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Some remarks on $f-$structures with parallelizable kernel

by
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Abstract

We consider a $(k, \mu)-$type curvature condition on metric $f-$structures with parallelizable kernel. Under some additional assumptions it imposes some restrictions on the manifold, particulary some integrable distributions arise in the tangent bundle. We provide a class of examples

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1 Introduction

The study of the curvature of metric $f-$structures with parallelizable kernel, viewed as generalizations of almost contact manifolds is our interest in the present paper. In recent years, a very extensive research has been done in contact geometry, see [1] and references therein, almost S-structures [8], [5], [12], almost C-structures [6], [11]. In particular almost S-structures satisfying the $(k, \mu)-$nullity condition were defined and studied in [5], [12].

2 Preliminaries

Let M^{2n+s} be a smooth manifold equipped with a (1-1) tensor φ of the tangent bundle such that $\varphi^3 + \varphi = 0$, see [7], with $dim \mathrm{Im}\varphi_x \equiv 2n, \forall x \in M$. Suppose further that there exist s global vectorfields $\xi_1, ..., \xi_s$ and s $1-$forms $\eta_1, ..., \eta_s$ s.t. $\varphi(\xi_j) = 0,\ \eta_j \circ \varphi = 0, \varphi^2 = -id + \sum\limits_{r=1}^{s} \eta_r \otimes \xi_r, \eta_j(\xi_i) = \delta_{ji}, (j, i = \overline{1, s}\ )$. One says that $\mathrm{M}^{2n+s}(\xi_j, \eta_j, \varphi)$ is a $f-$structure with parallelizable kernel, see [8].There are then compatible metrics on M, i.e. verifying $g(\varphi A, \varphi B) = g(A, B) - \sum \eta_j(A)\eta_j(B)$. A $2-$ form is given by $\Phi(A, B) = g(A, \varphi B)$, where A, B are arbitrary vectorfields on M. Denote by D the distribution orthogonal to the ξ's; $h_i := -\frac{1}{2} L_{\xi_i}(\varphi)$.

Lemma 2.1. *Let M be a Riemannian manifold with a $f-$structure with parallelizable kernel. For $X, Y \in \Gamma D$, for $i, j, r, \theta \in \{1, ...s\}$ one gets*

$$g(h_j(X), Y) = g(X, h_j(Y)) \Leftrightarrow d\Phi(\xi_j, X, Y) = d\Phi(\xi_j, \varphi X, \varphi Y) \quad (1)$$

$$h_j(\xi_i) = 0 \Leftrightarrow d\Phi(\xi_i, \xi_j, .) \equiv 0 \quad (2)$$

$$d\Phi(\xi_i, \xi_j, \xi_\theta) = 0 \quad (3)$$

$$g(h_j(\xi_r), C) = g(h_j(C), \xi_r) \quad (4)$$

if and only if each term in the following 2 identities is equal to another

$$g((L_{\xi_j}\varphi)(\xi_r), C) = g([\xi_j, \xi_r], \varphi C) = -d\Phi(\xi_j, \xi_r, C)$$

$$g((L_{\xi_j}\varphi)(C), \xi_r) = g([\xi_j, \varphi C], \xi_r) = -\eta_r(\xi_j, \varphi C) = -d\eta_r(\xi_j, \varphi C)$$

$$g(h_i(\xi_r), \xi_i) = g(h_i(\xi_i), \xi_r) = 0 \quad (5)$$

$$(h_j\varphi + \varphi h_r)(\xi_\theta) = 0 \Leftrightarrow \varphi^2[\xi_\theta, \xi_r] = 0 \Leftrightarrow \varphi[\xi_\theta, \xi_r] = 0 \quad (6)$$

$$(h_j\varphi + \varphi h_\theta)(X) = 0 \Leftrightarrow \left\{ \begin{array}{l} d\eta_r(X, \xi_j) = 0 \forall r = \overline{1, s} \\ g([\xi_j - \xi_\theta, X], Y) = \\ g([\xi_j - \xi_\theta, \varphi X], \varphi Y) \forall Y \in \Gamma D \end{array} \right\} \quad (7)$$

Proof: These are straightforward, for some detailed calculations see [9]. □

3 The generalized $(k, \mu)-$nullity condition

Suppose that on the manifold there exist functions $k_{\theta j}, \mu_{\theta j} \in F(M)$ s.t.

$$R(A, B)\xi_j = \sum_\theta k_{\theta j}(\eta_\theta(A)\varphi^2(B) - \eta_\theta(B)\varphi^2(A)) +$$

$$+ \sum_\theta \mu_{\theta j}(\eta_\theta(B)h_\theta(A) - \eta_\theta(A)h_\theta(B))(*)$$

with Riemannian curvature tensor

$$R(A, B)C = \nabla_A\nabla_B C - \nabla_B\nabla_A C - \nabla_{[A,B]}C;$$

A, B, C are any vectorfields on M. We say then that M satisfies on the generalized $(k, \mu)-$ nullity condition. Conditions of this type were considered and studied by Blair, Koufogiorgos, Papantoniu [10], Boeckx [3] for contact metric manifolds and more recently by Cappelletti Montano, di Terlizzi [5] for almost S-manifolds; see also Cabrerizo, Fernandez, Fernandez [4].

Then it follows at once $R(\xi_r, \xi_i)\xi_j = -\frac{1}{2}(\mu_{ij} + \mu_{rj})\varphi[\xi_r, \xi_i]$, $R(X, Y)\xi_j = 0$; $g(R(\xi_j, \xi_r)X, Y) = g(R(\xi_r, \xi_i)\xi_j, \xi_\theta) = 0$, $R(X, \xi_r)\xi_j = k_{rj}X + \mu_{rj}h_r(X)(**)$ where X, Y are any sections in the distribution D.

The symmetry $g(R(X,\xi_r)\xi_j,\xi_\theta)=-g(R(\xi_j,\xi_\theta)\xi_r,X)$ gives

$$\frac{1}{2}\mu_{rj}d\eta_\theta(\varphi X,\xi_r)=\frac{1}{2}(\mu_{jr}+\mu_{\theta r})g(\varphi\left[\xi_j,\xi_\theta\right],X),\mu_{rj}d\eta_j(\varphi X,\xi_r)=0$$

The symmetry $g(R(X,\xi_j)Y,\xi_r)=g(R(Y,\xi_r)X,\xi_j)$ gives

$$g(k_{jr}X+\mu_{jr}h_j(X),Y)=g(k_{rj}Y+\mu_{rj}h_r(Y),X)$$

Bianchi identity $g(R(X,\xi_r)\xi_j,Y)+g(R(\xi_j,X)\xi_r,Y)=0$ gives

$$g(k_{rj}X+\mu_{rj}h_r(X),Y)=g(k_{jr}X+\mu_{jr}h_j(X),Y),\mu_{rj}g(h_r(X),Y)=\mu_{rj}(h_r(Y),X)$$

From now on assume that $\mu_{jr}+\mu_{\theta r}\neq 0\forall j,r,\theta\in\{1,...s\}$. It follows that $d\eta_j(\varphi X,\xi_r)=g(h_r(X),\xi_j)=g(\left[\xi_r,\varphi X\right],\xi_j)=g(\varphi\left[\xi_j,\xi_\theta\right],X)=0$:

Proposition 3.1. *Let M be a Riemannian manifold with compatible $f-$structure with parallelizable kernel satisfying the generalized $(k,\mu)-$ condition, as above. Suppose that $\mu_{jr}+\mu_{\theta r}\neq 0\forall j,r,\theta\in\{1,...s\}$ Then the distribution $ker\varphi$ generated by the vectorfields ξ is integrable.*

Now, $g(h_r(X),\xi_j)=g(h_r(\xi_j),X),g(h_r(Y),X)=g(h_r(X),Y),X,Y\in\Gamma D$ so that, using the Lemma,(1)-(7), h_i's are symmetric endomorphisms and $h_r\varphi+\varphi h_r=0$. Finally, $R(\xi_r,\xi_i)\xi_j=0$, so that the leaves of the foliation $ker\varphi$ are totally geodesic and flat; $L_Xg(\xi_j,\xi_r)=0$ is essentially required. Moreover $R(X,\xi_i)\xi_j=R(X,\xi_j)\xi_i$, from Bianchi identity, since now $g(R(\xi_i,\xi_j)X,A)=0$. From $R(X,\xi_i)\xi_j=k_{ij}X+\mu_{ij}h_i(X)$ one further gets $\varphi R(\varphi X,\xi_i)\xi_j=-k_{ij}X+\mu_{ij}h_i(X);k_{ji}=k_{ij};\mu_{ij}h_i=\mu_{ji}h_j$.

4 Example

Let $M=R^{2+s}$ (or possibly an open subset). For simplicity the case $s=2$ is exhibited here. The $1-$ forms $\eta_j=dx_j+ydz$ and vectorfields $\xi_j=\frac{\partial}{\partial x_j}$ are globally defined. Conventions on exterior differentiation are s.t., e.g. $(dy\wedge dz)(\frac{\partial}{\partial y},\frac{\partial}{\partial z})=1$; $d\eta(A,B)=A\eta(B)-B\eta(A)-\eta[A,B]$. Following [10], the tensors g, φ with respect to the frame $\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial y},\frac{\partial}{\partial z}$ are

$$g=\begin{pmatrix}1&0&0&-a\\0&1&0&-a\\0&0&1&-b\\-a&-a&-b&1+2a^2+b^2\end{pmatrix}$$

$$\varphi=\begin{pmatrix}0&0&-a&ab\\0&0&-a&ab\\0&0&-b&1+b^2\\0&0&-1&b\end{pmatrix}$$

where a, b are real functions that will satisfy on some conditions (notice the conventions in [10] are different). Now, $\varphi^2 = -id + \sum \eta_j \otimes \xi_j$ is verified iff $a = -y$, which also gives compatibility with the metric. The global orthonormal frame $\xi_1, \xi_2, \frac{\partial}{\partial y}, \varphi(\frac{\partial}{\partial y})$ will be appropriate to calculate the curvature of the manifold. The endomorphisms h verify

$$h_j(\frac{\partial}{\partial y}) = \frac{1}{2}\frac{\partial b}{\partial x_j}\frac{\partial}{\partial y}; h_j(\varphi\frac{\partial}{\partial y}) = -\frac{1}{2}\frac{\partial b}{\partial x_j}\varphi\frac{\partial}{\partial y}$$

Straightforward calculations express the curvature and taking

$$b = \varepsilon(z)y + \psi_1(z)x_1 + \psi_2(z)x_2 + \nu(z); \psi_l(z) = c_l \exp(-2E),$$

E being any primitive of the function ε and c_l being constant gives a Riemannian manifold M that satisfies on the generalized (k, μ)- condition. In the original example of [10], $s = 1$ and b is to be chosen s.t. $\varepsilon(z) = \frac{1}{4(1-z)}$ in an appropriate open subset of R^3.

5 The eigenspaces of h_i's

It is clear from $\mu_{ij}h_i = \mu_{ji}h_j, \mu_{ij} \neq 0$ that all h_i's have the same eigenspaces, let denote those $V^0 \oplus \ker\varphi, V^l, V^{-l} = \varphi V^l$, corresponding to eigenvalues $0, \lambda_i^l, -\lambda_i^l$ of h_i, $l \leq n$. Second Bianchi identity for $\sum(\underset{Y}{\nabla} R)(X, \xi_i)\xi_j$ takes into consideration(**), then $g([\xi_i, Y], \xi_j) = 0$, $g\left(\underset{\xi_i}{\nabla}\xi_j, X\right) = 0$, $g\left(\xi_i, \underset{Y}{\nabla}\xi_j\right) = g\left(\xi_i, \underset{\xi_j}{\nabla}Y\right) = 0$, so further

$$-\left(R\left(\underset{Y}{\nabla}X, \xi_i\right)\xi_j + R\left(\xi_i, \underset{X}{\nabla}Y\right)\xi_j\right) = -\left(R\left([Y,X], \xi_i\right)\xi_j\right) =$$

$$-\left(k_{ij}\pi^D[Y,X] + \mu_{ij}h_i\pi^D[Y,X]\right);$$

$$-\left(R\left(X, \underset{Y}{\nabla}\xi_i\right)\xi_j + R\left(\underset{X}{\nabla}\xi_i, Y\right)\xi_j + R\left(\underset{\xi_i}{\nabla}Y, X\right)\xi_j + R\left(Y, \underset{\xi_i}{\nabla}X\right)\xi_j\right) = 0;$$

$$-R(X,\xi_i)\underset{Y}{\nabla}\xi_j - R(\xi_i, Y)\underset{X}{\nabla}\xi_j - R(Y,X)\underset{\xi_i}{\nabla}\xi_j =$$

$$= R\left(\xi_i, \underset{Y}{\nabla}\xi_j\right)X - R\left(\xi_i, \underset{X}{\nabla}\xi_j\right)Y.$$

In particular, for $Y = U, X = V$ sections in the eigendistribution D^l denoting generic $c = k_{ij} + \mu_{ij}\lambda_i^l$, second Bianchi identity called above gives

$$U(c)V - V(c)U + c[U,V] - \left(k_{ij}\pi^D[U,V] + \mu_{ij}h_i\pi^D[U,V]\right) +$$
$$R\left(\xi_i, \underset{U}{\nabla}\xi_j\right)V + R\left(\underset{V}{\nabla}\xi_j, \xi_i\right)U = 0$$

Taking now $D-$component, one gets:

$$U(c)V - V(c)U + \mu_{ij}\left(\lambda\pi^{^D}[U,V] - h_i\pi^D[U,V]\right) = 0$$

Consequently $\lambda\pi^{^D}[U,V] - h_i\pi^D[U,V] \in \Gamma D^l$, which actually is $\pi^D[U,V] \in \Gamma D^l$, since h are diagonalizable. Observe that if the 2-forms $d\eta_j$ are all proportional to Φ then $g([U,V],\xi_r) = 0$, for $l \neq 0$, being any two different D^l ortoghonal to each other; one gets

Theorem 5.1. *Let M be a Riemannian manifold with compatible $f-$structure with parallelizable kernel satisfying the generalized $(k,\mu)-$condition. Suppose that each of the 2-forms $d\eta_j$ are proportional to Φ. Then the eigendistributions D^l, $l \neq 0$ are integrable.*

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