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On p-open mappings

by
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Abstract

p-open mappings between topological spaces are those preserving preopen sets. Restrictions of this type of mappings are considered. Preimages of semi-connected spaces and images of S-closed spaces are investigated. Also, some observations concerning a.c.H. mappings are provided.

Key Words: Preopen, semi-open, semi-preopen sets; a.c.H., semi-continuous, almost open mappings; semi-connected, S-closed spaces.
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1 Introduction

In 1984 Mashhour et al. [23, Definition 3.2] have introduced the notion of $\mathcal{M}$-*preopenness*: a mapping is $\mathcal{M}$-preopen if images of preopen sets are preopen. Later, Janković [16] referred to this property as to the p-*openness*. We shall use the latter up to its briefness. In this paper, characterizations of p-open mappings are given and restrictions of those mappings are investigated. Preimages of semi-connected spaces and images of $\mathcal{S}$-closed spaces are considered. No separation axioms are assumed.

2 Preliminaries

Throughout this paper, (X,τ) and (Y,σ) denote topological spaces. For $A \subset X$, (A,τ_A) stands for the topological subspace of (X,τ). Fix an (X,τ). The *closure* of $S \subset X$ and the *interior* of S (both in (X,τ)) are denoted with $\mathrm{cl}\,(S)$ and $\mathrm{int}\,(S)$ respectively. For a subspace (A,τ_A), $A \supset S$, these will be denoted with $\mathrm{cl}_A(S)$ and $\mathrm{int}_A(S)$ respectively. A subset $S \subset X$ is said to be *regular open* (resp. *regular closed*; α-*open* [25]; *semi-open* [17]; *semi-closed* [7]; *preopen* [20]; *preclosed* [3]; *semi-preopen* [3] (equiv. β-*open* [1]); *semi-regular* [9] (equiv. *regular semi-open* [5])) in (X,τ) if $S = \mathrm{int}\,(\mathrm{cl}\,(S))$ (resp. $S = \mathrm{cl}\,(\mathrm{int}\,(S))$; $S \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S)))$; $S \subset$

$\mathrm{cl}\,(\mathrm{int}\,(S))$; $\mathrm{int}\,(\mathrm{cl}\,(S)) \subset S$; $S \subset \mathrm{int}\,(\mathrm{cl}\,(S))$; $\mathrm{cl}\,(\mathrm{int}\,(S)) \subset S$; $S \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S)))$; $\mathrm{int}\,(\mathrm{cl}\,(S)) \subset S \subset \mathrm{cl}\,(\mathrm{int}\,(S)))$.

A subset S of (X, τ) is called *generalized closed* [19] if $\mathrm{cl}\,(S) \subset U$ whenever $S \subset U$ and $U \in \tau$.

The family of all regular open (resp. closed; regular closed; α-open; semi-open; semi-closed; preopen; preclosed; semi-preopen; semi-regular; generalized closed) subsets of an (X, τ) will be denoted with $\mathrm{RO}\,(X, \tau)$ (resp. $\mathrm{c}\,(X, \tau)$; $\mathrm{RC}\,(X, \tau)$; τ^{α}; $\mathrm{SO}\,(X, \tau)$; $\mathrm{SC}\,(X, \tau)$; $\mathrm{PO}\,(X, \tau)$; $\mathrm{PC}\,(X, \tau)$; $\mathrm{SPO}\,(X, \tau)$; $\mathrm{SR}\,(X, \tau)$; $\mathrm{Gc}\,(X, \tau)$). A subset $S \in \mathrm{SO}\,(X, \tau)$ if and only if there exists an $U \in \tau$ such that $U \subset S \subset \mathrm{cl}\,(U)$ [17].

The following (proper, in general) inclusions are well-known: $\tau \subset \tau^{\alpha} = \mathrm{SO}\,(X, \tau) \cap \mathrm{PO}\,(X, \tau)$ ([34, Lemma 3.1] for the equality); $\mathrm{SO}\,(X, \tau) \cup \mathrm{PO}\,(X, \tau) \subset \mathrm{SPO}\,(X, \tau)$ [3, Theorem 2.2 & Example 2.3]. The family τ^{α} forms a topology on X.

The intersection of all semi-closed (resp. preclosed) subsets of (X, τ) containing $S \subset X$ is called the *semi-closure* (resp. the *preclosure*) of S and is denoted with $\mathrm{scl}\,(S)$ (resp. $\mathrm{pcl}\,(S)$). The union of all preopen subsets of (X, τ) contained in $S \subset X$ is called the *preinterior* of S and is denoted with $\mathrm{pint}\,(S)$. It is known that $S \in \mathrm{SC}\,(X, \tau)$ (resp. $S \in \mathrm{PC}\,(X, \tau)$; $S \in \mathrm{PO}\,(X, \tau)$) if and only if $S = \mathrm{scl}\,(S)$ (resp. $S = \mathrm{pcl}\,(S)$; $S = \mathrm{pint}\,(S)$).

A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be *α-continuous* [22, 31] (resp. *semi-continuous* (briefly *s.c.*) [17]; an *R-map* [6]; *irresolute* [8]) if the preimage $f^{-1}(V)$ is in τ^{α} (resp. in $\mathrm{SO}\,(X, \tau)$; $\mathrm{RO}\,(X, \tau)$; $\mathrm{SO}\,(X, \tau)$) for every V from σ (resp. from σ; $\mathrm{RO}\,(Y, \sigma)$; $\mathrm{SO}\,(Y, \sigma)$). In 1966, Husain [14] has introduced *almost continuous* mappings (briefly *a.c.H.*). Mashhour et al. [20] has shown that a.c.H. for an $f : (X, \tau) \to (Y, \sigma)$ coincides with the so-called precontinuity of f: $f^{-1}(V) \in \mathrm{PO}\,(X, \tau)$ for every $V \in \sigma$. Rose [37, Theorem 6] proved that an f is a.c.H. iff $f(\mathrm{cl}\,(U)) \subset \mathrm{cl}\,(f(U)$ for every $U \in \tau$. Janković [16, Proposition 3.1] observed that an f is a.c.H. iff $f(\mathrm{cl}\,(U)) \subset \mathrm{cl}\,(f(U)$ for every $U \in \mathrm{SO}\,(X, \tau)$.

In 1967, Wilansky [43] introduced *almost open* mappings (briefly: *a.o.W.*). Rose [37, Theorem 11] showed that an f is a.o.W. iff $f(\mathrm{cl}\,(U)) \subset \mathrm{int}\,(\mathrm{cl}\,(f(U))$ for every $U \in \tau$ (i.e., iff f is preopen [20]).

A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be *almost open in the sense of Singals* [39] (*a.o.S.*, for short) if $f(U) \in \sigma$ for every $U \in \mathrm{RO}\,(X, \tau)$.

A space (X, τ) is said to be *extremally disconnected* (briefly *e.d.*) (resp. *submaximal*) if $\mathrm{cl}\,(U) \in \tau$ for every $U \in \tau$ (resp. if for every dense $U \subset X$ we have $U \in \tau$).

3 Characterizations

Definition 1. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be* p-***open*** *if $f(U) \in \mathrm{PO}\,(Y, \sigma)$ for every $U \in \mathrm{PO}\,(X, \tau)$.*

Theorem 1. *For an $f : (X, \tau) \to (Y, \sigma)$ the following are equivalent:*

(**a**) *f is* p-*open.*

(**b**) *For every $S \subset Y$ and every $F \in \mathrm{PC}\,(X, \tau)$ with $f^{-1}(S) \subset F$ there exists a $G \in \mathrm{PC}\,(Y, \sigma)$ such that $S \subset G$ and $f^{-1}(G) \subset F$.*

(**c**) *For every $B \subset Y$, $f^{-1}(\mathrm{cl}\,(\mathrm{int}\,(B))) \subset \mathrm{pcl}\,\big(f^{-1}(B)\big)$.*

(**d**) *For every $A \subset X$, $f(\mathrm{pint}\,(A)) \subset \mathrm{int}\,(\mathrm{cl}\,(f(A)))$.*

(**e**) *For every $V \in \sigma$, $f^{-1}(\mathrm{cl}\,(V)) \subset \mathrm{pcl}\,\big(f^{-1}(V)\big)$.*

(**f**) *For every $V \in \mathrm{SO}\,(Y, \sigma)$, $f^{-1}(\mathrm{cl}\,(V)) \subset \mathrm{pcl}\,\big(f^{-1}(V)\big)$.*

(**g**) *For every $V \in \sigma^{\alpha}$, $f^{-1}(\mathrm{cl}\,(V)) \subset \mathrm{pcl}\,\big(f^{-1}(V)\big)$.*

Proof: (**a**)$\Rightarrow$(**b**). Let $S \subset Y$ and $F \in \mathrm{PC}\,(X, \tau)$ be such that $f^{-1}(S) \subset F$. Put $G = Y \setminus f(X \setminus F)$. Since f is p-open, $G \in \mathrm{PC}\,(Y, \sigma)$, and since $f^{-1}(B) \subset F$ we get $f(X \setminus F) \subset f\big(f^{-1}(Y) \setminus f^{-1}(S)\big) \subset Y \setminus S$. So, $S \subset G$. One easily checks that $f^{-1}(G) \subset F$.
(**b**)$\Rightarrow$(**c**). Let $B \subset Y$. Put $F = \mathrm{pcl}\,\big(f^{-1}(B)\big) \in \mathrm{PC}\,(X, \tau)$. Clearly, $f^{-1}(B) \subset F$. By assumption there exists a set $G \in \mathrm{PC}\,(Y, \sigma)$ with $B \subset G$ and such that $f^{-1}(G) \subset F$. Thus, we have $\mathrm{cl}\,(\mathrm{int}\,(B)) \subset G$ and $f^{-1}(\mathrm{cl}\,(\mathrm{int}\,(B)) \subset \mathrm{pcl}\,\big(f^{-1}(B)\big)$.
(**c**)$\Rightarrow$(**d**). Let $A \subset X$. Putting $B = Y \setminus f(A)$ we obtain

$$f^{-1}\big(Y \setminus \mathrm{int}\,(\mathrm{cl}\,(f(A))) \subset X \setminus f^{-1}(f(A)) \subset X \setminus \mathrm{pint}\,\big(f^{-1}(f(A))\big).$$

Hence $\mathrm{pint}\,(A) \subset \mathrm{pint}\,\big(f^{-1}(f(A))\big) \subset f^{-1}(\mathrm{int}\,(\mathrm{cl}\,(f(A)))$. Therefore we get $f(\mathrm{pint}\,(A)) \subset \mathrm{int}\,(\mathrm{cl}\,(f(A)))$.
(**d**)$\Rightarrow$(**a**). Let $S \in \mathrm{PO}\,(X, \tau)$. Then $S = \mathrm{pint}\,(S)$ and by our supposition we obtain $f(S) = f(\mathrm{pint}\,(S) \subset \mathrm{int}\,(\mathrm{cl}\,(f(S)))$. So, $f(S) \in \mathrm{PO}\,(Y, \sigma)$.
(**a**)$\Leftrightarrow$(**e**). Clear by (**c**).
(**c**)$\Leftrightarrow$(**f**). Use [27, Lemma 2].
(**f**)$\Rightarrow$(**g**) and (**g**)$\Rightarrow$(**e**). Obvious. □

It is clear that every p-open mapping is a.o.W., but the converse is not true, in general, see the following example.

Example 1. *Let $X = \{a, b, c\} = Y$, $\tau = \big\{\emptyset, X, \{a\}, \{a, b\}\big\}$, and $\sigma = \big\{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\big\}$. Define a mapping $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = f(b) = a$, $f(c) = b$. Then f is a.o.W. and not* p-*open since $f(\{a, c\}) = \{a, b\} \notin \mathrm{PO}\,(Y, \sigma)$.*

Almost openness in the sense of Singals (or openness) and p-openness are independent notions.

Example 2. (**a**). *Let* $X = \{a, b\} = Y$, $\tau = \{\emptyset, X\}$, *and* $\sigma = \{\emptyset, Y, \{a\}\}$. *Let* f *be the identity on* X. *Then* f *is open (hence a.o.S.) and not* p-*open.*
(**b**). *Let* $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}\}$, *and* $\sigma = \{\emptyset, Y, \}$. *Let* f *be the identity mapping. Then* f *is* p-*open and not a.o.S. (so not open).*

Lemma 1. *[28, Theorem 1]. A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is s.c. if and only if* $f(\mathrm{int}\,(\mathrm{cl}\,(A))) \subset \mathrm{cl}\,(f(A))$ *for every* $A \subset X$.

Theorem 2. *If an* $f : (X, \tau) \to (Y, \sigma)$ *is a.o.S. and s.c., then it is* p-*open (and irresolute [30, Theorem 1.12]).*

Proof: Let $U \in \mathrm{PO}\,(X, \tau)$. Applying [32, Theorem 2.1] we obtain $f(U) \subset f(\mathrm{int}\,(\mathrm{cl}\,(U))) \subset \mathrm{int}\,(f(\mathrm{int}\,(\mathrm{cl}\,(U))))$, because $\mathrm{int}\,(\mathrm{cl}\,(U)) \in \mathrm{SC}\,(X, \tau)$. But f is s.c., thus by Lemma 1, $f(U) \in \mathrm{PO}\,(Y, \sigma)$. □

4 p-a.c.H. and a.c.H. mappings

Definition 2. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* **p-*a.c.H.*** *if* $f(\mathrm{cl}\,(S)) \subset \mathrm{cl}\,(f(S))$ *for every* $S \in \mathrm{PO}\,(X, \tau)$.

It is well-known that if a mapping $f : (X, \tau) \to (Y, \sigma)$ is α-continuous, then $f(\mathrm{cl}\,(S)) \subset \mathrm{cl}\,(f(S))$ for every $S \in \mathrm{PO}\,(X, \tau)$ [22, Corollary 1.1(i)]. Thus each α-continuous mapping is p-a.c.H. On the other hand, every p-a.c.H. mapping is a.c.H. [37, Theorem 6], but the converse for this implication does not hold.

Example 3. *Let* $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{a, b\}\}$, *and* $\sigma = \{\emptyset, Y, \{b, c\}\}$. *Let* f *be the identity on* X. *Then* f *is a.c.H. and not* p-*a.c.H. See also [24, Example 3.1].*

Problem 1. *Find a* p-*a.c.H. mappng which is not* α-*continuous.*

Lemma 2. *[2, Lemma 3.1]. An* $A \in \mathrm{PO}\,(X, \tau)$ *if and only if there exists a* $G \in \tau$ *with* $A \subset G \subset \mathrm{cl}\,(A)$.

Theorem 3. *Let a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be open and* p-*a.c.H. Then* f *is* p-*open.*

Proof: Let $A \in \mathrm{PO}\,(X, \tau)$. Then, by Lemma 2 we have $A \subset G \subset \mathrm{cl}\,(A)$ for a certain $G \in \tau$. Hence, we obtain $f(A) \subset f(G) \subset f(\mathrm{cl}\,(A)) \subset \mathrm{cl}\,(f(A))$ where $f(G) \in \sigma$. Therefore $f(A) \in \mathrm{PO}\,(Y, \sigma)$. □

Corollary 1. *If a mapping is open and continuous, then it is* p-*open.*

Example 4. (**a**). *Let* $X = \{a, b\}$, $Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}\}$, *and*

$$\sigma = \{\emptyset, Y, \{c\}, \{a, b\}, \{a, b, c\}\}.$$

Let $f : (X, \tau) \to (Y, \sigma)$ *be the identity on* X. *Then* f *is* p-*a.c.H. and not open.*
(**b**). *Let* $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. *The identity on* X *is* p-*a.c.H. and open.*
(**c**). *Let* $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X\}$, *and* $\sigma = \{\emptyset, Y, \{c\}, \{a, b\}, \{a, b, c\}\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be the identity on* X. *Then* f *is open and not* p-*a.c.H.*

Problem 2. *In 1973, Neubrunnová [24, Examples 3.1&3.2] has shown that a.c.H. and s.c. are independent of each other. Indicate a* p-*a.c.H. mapping which is not s.c.*

Theorem 4. *For an* $f : (X, \tau) \to (Y, \sigma)$ *the following are equivalent:*

(**a**) f *is* α-*continuous.*

(**b**) f *is s.c. and a.c.H.*

(**c**) f *is s.c. and* p-*a.c.H.*

Proof: (**a**)$\Leftrightarrow$(**b**). [34, Theorem 3.2].
(**b**)$\Rightarrow$(**c**). By Lemma 1 we have $f(\mathrm{int}\,(\mathrm{cl}\,(S))) \subset \mathrm{cl}\,(f(A))$ for every subset $A \subset X$. Hence, $\mathrm{cl}\,\big(f(\mathrm{int}\,(\mathrm{cl}\,(S)))\big) \subset \mathrm{cl}\,(f(A))$ and $f(\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S)))) \subset \mathrm{cl}\,\big(f(\mathrm{int}\,(\mathrm{cl}\,(S)))\big)$, because f is a.c.H. Let $A \in \mathrm{PO}\,(X, \tau)$. Clearly, $f(\mathrm{cl}\,(A)) \subset \mathrm{cl}\,(f(A))$.
(**c**)$\Rightarrow$(**b**). Obvious. □

Corollary 2. *Assume a mapping* $f : (X, \tau) \to (Y, \sigma)$ *is s.c. Then* f *is* p-*a.c.H. iff it is a.c.H.*

Remark 1. *An a.c.H. mapping* $f : (X, \tau) \to (Y, \sigma)$ *is* p-*a.c.H. if and only if* $\mathrm{SO}\,(X, \tau) = \mathrm{PO}\,(X, \tau)$ *[16, Proposition 3.1]. On the other hand,* $\mathrm{PO}\,(X, \tau) = \mathrm{SO}\,(X, \tau)$ *if and only if* (X, τ) *is e.d. and* (X, τ^α) *is submaximal (see [16, Proposition 4.1] and [13, Theorem 4] respectively).*

The conditions '(X, τ) is e.d.' and '(X, τ^α) is submaximal' do not depend on the other.

Example 5. (**a**). *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{\emptyset, X, \{a, b\}\}$. *Then* (X, τ) *is e.d. and* (X, τ^α) *is not submaximal.*
(**b**). *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. *Then* (X, τ) *is not e.d. but* (X, τ^α) *is submaximal (we have* $\tau = \tau^\alpha$*).*

Corollary 3. *An a.c.H. mapping* $f : (X, \tau) \to (Y, \sigma)$ *is* p-*a.c.H. if and only if the space* (X, τ) *is e.d. and* (X, τ^α) *is submaximal.*

Theorem 5. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is a.c.H. and a.o.S., then $f(S) \in \mathrm{SO}(Y, \sigma)$ for every $S \in \mathrm{SR}(X, \tau)$.*

Proof: Take an $S \in \mathrm{SR}(X, \tau)$. With [37, Theorem 6] and [32, Theorem 2.1] we have $f(S) \subset f(\mathrm{cl}(\mathrm{int}(S))) \subset \mathrm{cl}\big(f(\mathrm{int}(S))\big) \subset \mathrm{cl}(\mathrm{int}(f(S)))$. □

An analogous result can be obtained when we replace a.o.S. by semi-openness [4] (see [30, Theorem 2.5]). Semi-openness and a.o.S. are independent notions [30, p.315]. It is worth to compare Theorem 5 with [32, Theorem 4.2].

5 Restrictions

Noiri has shown that the restriction of an a.o.W. mapping to a closed subdomain, is not necessarily a.o.W. [32, Example 3.2]. A similar assertion can be proved for p-open mappings.

Example 6. *Let $X = \{a, b, c\}$ and $\tau = \big\{\emptyset, X, \{a, b\}\big\}$. Then*

$$\mathrm{PO}(X, \tau) = \big\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\big\}$$

and $\{c\}$ is a closed set. The identity mapping on X is p-open while its restriction to $\{c\}$ is not.

Also, the restriction of a p-open mapping to a semi-open set may be not p-open.

Example 7. *Let $\mathbb{R}$ be the space of reals endowed with the Euclidean topology τ_e. Consider any interval $I = [a, b)$, $a < b$, $a, b \in \mathbb{R}$. Obviously $I \in \mathrm{SO}(\mathbb{R}, \tau_e) \setminus \big(\mathrm{PO}(\mathbb{R}, \tau_e) \cup \mathrm{c}(\mathbb{R}, \tau_e)\big)$. The identity on $\mathbb{R}$ is p-open but its restriction to I is not.*

Lemma 3. *[21, Lemma 2.2]. If $U \in \mathrm{PO}(X, \tau)$ and $V \in \mathrm{PO}(U, \tau_U)$, then $V \in \mathrm{PO}(X, \tau)$.*

Theorem 6. *Let (X, τ) and (Y, σ) be any spaces and a mapping $f : (X, \tau) \to (Y, \sigma)$ be p-open. If $U \in \mathrm{PO}(X, \tau)$ then the restriction $f \restriction U$ is p-open too.*

Proof: Follows by Lemma 3. □

The two examples below present some other cases when a restriction of a p-open mapping is not p-open.

Example 8. *Consider $(\mathbb{R}, \tau_e)$, the identity mapping on $\mathbb{R}$, and the subset $A = [0, 1] \cup ((1, 2] \cap \mathbb{Q})$, where $\mathbb{Q}$ stands for the set of rationals. Clearly, $A \in \mathrm{SPO}(\mathbb{R}, \tau_e) \setminus \big(\mathrm{PO}(\mathbb{R}, \tau_e) \cup \mathrm{SO}(\mathbb{R}, \tau_e) \cup \mathrm{Gc}(\mathbb{R}, \tau_e)\big)$. One easily checks that the restriction of identity to A is not p-open.*

Example 9. *For* $(\mathbb{R}, \tau_e)$ *and the identity mapping on* $\mathbb{R}$ *consider the set* $A = [0,1) \cup \{2\}$. *We have* $A \notin \mathrm{SPO}\,(\mathbb{R}, \tau_e)$ *and the restriction of identity to* A *is not* p*-open.*

Lemma 4. *Let* $A \subset X$ *be an arbitrary open subset of* (X, τ) *and let* $S \subset A$. *If* $S \in \mathrm{PO}\,(X, \tau)$ *then* $S \in \mathrm{PO}\,(A, \tau_A)$.

Proof: Let $S \in \mathrm{PO}\,(X, \tau)$. By [3, Theorem 1.5(f)] we get $S \supset \mathrm{pint}_A(S) = S \cap \mathrm{int}_A(\mathrm{cl}_A(S)) \supset S \cap \mathrm{int}\,(A \cap \mathrm{cl}\,(S)) = S \cap A \cap \mathrm{int}\,(\mathrm{cl}\,(S)) = S$. Thus $S = \mathrm{pint}_A(S)$ and $S \in \mathrm{PO}\,(A, \tau_A)$. □

Corollary 4. *Let* (X, τ) *be arbitrary and an* $A \in \tau$. *Then* $S \in \mathrm{PO}\,(A, \tau_A)$ *if and only if* $S \in \mathrm{PO}\,(X, \tau)$.

Proof: Lemmas 3 and 4. □

Theorem 7. *Let a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be* p*-open and let for an* $A \in \mathrm{PO}\,(X, \tau)$, $f(A) \in \sigma$. *Then the surjection* $g_A : (A, \tau_A) \to (f(A), \sigma_{f(A)})$, *where* $g_A(x) = f(x)$, $x \in A$, *is* p*-open.*

Proof: Follows from Lemmas 3 and 4. □

Corollary 5. *Let a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be open and* p*-open. Then, for any* $A \in \tau$ *the surjection* g_A *from Theorem 7 is* p*-open.*

Theorem 8. *Let a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be an open injection and let* $A \in \tau$ *be arbitrary. Then, the bijection* g_A *from Theorem 7 is open.*

Lemma 5. *Let an* A *be arbitrary subset of a space* (X, τ). *Then* $A \cap \mathrm{pcl}\,(S) \subset \mathrm{pcl}_A(S)$ *for every* $S \subset A$.

Proof: Let $S \subset A \subset X$. Applying [19, Theorem 1.5(e)] we infer what follows:

$$\mathrm{pcl}_A(S) = S \cup \mathrm{cl}_A(\mathrm{int}_A(S)) =$$

$$= S \cup \big(A \cap \mathrm{cl}\,(\mathrm{int}_A(S))\big) \supseteq A \cap \big(S \cup \mathrm{cl}\,(\mathrm{int}\,(S))\big) = A \cap \mathrm{pcl}\,(S).$$

□

Remark 2. *Recall that Dontchev et al. [10] have proved that if* $A \in \mathrm{PO}\,(X, \tau)$ *and* $S \in \mathrm{PO}\,(A, \tau_A)$ *then* $\mathrm{pcl}\,(S) \subset \mathrm{pcl}_A(S)$.

Theorem 9. *If a mapping* $f : (X, \tau) \to (Y, \sigma)$ *is* p*-open and* $B \in \mathrm{SO}\,(Y, \sigma)$, *then* $f \restriction f^{-1}(B) : f^{-1}(B) \to B$ *is* p*-open*

Proof: We shall use the characterization of p-openness stated in Theorem 1(**f**). Let a $V \in \mathrm{SO}(B, \sigma_B)$. Since $B \in \mathrm{SO}(Y, \sigma)$, $V \in \mathrm{SO}(Y, \sigma)$ [26, Theorem 5], and hence with p-openness of f we obtain $f^{-1}(\mathrm{cl}(V)) \subset \mathrm{pcl}(f^{-1}(V))$. So, using Lemma 5 we calculate as follows:

$$\left(f \restriction f^{-1}(B)\right)^{-1}(\mathrm{cl}_B(V)) = f^{-1}(B \cap \mathrm{cl}(V)) \subseteq f^{-1}(B) \cap \mathrm{pcl}\left(f^{-1}(V)\right) \subset$$
$$\subset \mathrm{pcl}_{f^{-1}(B)}\left(f^{-1}(V)\right) = \mathrm{pcl}_{f^{-1}(B)}\left(\left(f \restriction f^{-1}(B)\right)^{-1}(V)\right).$$

Therefore $f \restriction f^{-1}(B)$ is p-open. □

Lemma 6. *[12, Theorem 2.4]. If $S \in \mathrm{SO}(X, \tau)$ then $\mathrm{pcl}(S) = \mathrm{cl}(S)$.*

Let us observe, that a shorter proof of Lemma 6 one could have obtained with [3, Theorem 1.5(e)] and [27, Lemma 2].

Lemma 7. *Let a mapping $f : (X, \tau) \to (Y, \sigma)$ be s.c. and let $V \in \sigma$ be arbitrary. Then $f \restriction f^{-1}(V) : f^{-1}(V) \to V$ is s.c. too.*

Proof: Let an $U \in \sigma_V$. Then $U \in \sigma$. By hypothesis we have $f^{-1}(U) \in \mathrm{SO}(X, \tau)$. Hence $f^{-1}(U) \in \mathrm{SO}\left(f^{-1}(V), \tau_{f^{-1}(V)}\right)$, because $f^{-1}(V) \in \mathrm{SO}(X, \tau)$ [26, Theorem 5]. Therefore, the mapping $f \restriction f^{-1}(V)$ is s.c. □

Theorem 10. *Let an $f : (X, \tau) \to (Y, \sigma)$ be s.c. and let $\{V_\alpha : \alpha \in \nabla\}$ be a cover of Y. If mappings $f \restriction f^{-1}(V_\alpha) : f^{-1}(V_\alpha) \to V_\alpha$ are p-open for every $\alpha \in \nabla$, then f is p-open.*

Proof: Let $V \in \sigma$ be arbitrary. Put $f_\alpha = f \restriction f^{-1}(V_\alpha)$ and $U_\alpha = f^{-1}(V_\alpha)$ for every $\alpha \in \nabla$. Each subset $V \cap V_\alpha$ is open in the space $(V_\alpha, \sigma_{V_\alpha})$, thus with the assumption and Theorem 1(**e**) we get $f_\alpha^{-1}\left(\mathrm{cl}_{V_\alpha}(V \cap V_\alpha)\right) \subset \mathrm{pcl}_{U_\alpha}\left(f_\alpha^{-1}(V \cap V_\alpha)\right)$ for every $\alpha \in \nabla$. Since $V_\alpha \in \sigma$ for any $\alpha \in \nabla$, we obtain what follows:

$$f^{-1}\left(\mathrm{cl}_\sigma(V)\right) = \bigcup_{\alpha \in \nabla} f^{-1}\left(V_\alpha \cap \mathrm{cl}_\sigma(V)\right) \subset$$
$$\subset \bigcup_{\alpha \in \nabla} f_{V_\alpha}^{-1}\left(\mathrm{cl}_{V_\alpha}(V_\alpha \cap V)\right) \subset \bigcup_{\alpha \in \nabla} \mathrm{pcl}_{U_\alpha}\left(f_\alpha^{-1}(V_\alpha \cap V)\right).$$

From Lemma 7 and from our hypothesis we infer that $f_\alpha^{-1}(V_\alpha \cap V) \in \mathrm{SO}(U_\alpha, \tau_{U_\alpha})$ for every $\alpha \in \nabla$, and so $\mathrm{pcl}_{U_\alpha}\left(f_\alpha^{-1}(V_\alpha \cap V)\right) = \mathrm{cl}_{U_\alpha}\left(f_\alpha^{-1}(V_\alpha \cap V)\right)$ (Lemma 6). Then, we obtain

$$f^{-1}\left(\mathrm{cl}_\sigma(V)\right) \subset \bigcup_{\alpha \in \nabla} \mathrm{cl}\left(f_\alpha^{-1}(V_\alpha \cap V)\right) \subset \mathrm{cl}\left(f^{-1}(V)\right) = \mathrm{pcl}\left(f^{-1}(V)\right),$$

because $f^{-1}(V) \in \mathrm{SO}(X, \tau)$. Therefore, by Theorem 1(**e**), f is p-open. □

Corollary 6. *Let $\{V_\alpha : \alpha \in \nabla\}$ be a cover of a space (Y, σ) and let a mapping $f : (X, \tau) \to (Y, \sigma)$ be s.c. Then f is* p*-open if and only if $f \restriction f^{-1}(V_\alpha) : f^{-1}(V_\alpha) \to V_\alpha$ are* p*-open for every $\alpha \in \nabla$.*

Proof: It directly follows from Theorems 8 and 9. □

The next result is worth being noticed.

Theorem 11. *For any space (X, τ), if $S \subset A \subset X$, $A \in \mathrm{SO}(X, \tau)$, and $S \in \mathrm{SO}(A, \tau_A)$, then*

$$\mathrm{pcl}_A(S) = A \cap \mathrm{pcl}(S). \tag{1}$$

Proof: Let $S \subset A \subset X$. By [3, Theorem 1.5(e)] we have $\mathrm{pcl}_A(S) = S \cup \mathrm{cl}_A(\mathrm{int}_A(S)) = S \cup \mathrm{cl}_A(S) = \mathrm{cl}_A(S)$, because $S \in \mathrm{SO}(A, \tau_A)$ [27, Lemma 2]. Since $S \in \mathrm{SO}(X, \tau)$ [26, Theorem 5], $\mathrm{cl}(S) = \mathrm{pcl}(S)$ (by Lemma 6). Therefore (1) holds. □

Remark 3. *Dontchev et al. [10] proved that if $A \in \mathrm{SO}(X, \tau)$, then $\mathrm{pcl}_A(S) \subset \mathrm{pcl}(S)$ for any $S \subset A \subset X$.*

6 $\mathcal{S}$-connectedness of spaces

Definition 3. *A topological space (X, τ) is said to be **semi-connected** [35] (resp. **preconnected** [36]) if X cannot be expressed as the union of two nonempty semi-open (resp. preopen) subsets of (X, τ).*

Semi-connectedness and preconnectedness of a space are independent notions [15, Examples 21.&2.2]. It is clear that if the range of any open bijection (equiv. any closed bijection) is connected, then its domain is connected too. Similarly, if the range of a p-open bijection is preconnected, then the domain is preconnected too. In this section we study conditions under which the domain of an open or a p-open bijection is semi-connected if the range is semi-connected.

Lemma 8. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is open and a.c.H. then the set $\mathrm{cl}\left(f(\mathrm{cl}(A))\right) \in \mathrm{SO}(Y, \sigma)$ for every $A \in \mathrm{PO}(X, \tau)$.*

Proof: Take any $A \in \mathrm{PO}(X, \tau)$. By Lemma 2 there exists a $G \in \tau$ such that $A \subset G \subset \mathrm{cl}(A)$. Since f is a.c.H., we get what follows:

$$f(G) \subset f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{cl}\left(f\left(\mathrm{int}(\mathrm{cl}(A))\right)\right) \subset$$
$$\subset \mathrm{cl}(f(\mathrm{cl}(A))) = \mathrm{cl}(f(\mathrm{cl}(G))) \subset \mathrm{cl}(f(G)).$$

Thus $\mathrm{cl}(f(\mathrm{cl}(A))) \in \mathrm{SO}(Y, \sigma)$, since $f(G) \in \sigma$ [17, Definition 1]. □

With the proposition below we complete characterizations of semi-connected spaces; for some other the reader is referred to [11, 18, 29, 38, 42].

Theorem 12. *For any (X,τ), the following are equivalent*

(**a**) *(X,τ) is semi-connected.*

(**b**) *$\mathrm{scl}_\tau(A) = X$ for each nonempty $A \in \mathrm{PO}\,(X,\tau)$.*

(**c**) *$\mathrm{cl}_\tau(A) = X$ for each nonempty $A \in \mathrm{PO}\,(X,\tau)$.*

(**c'**) *$\mathrm{cl}_{\tau^\alpha}(A) = X$ for each nonempty $A \in \mathrm{PO}\,(X,\tau)$.*

(**d**) *$\mathrm{scl}_\tau(U) = X$ for each nonempty $U \in \mathrm{SO}\,(X,\tau)$.*

(**e**) *$\mathrm{cl}_\tau(U) = X$ for each nonempty $U \in \mathrm{SO}\,(X,\tau)$.*

(**e'**) *$\mathrm{cl}_{\tau^\alpha}(U) = X$ for each nonempty $U \in \mathrm{SO}\,(X,\tau)$.*

Proof: (**a**)⇒(**b**). Suppose for some nonempty $A \in \mathrm{PO}\,(X,\tau)$ we have $\mathrm{scl}\,(A) \neq X$. With [16, Proposition 2.7(a)] (or with [3, Theorem 1.5(a)]) we get $\emptyset \neq \mathrm{int}\,(\mathrm{cl}\,(A)) \neq X$. So, we obtain that $X = \mathrm{int}\,(\mathrm{cl}\,(A)) \cup \mathrm{cl}\,(\mathrm{int}\,(X \setminus A))$, where the nonempty sets $\mathrm{int}\,(\mathrm{cl}\,(A))$, $\mathrm{cl}\,(\mathrm{int}\,(X \setminus A))$ belong to $\mathrm{SO}\,(X,\tau)$. Therefore (X,τ) is not semi-connected, a contradiction.
(**b**)⇒(**a**). Suppose an (X,τ) fulfills (**b**), but it is not semi-connected. Hence $X = S_1 \cup S_2$, where $S_1 \neq \emptyset \neq S_2$, $S_1, S_2 \in \mathrm{SO}\,(X,\tau)$, and $S_1 \cap S_2 = \emptyset$. By [24, Lemma 3.5] we have

$$\begin{aligned}\emptyset = \mathrm{int}\,(\mathrm{cl}\,(S_1 \cap S_2)) = \mathrm{int}\,\mathrm{cl}\,(S_1)) \cap \mathrm{int}\,\mathrm{cl}\,(S_2)) = \\ = \mathrm{scl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1))) \cap \mathrm{scl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_2))).\end{aligned} \tag{2}$$

We shall show that $\mathrm{int}\,(\mathrm{cl}\,(S_i)) \neq \emptyset$, $i = 1, 2$. Suppose not. Thus, from [3, Theorem 1.5(a)] we infer that $\mathrm{scl}\,(S_i) = S_i$, i.e., $S_i \in \mathrm{SC}\,(X,\tau)$, $i = 1, 2$. With a dual equality to that of [27, Lemma 2] we obtain $\mathrm{int}\,(\mathrm{cl}\,(S_i)) = \mathrm{int}\,(S_i)$, and so $\mathrm{int}\,(S_i) = \emptyset$, $i = 1, 2$. A contradiction since S_i's are nonempty semi-open subsets of (X,τ) [7, Remark 1.2]. Finally, from (2) we get $\emptyset = X$. So, (**b**)⇒(**a**) holds.
The implications (**b**)⇒(**c**) and (**d**)⇒(**e**) are clear. The property from (**d**) has been established in [29, Theorem 3.1].
The implications (**c**)⇒(**b**) and (**e**)⇒(**d**) are obvious by [3, Theorem 1.5].
The equivalence (**c**)⇔(**c'**) (resp. (**e**)⇔(**e'**)) is an immediate consequence of the conditions (**b**) and (**c**) (resp. (**d**) and (**e**)). □

Theorem 13. *Let a bijection $f : (X,\tau) \to (Y,\sigma)$ be open,* p*-open, and a.c.H. If (Y,σ) is semi-connected then (X,τ) is semi-connected.*

Proof: Suppose (X,τ) is not semi-connected. Then, by Theorem 12(**c**), there exists a nonempty $A \in \mathrm{PO}(X,\tau)$ with $\mathrm{cl}(A) \neq X$. Put $V = f(\mathrm{cl}(A)) \in \mathrm{c}(Y,\sigma)$. Using the characterization of p-openness from Theorem 1(**f**) we obtain

$$f^{-1}(\mathrm{cl}(V)) \subset \mathrm{pcl}\left(f^{-1}(f(\mathrm{cl}(A)))\right) = \mathrm{pcl}(\mathrm{cl}(A)) = \mathrm{cl}(A),$$

because $V \in \mathrm{SO}(Y,\sigma)$ (Lemma 8). Thus $\mathrm{cl}(V) \neq Y$, whence with Theorem 12(**e**) we get (Y,σ) is not semi-connected. □

7 Images of $\mathcal{S}$-closed spaces

Definition 4. *A topological space (X,τ) is said to be **$\mathcal{S}$-closed** [40] if every semi-open cover of X has a finite subcollection whose members have closures covering X.*

Lemma 9. *Let a mapping $f : (X,\tau) \to (Y,\sigma)$ be a.o.S. and a.c.H. Then,*

$$f(\mathrm{pcl}(S)) \subset \mathrm{pcl}(f(S))$$

for every $S \in \mathrm{SC}(X,\tau)$.

Proof: Take an $S \in \mathrm{SC}(X,\tau)$. With hypothesis, [3, Theorem 1.5(e)], and [32, Theorem 2.1] we have what follows

$$\begin{aligned} f(\mathrm{pcl}(S)) = f\left(S \cup \mathrm{int}(\mathrm{cl}(S))\right) &\subset f(S) \cup \mathrm{cl}(f(\mathrm{int}(S))) \subset \\ &\subset f(S) \cup \mathrm{cl}(\mathrm{int}(f(S))) = \mathrm{pcl}(f(S)). \end{aligned}$$

□

Theorem 14. *Let a surjection $f : (X,\tau) \to (Y,\sigma)$ be an α-continuous, a.o.S. R-map. If (X,τ) is $\mathcal{S}$-closed then for each cover $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}(Y,\sigma)$ of Y there exists a finite subset $\nabla_0 \subset \nabla$ with $\bigcup_{\alpha\in\nabla_0} \mathrm{cl}(V_\alpha) = Y$.*

Proof: Let a family $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}(Y,\sigma)$ cover Y. Since f is an R-map,

$$\{f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(V_\alpha)))) : \alpha \in \nabla\} \subset \mathrm{RC}(X,\tau)$$

is a cover of X. The space (X,τ) is $\mathcal{S}$-closed, hence by [5, Theorem 2] there exists a finite subset $\nabla_0 \subset \nabla$ such that $X = \bigcup_{\alpha\in\nabla_0} f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(V_\alpha))))$. Since f is s.c. (and also a.c.H. [34, Theorem 3.2]) and a.o.S., it is p-open by Theorem 2. Thus, applying Theorem 1 we obtain

$$X = \bigcup_{\alpha\in\nabla_0} \mathrm{pcl}\left(f^{-1}(\mathrm{int}(\mathrm{cl}(V_\alpha)))\right).$$

But, $f^{-1}(\mathrm{int}\,(\mathrm{cl}\,(V_\alpha))) \in \mathrm{RO}\,(X,\tau) \subset \mathrm{SC}\,(X,\tau)$ for every $\alpha \in \nabla_0$, whence with Lemma 2 we have

$$Y = \bigcup_{\alpha\in\nabla_0} \mathrm{pcl}\,\big(f\big(f^{-1}(\mathrm{int}\,(\mathrm{cl}\,(V_\alpha)))\big)\big) = \bigcup_{\alpha\in\nabla_0} \mathrm{pcl}\,(\mathrm{cl}\,(V_\alpha)) = \bigcup_{\alpha\in\nabla_0} \mathrm{cl}\,(V_\alpha).$$

□

Example 10. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b,c\}, \{a,b\}\}$, $Y = \{a,b\}$, and $\sigma = 2^Y$. Define $f : (X,\tau) \to (Y,\sigma)$ via $f(a) = f(b) = a$, $f(c) = b$. Then f is an R-map, it is α-continuous (in fact, even continuous), a.o.S., and surjective.*

The examples below show that 'R-mapness' and 'α-continuity' are independent of each other.

Example 11. (**a**). *Let $X = \{a,b,c\} = Y$, $\tau = \{\emptyset, X, \{a,b\}\}$, and $\sigma = \{\emptyset, Y, \{a\}\}$. The identity mapping $f : (X,\tau) \to (Y,\sigma)$ is an R-map and it is not α-continuous (in fact, even not continuous).*
(**b**). *Observe that in* (**a**), *if we put $f(a) = a$, $f(b) = f(c) = b$, it will give us an α-continuous (discontinuous) R-map which is neither surjective nor injective.*

Example 12. (**a**). *Let $X = \{a,b,c,d\} = Y$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$, and $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a,b\}\}$. The identity mapping $f : (X,\tau) \to (Y,\sigma)$ is α-continuous (obviously even continuous), but it is not an R-map, because $f^{-1}(\{a\}) \notin \mathrm{RO}\,(X,\tau)$.*
(**b**). *Let (X,τ) and (Y,σ) be as above. Define $f : (X,\tau) \to (Y,\sigma)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = f(d) = c$. It is seen that f is α-continuous (continuous), not an R-map, and neither a surjective nor an injective mapping.*

Remark 4. *Notice that Examples 11 and 12 complete* Diagram *from [33, p.249] related to 'R-mapness' and 'continuity'.*

In [41, Theorem 3.5] Thompson has proved that irresolute surjections preserve the '$\mathcal{S}$-closed covering property'.

Example 13. (**a**). *Let $X = \{a,b,c\} = Y$, $\tau = \{\emptyset, X, \{a,c\}\}$, and $\sigma = \{\emptyset, Y, \{b\}\}$. Then the identity mapping $f : (X,\tau) \to (Y,\sigma)$ is an R-map and is not irresolute.*
(**b**). *The identity mapping from [34, Example 3.11] is irresolute but it is not an R-map.*

Thus, R-mapness and irresoluteness are independent of each other. Recall that α-continuity and irresoluteness are also independent notions [34, Example 3.11&Theorem 3.12].

Theorem 15. *Let a surjection $f : (X,\tau) \to (Y,\sigma)$ be irresolute, a.c.H., and a.o.W. If (X,τ) is $\mathcal{S}$-closed, then for each cover $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}\,(Y,\sigma)$ of Y, there exists a finite subset $\nabla_0 \subset \nabla$ such that $Y = \bigcup_{\alpha\in\nabla_0} \mathrm{cl}\,(V_\alpha)$.*

Proof: Let a family $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}\,(Y, \sigma)$ cover Y. Since f is irresolute, the family $\{f^{-1}(\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(V_\alpha)))) : \alpha \in \nabla\} \subset \mathrm{SO}\,(X, \tau)$ covers X. By its $\mathcal{S}$-closedness there exists a finite subset $\nabla_0 \subset \nabla$ with

$$X = \bigcup_{\alpha \in \nabla_0} \mathrm{cl}\left(f^{-1}(\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(V_\alpha))))\right).$$

But f is a.o.W., whence by [37, Theorem 11] we obtain

$$X = \bigcup_{\alpha \in \nabla_0} \mathrm{cl}\left(f^{-1}(\mathrm{int}\,(\mathrm{cl}\,(V_\alpha)))\right).$$

Thus, from [16, Proposition 3.1(c)] we get

$$Y = \bigcup_{\alpha \in \nabla_0} f\big(\mathrm{cl}\left(f^{-1}(\mathrm{int}\,(\mathrm{cl}\,(V_\alpha)))\right)\big) = \bigcup_{\alpha \in \nabla_0} \mathrm{cl}\,(V_\alpha).$$

□

Recall that 'a.o.W.' and 'a.o.S.' are independent notions [30, p.315]. Also, irresoluteness and 'a.c.H.' are independent of each other, see the example below.

Example 14. (**a**). *Let* $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{a, b\}\}$, *and* $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$ *and let* f *be the identity on* X. *Then* f *is a.c.H. and not irresolute.*
(**b**). *Let* $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$, *and* $\sigma = \{\emptyset, Y, \{a, b\}\}$. *Then the identity on* X *is irresolute but not a.c.H.*

Concluding the comparison of Theorems 14 and 15, we point out that the converse to an obiovus implication α*-continuity* $\Rightarrow$ *a.c.H.* fails in general.

Example 15. *Let* $X = \{a, b\} = Y$, $\tau = \{\emptyset, X\}$, *and* $\sigma = \{\emptyset, Y, \{a\}\}$. *Then the identity on* X *is a.c.H. but it is not* α*-continuous.*

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ., **12** (1983), 77–90.

[2] M. E. Abd El-Monsef, E. F. Lashien, A. A. Nasef, *Remarks on the $\star$-topology mappings*, Tamkang J. Math., **24**(1) (1993), 9–22.

[3] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24–32.

[4] N. Biswas, *On some mappings in topological spaces*, Bull. Cal. Math. Soc., **61** (1969), 127–135.

[5] D. E. Cameron, *Properties of $\mathcal{S}$-closed spaces*, Proc. Amer. Math. Soc., **72** (1978), 581–586.

[6] D. A Carnahan, *Some properties related to compactness in topological spaces*, Ph.D. thesis, University of Arkansas, 1973.

[7] C. G. Crossley, S. K. Hildebrand, *Semi-closure*, Texas J. Sci., **22**(2-3) (1971), 99–112.

[8] C. G. Crossley, S. K. Hildebrand, *Semi-topological properties*, Fundamenta Math., **74** (1972), 233–254.

[9] G. Di Maio, T. Noiri, *On* s*-closed spaces*, Indian J. Pure Appl. Math., **18**(3) (1987), 226–233.

[10] J. Dontchev, M. Ganster, T. Noiri, *On* p*-closed spaces*, Internat. J. Math. & Math. Sci., **24** (2000), 203–212.

[11] Z. Duszyński, *On some concepts of weak connectedness of topological spaces*, Acta Math. Hungar., **110**(1-2) (2006), 69–78.

[12] S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour, T. Noiri, *On* p*-regular spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **27(75)**(4) (1983), 311–315.

[13] M. Ganster, *Preopen sets and resolvable spaces*, Kyungpook Math. J., **27**(2) (1987), 135–143.

[14] T. Husain, *Almost continuous mappings*, Prace Mat., **10** (1966), 1–7.

[15] S. Jafari, T. Noiri, *Properties of β-connected spaces*, Acta Math. Hungar., **101**(3) (2003), 227–236.

[16] D. S. Janković, *A note on mappings of extremally disconnected spaces*, Acta Math. Hungar., **46**(1-2) (1985), 83–92.

[17] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.

[18] N. Levine, *Dense topologies*, Amer. Math. Monthly, **75** (1968), 847–853.

[19] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Math. Palermo, **19**(2) (1970), 89–96.

[20] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. and Phys. Soc. Egypt, **53** (1982), 47–53.

[21] A. S. Mashhour, I. A. Hasanein, S. N. El-Deeb, *A note on semi-continuity and precontinuity*, Indian J. Pure Appl. Math., **13**(10) (1982), 1119–1123.

[22] A. S. Mashhour, I. A. Hasanein, S. N. El-Deeb, *α-continuous and α-open mappings*, Acta Math. Hungar., **41**(3-4) (1983), 213–218.

[23] A. S. Mashhour, M. E. Abd El-Monsef, I. A. Hasanein, *On pretopological spaces*, Bull. Math. Soc. Math. R. S. Roumanie, **28(76)**(1) (1984), 39–45.

[24] A. Nebrunnová, *On certain generalizations of the notion of continuity*, Mat. Čas., **23**(4) (1973), 374–380.

[25] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math., **15** (1965), 961–970.

[26] T. Noiri, *Remarks on semi-open mappings*, Bull. Cal. Math. Soc., **65** (1973), 197–201.

[27] T. Noiri, *On semi-continuous mappings*, Lincei-Rend. Sc. fis. mat. e nat., **54** (1973), 210–214.

[28] T. Noiri, *A note on semi-continuous mappings*, Lincei-Rend. Sc. fis. mat. e nat., **55** (1973), 400–403.

[29] T. Noiri, *A note on hyperconnected sets*, Mat. Vesnik, **3**(16)(31) (1979), 53–60.

[30] T. Noiri, *Semi-continuity and weak-continuity*, Czechoslovak Math. J., **31**(**106**) (1981), 314–321.

[31] T. Noiri, *A function which preserves connected spaces*, Čas. pěst. mat., **107** (1982), 393–396.

[32] T. Noiri, *Almost-open functions*, Indian J. Math., **25**(1) (1983), 73–79.

[33] T. Noiri, *Super-continuity and some strong forms of continuity*, Indian J. Pure Appl. Math., **15**(3) (1984), 241–250.

[34] T. Noiri, *On α-continuous functions*, Čas. pěst. mat., **109** (1984), 118–126.

[35] V. Pipitone, G. Russo, *Spazi semiconnessi a spazi semiaperti*, Rend. Circ. Mat. Palermo (2), **24** (1975), 273–285.

[36] V. Popa, *Properties of H-almost continuous functions*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.), **31**(79) (1987), 163–168.

[37] D. A. Rose, *Weak continuity and almost continuity*, Internat. J. Math. & Math. Sci., **7**(2) (1984), 311–318.

[38] A. K. Sharma, *On some properties of hyperconnected spaces*, Mat. Vesnik, **1**(14)(29) (1977), 25–27.

[39] M. K. Singal, Asha Rani Singal, *Almost-continuous mappings*, Yokohama Math. J., **16** (1968), 63–73.

[40] T. Thompson, *$\mathcal{S}$-closed spaces*, Proc. Amer. Math. Soc., **60** (1976), 335–338.

[41] T. Thompson, *Semicontinuous and irresolute images of $\mathcal{S}$-closed spaces*, Proc. Amer. Math. Soc., **66**(2) (1977), 359–362.

[42] T. Thompson, *Characterizations of irreducible spaces*, Kyungpook Math. J., **21**(2) (1981), 191–194.

[43] A. Wilansky, *Topics in functional analysis*, Lecture Notes in Mathematics, vol. **45**, Springer-Verlag 1967.

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