}}\bar{Y}$ and $\frac{d}{ds}$ is an unitary real vector field, it is deduced that (1.7) can be rewritten as:

$$k_1\mathcal{N}_1 = k'_1\mathcal{N}'_1 + k''_1\mathcal{N}''_1 \text{ with } \overline{k'_1\mathcal{N}'_1} = k''_1\mathcal{N}''_1 \text{ and } \mid k'_1 \mid = \mid k''_1 \mid.$$

We made this short digression because in [A-P], p. 101, $D_{T^h+\bar{T}^h}T^h = 0$ is a necessary and sufficient condition such that c be a geodesic curve in a weakly Kähler Finsler space. Corroborating the above discussion with (1.5), we infer

Proposition 1.2. *i)* $k_1 = 0$ *if and only if* c^* *is an affine horizontal curve, i.e.* $z''^i = 0$ *and* $L^i_{jk}z'^j z'^k = 0$.

ii) If (M, F) *is an weakly Kähler Finsler space and* $k_1' = 0$, *then* c^* *is a geodesic curve with respect to Chern-Finsler connection.*

Assuming that $k_1 \neq 0$ on an interval $(-\varepsilon, \varepsilon)$, the vector $\mathcal{N}_1 = \frac{1}{k_1} D_{\frac{d}{ds}} \frac{d}{ds}$ will be called the *first principal normal* of the curve c^* in (M, F) and k_1 the *first principal curvature function.*

Now, recall that in (1.7) we assumed that $\tilde{G}(\mathcal{N}_1, \mathcal{N}_1) = 1$ and we saw that $\mathcal{N}_1 \perp \frac{d}{d\theta}$, i.e. $\tilde{G}(\mathcal{N}_1, \frac{d}{d\theta}) = 0$. From the first assertion and $\left(D_{\frac{d}{ds}} \tilde{G}\right)(\mathcal{N}_1, \mathcal{N}_1) = 0$ is obtained immediately $\tilde{G}(D_{\frac{d}{ds}} \mathcal{N}_1, \mathcal{N}_1) = 0$, which means that $D_{\frac{d}{ds}} \mathcal{N}_1$ is orthogonal to $\mathcal{N}_1$. The second assertion and $\left(D_{\frac{d}{ds}} \tilde{G}\right)(\mathcal{N}_1, \frac{d}{d\theta}) = 0$ yields $\tilde{G}(D_{\frac{d}{ds}} \mathcal{N}_1, \frac{d}{d\theta}) = -k_1$. Thus, $D_{\frac{d}{ds}} \mathcal{N}_1$, in the points of c^*, admits a decomposition by $\frac{d}{d\theta}$ and by an orthogonal direction to $\mathcal{N}_1$,

$$D_{\frac{d}{ds}} \mathcal{N}_1 = -k_1 \frac{d}{d\theta} + \widetilde{\mathcal{N}_2}. \tag{1.8}$$

Let $k_2 = \| \widetilde{\mathcal{N}_2} \|_{\tilde{G}} = \| D_{\frac{d}{ds}} \mathcal{N}_1 + k_1 \mathcal{N}_1 \|_{\tilde{G}}$, called the *second principal curvature.*

Next, if $k_2(s) \neq 0$ on a small $(-\varepsilon, \varepsilon)$, let us define $\mathcal{N}_2 := \frac{1}{k_2} \widetilde{\mathcal{N}_2}$, and consequently we have

$$D_{\frac{d}{ds}} \mathcal{N}_1 = -k_1 \mathcal{N}_0 + \mathcal{N}_2,$$

where we set $\mathcal{N}_0 := \frac{d}{d\theta}$. Note that $\mathcal{N}_2$ is an unitary vector field, orthogonal to $\mathcal{N}_1$.

Further on the algorithm is classical ([An, B-F]). Inductively, for $i < n$ and $k_1, k_2, ..., k_i \neq 0$ on $(-\varepsilon, \varepsilon)$ is obtained

$$D_{\frac{d}{ds}} \mathcal{N}_{i-1} = -k_{i-1} \mathcal{N}_{i-2} + k_i \mathcal{N}_i \ ,$$

and for $i = n$ is obtained that $D_{\frac{d}{ds}} \mathcal{N}_n = -k_n \mathcal{N}_{n-1}$.

The curvature functions k_i are deduced inductively from $k_{i+1} = \| \widetilde{\mathcal{N}}_{i+1} \|_{\tilde{G}} = \| D_{\frac{d}{ds}} \mathcal{N}_i + k_i \mathcal{N}_i \|_{\tilde{G}}$. In order to obtain an explicit writing for k_i, first observe that for any projectable vector $X = X^j(s)\dot{\partial}_j$ is obtained $D_{\frac{d}{ds}} X = (\frac{dX^i}{ds} + L^i_{jk} X^j z'^k)\dot{\partial}_i + \frac{d\bar{X}^i}{ds}\dot{\partial}_{\bar{\imath}}$ and hence, tacking into account that $X = \mathcal{N}_i$ is projectable, it is deduced a recurrent formula for k_i starting with $k_1 = \| (z''^i + N^i_0)\dot{\partial}_i + \bar{z}''^i \dot{\partial}_{\bar{\imath}} \|_{\tilde{G}}$.

The above algorithm furnishes an orthonormal frame $\{\mathcal{N}_0, \mathcal{N}_1, ..., \mathcal{N}_n\}$ along the points of c^* and called the *Frenet frame* of the complex Finsler space (M, F).

If k_i is identically zero for an $i < n$, then $D_{\frac{d}{ds}}\mathcal{N}_{i-1} = -k_{i-1}\mathcal{N}_{n-i}$ and the Frenet frame reduces to $\{\mathcal{N}_0, \mathcal{N}_1, ..., \mathcal{N}_{i-1}\}$.

Finally, using the general theory of complex system equations, the following theorem is proved through analogy with that from real Finsler spaces ([B-F]).

Theorem 1.1. *Let (M,F) be a complex Finsler space and let (z_0,η_0) be a fixed point of $T'M$. If $\{\mathcal{V}_0, \mathcal{V}_1, ..., \mathcal{V}_n\}$ is an orthonormal frame in Re $VT'M$ and $k_1, k_2, ..., k_n : (-\varepsilon, \varepsilon) \to \mathbb{R}^+$ are positive smooth functions, then there exists a unique curve $c : s \to (z^i(s))$, $s \in (-\varepsilon, \varepsilon)$, with $z^i(0) = z_0^i$, such that k_i are its curvature functions and $\mathcal{N}_i(0) = \mathcal{V}_i$ along the points of c^*, $\forall i = \overline{1,n}$.*

2 The geometry of a Riemannian surface in a complex Finsler space

It is well known that complex Finsler geometry has been motivated by the Kobayashi metric which is the image on $T'M$ of the Poicaré metric via holomorphic maps. The study of holomorphic maps in a complex Finsler space, which are complex curves from a geometrical point of view, impetuously imposed. Recently Nishikawa, [Ni], studied some geometrical aspects (first variations, critical points, etc.) for a holomorphic map in a complex Finsler space.

Our aim is to make here an intrinsic study of a holomorphic map, regarded as a complex curve of the complex Finsler space. A complex curve is a one dimensional holomorphic subspace and since $T'M$ is a complex manifold it will be called also a Riemannian surface of the complex Finsler space.

Further on we use our terminology from the general theory of holomorphic subspaces. For details see [Mu].

Let M be a complex manifold, $\dim M = n$, $(z^k)_{k=\overline{1,n}}$ coordinates in a local chart and $c : t \to z^k(t)$, $t \in [a,b]$, a curve on M, $\eta^k(t) = dz^k/dt$ the tangent vector in $z(t)$. If c varies arbitrary in the class of smooth curves, then (z^k, η^k) define local coordinates in a chart on $T'M$, the holomorphic tangent bundle of M.

Now, let (M,F) be a complex Finsler space. We saw in the preceding section that the length arc s defines a real parameter on the curves of (M,F) and this is an inconvenient for the study of complex curves.

Next we will consider $\widetilde{M}$ a Riemannian surface of M, that means $\dim_C \widetilde{M} = 1$, and $i : \widetilde{M} \hookrightarrow M$ the inclusion map. Let w be the local coordinate in a chart of $\widetilde{M}$ and thus $i : w \to z^k(w)$ will be a local parametrization of $\widetilde{M}$. Since $\widetilde{M}$ is a Riemannian surface it is a holomorphic subspace, and hence $\frac{dz^k}{d\bar{w}} = \frac{d\bar{z}^k}{dw} = 0$.

If $c : t \to w(t)$ is in $\widetilde{M}$, by $\theta = \frac{dw}{dt}$ we denote its tangent vector. Then, the complexified tangent inclusion map $i_{*,\mathbf{C}} : T'\widetilde{M} \to T'M$, with $i_{*,\mathbf{C}}(w,\theta) = (z(w), \eta(w,\theta))$, has the following local expression:

$$z^k = z^k(w) \ ; \ \eta^k = \theta\frac{dz^k}{dw}. \tag{2.1}$$

$\tilde{F}(w,\theta) = F(z(w),\eta(w,\theta))$ defines a complex Finsler function on $\widetilde{M}$. It follows that $(\widetilde{M},\tilde{F})$ is a one dimensional holomorphic subspace of (M,F) [Mu, Mu1, Mu2]. Henceforth we will apply the geometry of holomorphic subspaces to our particular situation $\dim \widetilde{M} = 1$.

Let $T_C T'\widetilde{M}$ and $T_C T'M$ be the corresponding complexified tangent spaces. From (2.1) we obtain the links between their local frames

$$\frac{\partial}{\partial w} = B^i \frac{\partial}{\partial z^i} + \dot{B}^i_0 \frac{\partial}{\partial \eta^i} \ ; \ \frac{\partial}{\partial \theta} = B^i \frac{\partial}{\partial \eta^i}, \tag{2.2}$$

where we kept our notations from general theory,

$$B^i = \frac{dz^i}{dw} \text{ and } \dot{B}^i_0 = \frac{d^2 z^i}{dw^2}\theta. \tag{2.3}$$

By conjugation everywhere in these formulas we obtain the links between $\{\frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial \bar{\theta}}\}$ and $\{\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{\eta}^i}\}$.

Let us consider $VT'\widetilde{M} = \{\frac{\partial}{\partial \theta}\}$ and $VT'M = \{\frac{\partial}{\partial \eta^i}\}_{i=\overline{1,n}}$ the corresponding vertical distributions, and $VT'\widetilde{M}^{\perp}$ an orthogonal distribution to $VT'\widetilde{M}$ in $VT'M$ with respect to Hermitian metric structure $G^v = g_{i\bar{j}} d\eta^i \otimes d\bar{\eta}^j$. By $\{N_a = B^i_a \frac{\partial}{\partial \eta^i}\}_{a=\overline{2,n}}$ is denoted a basis in $VT'\widetilde{M}^{\perp}$, which we can choose to be orthonormal. Thus we have, $g_{i\bar{j}} B^i B^{\bar{j}}_{\bar{a}} = 0$ and $g_{i\bar{j}} B^i_a B^{\bar{j}}_{\bar{b}} = \delta_{ab}$.

The frame $\mathcal{R} = [B^i \ B^i_a]$ is a moving frame along $\widetilde{M}$ and let $\mathcal{R}^{-1} = [B_i \ B^a_i]$ be the inverse matrix of $\mathcal{R}$. It follows that

$$B_i B^i = 1 \ ; \ B_i B^i_a = 0 \ ; \ B^a_i B^i = 0 \ ; \ B^a_i B^i_b = \delta^a_b. \tag{2.4}$$

It can be easily checked that $\frac{\partial}{\partial \eta^i} = B_i \frac{\partial}{\partial \theta} + B^a_i N_a$ and also that we have the following links between the dual bases

$$dz^i = B^i dw \ ; \ d\eta^i = \dot{B}^i_0 dw + B^i d\theta. \tag{2.5}$$

Let N^i_j be the coefficients of the Chern-Finsler complex nonlinear connection, like in the above section, and $\delta\eta^i = d\eta^i + N^i_j dz^j$ be its adapted dual cobasis. An *induced* (*c.n.c*) on $\widetilde{M}$ is defined by $\delta\theta = B_i \delta\eta^i$, where $\delta\theta = d\theta + N dw$. From the general theory, [Mu], p. 130, it results

Proposition 2.1. $N = B_i(\ \dot{B}^i_0 + B^j N^i_j)$.

For the adapted frames $\frac{\delta}{\delta w} = \frac{\partial}{\partial w} - N \frac{\partial}{\partial \theta}$ and $\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N^j_i \frac{\partial}{\partial \eta^j}$, according to Proposition 5.4.2 from [Mu], we have

Proposition 2.2. *i)* $dz^i = B^i dw \ ; \ \delta\eta^i = B^i \delta\theta + B^i_a M^a dw$;
ii) $\frac{\delta}{\delta w} = B^i \frac{\delta}{\delta z^i} + B^i_a M^a \frac{\partial}{\partial \eta^i} \ ; \ \frac{\partial}{\partial \theta} = B^i \frac{\partial}{\partial \eta^i}$, *where* $M^a = B^a_j(\dot{B}^j_0 + B^k N^j_k)$.

It is proved in [Mu] that the induced (*c.n.c.*) coincides with the intrinsic (*c.n.c.*) of $(\widetilde{M}, \tilde{F})$, that is $N = g^{-1}\partial^2\tilde{L}/\partial w\partial\bar{\theta}$, where $\tilde{L} = \tilde{F}^2$ and $g = g_{i\bar{j}}B^i\bar{B}^j$ is the induced metric on $(\widetilde{M}, \tilde{F})$.

Let $D\Gamma(N) = (L^i_{jk}, 0, C^i_{jk}, 0)$ be the Chern-Finsler linear connection like in the preceding section

$$L^i_{jk} = g^{\bar{m}i}\delta_k g_{j\bar{m}} \ \ ; \ \ C^i_{jk} = g^{\bar{m}i}\dot{\partial}_k g_{j\bar{m}}. \tag{2.6}$$

According to Theorem 5.4.4, the induced tangent connection of Chern-Finsler linear connection is $\tilde{D}\Gamma(\tilde{N}) = (L, 0, C, 0)$, where

$$L = g^{-1}\frac{\delta g}{\delta w} \ \ ; \ \ C = g^{-1}\frac{\partial g}{\partial \theta}. \tag{2.7}$$

Since $\tilde{N}$ is induced by N, the horizontal distribution spanned by $\frac{\delta}{\delta w}$ will be preserved by $\tilde{D}\Gamma(\tilde{N})$ and hence, on the sections of $T_C T'\widetilde{M}$ it holds the following decomposition

$$D_X Y = \tilde{D}_X Y + H(X, Y) \ \ , \forall X \in \Gamma(T_C T'\widetilde{M}) \ , \ Y \in \Gamma(V_C T'\widetilde{M}) \ , \tag{2.8}$$

where $D_X Y \in \Gamma(V_C T'\widetilde{M})$ and $H(X, Y) \in \Gamma(V_C T'\widetilde{M}^{\perp})$. The bilinear operator H is called the *second fundamental form.*

In adapted frames of Chern-Finsler (*c.n.c.*), let H be given by

$$H(\frac{\delta}{\delta w}, \frac{\partial}{\partial \theta}) = H^a N_a \ \text{ and } \ H(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}) = K^a N_a.$$

Direct calculus proves that $H(\frac{\delta}{\delta w}, \frac{\partial}{\partial \bar{\theta}}) = H(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \bar{\theta}}) = 0$ and

$$H^a = B^a_i(\dot{B}^i_0 + B^j B^k L^i_{jk} + B^j B^k_a M^a C^i_{jk}) \ ; \ K^a = B^a_i B^j B^k C^i_{jk}. \tag{2.9}$$

Further, in [Mu] for any normal vector field X we defined the normal induced connection $D^{\perp}\Gamma(\tilde{N})$ satisfying $D^{\perp}X^a = B^a_i DX^i$. Thus for $\forall X \in \Gamma(T_C T'\widetilde{M})$, $W \in \Gamma(V_C T'\widetilde{M}^{\perp})$, we have

$$D_X W = -A_W X + D^{\perp}_X W \ , \tag{2.10}$$

where $A_W X \in \Gamma(V_C T'\widetilde{M})$ and $D^{\perp}_X W \in \Gamma(V_C T'\widetilde{M}^{\perp})$.

The formulas (2.8) and (2.10) are the well known *Gauss-Weingarten formulas.*

If we set for simplicity $A_{N_a} := A_a$ and consider the following local expression for the *Weingarten operator* A

$$A_a(\frac{\delta}{\delta w}) = A_a\frac{\partial}{\partial \theta} \ \ ; \ \ A_a(\frac{\partial}{\partial \theta}) = V_a\frac{\partial}{\partial \theta},$$

then a direct computation proves that $A_a(\frac{\delta}{\delta \bar{w}}) = A_a(\frac{\partial}{\partial \bar{\theta}}) = 0$ and

$$\begin{aligned} A_a &= -\frac{\delta N^0_a}{\delta w} - B^k N^j_a L^0_{jk} - M^b B^k_b N^j_a C^0_{jk} \\ V_a &= -\frac{\partial N^0_a}{\partial \theta} - B^k N^j_a C^0_{jk}, \end{aligned}$$

where $N_a = N_a^j \frac{\partial}{\partial \theta}$.

The Gauss, Codazzi and Ricci equations of the Riemannian surface $(\widetilde{M}, \tilde{F})$ can be obtained directly from [Mu1], particularizing the general framework of holomorphic subspaces. We will leave out these technical elements here.

Further on we focus in some lines on the study of induced holomorphic sectional curvature. Recall that, [A-P, Mu], the holomorphic sectional curvature of a complex Finsler space $(M, L = F^2)$ is given by

$$K_F(z,\eta) = \frac{2}{L^2} G(\Omega(\chi, \bar{\chi})\chi, \bar{\chi}) \tag{2.11}$$

where G is the Hermitian metric structure, Ω is the curvature form of Chern-Finsler connection and $\chi = \eta^k \frac{\delta}{\delta z^k}$ is the horizontal lift of the radial vertical vector $\eta = \eta^k \frac{\partial}{\partial \eta^k}$.

K_F in η direction is written as a function of Ricci tensor as follows,

$$K_F(z,\eta) = \frac{2}{L^2} R_{\bar{j}k} \bar{\eta}^j \eta^k \text{ , where } R_{\bar{j}k} = -g_{l\bar{j}} \frac{\delta N_k^l}{\delta \bar{z}^h} \bar{\eta}^h. \tag{2.12}$$

In [Mu1] we found for a general holomorphic subspace $(\widetilde{M}, \tilde{F})$ the relationship between K_F and intrinsic holomorphic sectional curvature $\tilde{K}_{\tilde{F}}$. Particularizing this result it follows that $\tilde{K}_{\tilde{F}}$ in $\tilde{u} = (w, \theta)$, with θ a tangent direction of the Riemann surface, and K_F in $u = (z(w), \eta(w, \theta))$ are connected by

$$\tilde{K}_{\tilde{F}}(\tilde{u}) = K_F(u) - \frac{2}{L^2} B_{\bar{a}}^{\bar{l}} B_{\bar{h}}^{\bar{a}} \{Q_{k\bar{l}}^m + \rho_{k\bar{l}}^m\} g_{m\bar{j}} \bar{\eta}^j \bar{\eta}^h \eta^k \text{ ,} \tag{2.13}$$

where $Q_{k\bar{l}}^m = \frac{\delta}{\delta \bar{z}^l}(N_k^m)$ and $\rho_{k\bar{l}}^m = \frac{\partial}{\partial \bar{\eta}^l}(N_k^m)$.

For instance, if $(\widetilde{M}, \tilde{F})$ is a Riemannian surface of a locally Minkowsky space (M, F), then there exists a local chart in any point u in which $N_k^m = 0$ and consequently, in such charts, the intrinsic holomorphic curvature of $(\widetilde{M}, \tilde{F})$ coincide with that of (M, F).

Now, the last problem in discussion here is to determine the circumstances in which $(\widetilde{M}, \tilde{F})$ is totally geodesic immersed in (M, F). Roughly speaking, a *$c-$complex geodesic* of complex Finsler space (M, F) is a geodesic curve with respect to Chern-Finsler connection which corresponds via holomorphic maps to a geodesic curve on unitary disc Δ with respect to Poincaré metric. According to [A-P] a curve $\varphi : t \to z^k(t)$ on M is lifted, as we did at the beginning of this section, to a complex curve on (M, F) which is a $c-$complex geodesic (c is a real constant) iff (M, F) is weakly Kähler and along the points of the lifted curve the torsion of Chern-Finsler connection satisfies an additional condition. Recall that the weakly Kähler request means $g_{i\bar{l}} T_{jk}^i \eta^j \bar{\eta}^l = 0$, where $T_{jk}^i = L_{jk}^i - L_{kj}^i$ is the horizontal torsion.

In [Mu1] we proved that a holomorphic subspace $(\widetilde{M}, \tilde{F})$ is *complexly totally geodesic* immersed in (M, L), that is to say any $c-$ geodesic of $(\widetilde{M}, \tilde{F})$ is a

$c-$geodesic of (M,F), if and only if the weakly Kähler request for (M,F) is sent off to the holomorphic subspace $(\widetilde{M},\tilde{F})$ and $vT(\bar{h},h)$ torsion is one normal. In our particular case of Riemannian surfaces, the necessary and sufficient conditions such that $(\widetilde{M},\tilde{F})$ be complexly totally geodesic immersed in (M,F) are written as follow. The weakly Kähler condition for $(\widetilde{M},\tilde{F})$ with respect to induced tangent Chern-Finsler connection $\tilde{D}\Gamma(\tilde{N})$ means $g\tilde{T}\theta\bar{\theta}=0$. But $\tilde{T}=L-L=0$ and hence it follows that any Riemannian surface $(\widetilde{M},\tilde{F})$ is weakly Kähler. The second requirement is translate by $B_i\Theta^i_{j\bar{k}}\eta^j\bar{\eta}^k=0$ and it remains the essential condition for $(\widetilde{M},\tilde{F})$ be complexly totally geodesic immersed in (M,F). We point out, as it is deduced from the above, that the $c-$complex geodesic notion is also a special one. In a recent paper [Mu2], we studied a more general notion of geodesic complex curves in a holomorphic subspace by using the complex Berwald connection instead of Chern-Finsler connection. In the present study, such an approach needs a bit more space and for this reason here we leave out this topic.

Finally, let us point that in [Ni] is studied a similar problem concerning the complex geodesics of a Riemannian compact surface in a Finsler-Kähler space (M,F), called the harmonic maps of (M,F), starting from the metric $\langle\bar{\partial}c,\bar{\partial}c\rangle = g_{i\bar{j}}(w)\frac{dz^i}{dw}\overline{\frac{dz^i}{dw}}$ (where $g_{i\bar{j}}(w)=g_{i\bar{j}}(z(w),\eta(w,\theta))$ is the metric tensor of (M,F)) and from the variation of $\bar{\partial}-$energy $E_{\bar{\partial}}(c)=\int_{\tilde{M}}\langle\bar{\partial}c,\bar{\partial}c\rangle dV_{\widetilde{M}}$.

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